A Novel Vortex Method to Investigate Wind Turbine Near-Wake Characteristics

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We utilize vortex methods to identify possible mechanisms which cause vortex filaments to break-up in the wake behind a horizontal axis wind turbine (HAWT). This study was inspired by the flow characteristics observed in PIV measurements from an experiment conducted at the Atmospheric Boundary Layer (ABL) tunnel at ISU using a scaled model of an industry standard HAWT. Measurements revealed the presence of a tip vortex filament and a secondary vortex filament emanating from the mid-span location of the blade. The mid-span vortex has twice the circulation of the tip vortex filament and the vortices break-up at $X/D = 0.5$ axial station. Both filaments showed expansion with the wake and a merging effect between the filaments. Our investigation using a vortex-blob based method (2D) and a filament based method (3D) captured wake expansion of both filaments as well as the converging behavior, which could result in wake instability.

\textbf{Nomenclature}

\begin{align*}
\text{ABL} & = \text{Atmospheric boundary layer} \\
C_T & = \text{Thrust coefficient} \\
D & = \text{Diameter of the rotor} \\
\text{HAWT} & = \text{Horizontal axis wind turbine} \\
\text{PIV} & = \text{Particle image velocimetry} \\
\Psi & = \text{Stream function} \\
\omega & = \text{Vorticity} \\
\vec{u} & = \text{Velocity vector} \\
\nu & = \text{Kinematic Viscosity} \\
\Gamma_j & = \text{Circulation Strength} \\
\varepsilon & = \text{Smoothing Parameter} \\
\text{FMM} & = \text{Fast Multipole Methods} \\
\text{PSE} & = \text{Particle Strength Exchange} \\
\text{CS} & = \text{Core-Spreading} \\
\Delta t & = \text{Time step} \\
k & = \text{Wave number} \\
a & = \text{Radius of the filament} \\
m & = \text{Order of the modified Bessel functions} \\
n & = \text{Order of the vortex core model} \\
\rho, \theta & = \text{Polar coordinates} \\
K_{nm}^r, I_{nm}^r & = \text{Modified Bessel functions}
\end{align*}

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I. Introduction

Coherent structures in the wind turbine wake and their roles in the stability of the wake have been analyzed through many high fidelity grid based computational techniques. The wind turbine flow in the near wake region (X/D<1.0) contains very complex flow characteristics and the traditional grid based methods require substantial rotational turbulence modeling and upstream turbulence modeling. With the recent advances in computer technology, Lagrangian vortex methods have gained popularity in simulating a multitude of microscopic and macroscopic flow phenomenon. Vortex methods also present the user with advantages such as absence of numerical viscosity, easier formulation as the vorticity transport equation gets rid of pressure term, and the absence of complex grid generation. In this manuscript, we propose blob and filament based vortex-methods (2D and 3D) to model the near-wake vortex structures and the conditions leading up to the vortex “break-up” phenomenon due to mutually and internally induced strain fields.

This study was inspired by a wind turbine wake analysis study conducted at the Atmospheric Boundary Layer (ABL) wind tunnel at Iowa State University, using a scaled mode (1:350) of a 2 MW commercial wind turbine by TPI composites. Velocity fields in the near wake and far wake locations were measured by a high resolution digital PIV system. A phase-averaged measurement depicted in Figure 1, elucidated the presence of a secondary vortex filament emanating from 0.3D of the blade. The circulation strength in the mid-span vortex patch is twice as large as the tip vortex patches. The vortex patches break up around X/D = 0.5 axial station into turbulent eddies, which has not been observed in previous wind turbine wake studies which showed a dominating tip vortex propagating into far wake regions (X/D>1.0) before break-up. We suggest modeling this problem with a corrected rotor model to introduce tip and mid-span and hub vortices and observe the induced instabilities as the wake propagates downstream.

![Normalized Vorticity Plots based on PIV](image)

**Fig1**: Normalized Vorticity Plots based on PIV

A specific region of interest is highlighted in Figure 1(a) which clearly shows four discreet vortex patches present before the critical break-up point. An ensemble-averaged solution shows both filaments axially expanding and converging after X/D = 0.5 as highlighted in Figure 1(b) where interactions between filaments cause the formation of turbulent eddies in the shear layer. Dynamics of the helical vortex filaments are governed by advection, viscous diffusion and the stretching of the helical vortex filaments. A plethora of studies done previously managed to simulate the rotor wakes of helicopters and wind turbines using blob-based and filament based vortex methods. Filament based methods have utilized free-wake and prescribed wake models, while employing simplified rotor models such as lifting line, actuator line and actuator disk models. C. He and J. Zhao used a viscous vortex blob based method to simulate 3D flow structures shed by a helicopter rotor using the lifting line theorem, where the blade circulation was calculated using Kutta-Joukowski theorem. Near-wake downwash velocities from the solutions were compared to experimental data set with satisfactory agreement. Leishman studied the ground effect on helicopter wake using a filament based method, where a mirrored wake solution was used to simulate the ground plane. Leishman solves the free-wake method in an iterative and a time marching method and concludes that best results are obtained from the time-marching algorithm. H. Abedi used free-wake and prescribed wake models to simulate the propagation of a tip and hub vortex...
filaments which showed instabilities propagating to far wake. Abedi utilized the Vortex-Lattice-Method (VLM) based on the lifting line theorem, in which the blade surface is replaced by vortex panels. Vortex methods also allow the users to model flow instabilities, such as mutual induction, long-wave instability and short-wave instability which play dominant roles in rotor wake problems.

II. Methodology

Early work done in developing potential functions for velocity fields induced by helical filaments was investigated as a preliminary analysis. A potential function was developed for the stream function solution induced by a propagating filament using modified Bessel functions in Kapteyn-Kummer series. The stream function inside the helix is given by,

\[ \Psi = \frac{\Gamma}{4\pi k^2} - \frac{\Gamma a \rho}{\pi k^2} \sum_{m=1}^{N} K_m^2 \left( \frac{am}{k} \right) I_m \left( \frac{\rho m}{k} \right) \cos(m\theta) \quad \rho < a \quad (1) \]

While the stream function outside the filament is given by,

\[ \Psi = -\frac{\Gamma}{2\pi} \ln \rho - \frac{\Gamma a \rho}{\pi k^2} \sum_{m=1}^{N} K_m^2 \left( \frac{am}{k} \right) I_m \left( \frac{\rho m}{k} \right) \cos(m\theta) \quad \rho > a \quad (2) \]

This causes a singularity at the location of the filament (\( \rho = a \)) and should be omitted during the calculations. The super-position principle for stream functions can be used to derive an expression for multiple filaments where, \( a_1 < \rho < a_2 \) region between the helices will reveal the dynamics present. However, this approach doesn’t take the unsteady effects into account which prompted an investigation into the formulation of unsteady Lagrange methods. Lagrange particle methods has been explored initially by Alexander Chorin, where a vortex roll-up has been simulated using a de-singularized Biot-Savart relationships. In the recent work done, criterions such as particle strength exchange via viscous splitting techniques have been employed for an accurate modeling of viscous effects. Lagrange particle methods also present problem of non-physical velocity gradients induced by nodal concentrations which can be alleviated using “remeshing” techniques. Recent developments in Tree codes and Fast Multipole Methods (FMM) have reduced the computational costs due to direct integration from \( N^2 \) to \( N \log (N) \).

Governing dynamics of vortex methods in viscous, incompressible flow stem from the vorticity transport equation as given by,

\[ \frac{\partial \vec{\omega}}{\partial t} + (\vec{u} \cdot \nabla) \vec{\omega} = (\vec{\omega} \cdot \nabla) \vec{u} + \nu \nabla^2 \vec{\omega} \quad (3) \]

The first term on the right hand side stands for the stretching of vortices which is a three dimensional phenomenon and the second term governs viscous diffusion. The advection term \((\vec{u} \cdot \nabla) \vec{\omega}\) can be solved using a kernel function of varying order and the Biot-Savart law. For the purpose of this abstract we present the formulation of a two dimensional algorithm which will later be extended to an unsteady 3D vortex filament algorithm. The induced velocity components can be calculated using the Biot-Savart Kernels,

\[ \vec{u} = \sum_{j=1}^{N} K(r, \delta)(\vec{x} - \vec{x}_j) \times \vec{I}_j \quad (4) \]

Where \( \delta \) represents the width of the blob and \( r = |\vec{x} - \vec{x}_j| \). The order of the kernel function can be changed based on the simulation but for the purpose of this manuscript we have used a second order kernel (a Gaussian distribution) as given by,

\[ K(r, \delta)(\vec{x} - \vec{x}_j) = \frac{(\vec{x} - \vec{x}_j)}{2\pi \tau^2} \left( 1 - e^{-r^2/8\delta^2} \right) \quad (5) \]

The formulation of the kernel functions are subjected to change depending on the simulation run.
A variety of methods have been suggested to model viscous diffusion from core-spreading technique to Particle Strength Exchange (PSE) method. PSE method redistributes circulation strength among particles as advanced in time while conserving the total circulation in all the vortex blobs. The time derivative of $\Gamma$ can be presented as,

$$\frac{d\Gamma_p}{dt} = v e^{-2} \sum_q (\Gamma_q - \Gamma_p) \eta(\frac{x_p - x_q}{\epsilon})$$

(6)

Where $p$ denotes the particle of interest and the $q$ denotes the rest of the vortex blobs. The function $\eta$ can be determined as,

$$\eta(\bar{x}) = \frac{C}{1 + |\bar{x}|^2}$$

(7)

for $|\bar{x}| \leq 2$. If the $|\bar{x}| > 2$, the value of the function goes to zero. The constant $C$ has a value of 0.835. A simplistic approach based on core-spreading technique has been used in the present study as where the time dependent circulation can be calculated as,

$$\Gamma = \Gamma_0(1 - e^{4ut})$$

(8)

where, the initial circulation of a blob is given as $\Gamma_0$. However this approach contains numerical inconsistencies which can be alleviated via reducing the size of the vortex blobs.

Both advection and PSE equations are advanced in time using a 4th order Runge-Kutta scheme. In a three dimensional formulation presented in the final manuscript vortex stretching effects will also be taken into account. The final velocity for each particle at the end of a time step can be calculated as,

$$\bar{U} = \bar{U}_{\infty} + \bar{U}_{\text{induced}}$$

(9),

where, $\bar{U}_{\infty}$ represents the free-stream. The vorticity for each 2D vortex blob can be derived as,

$$\bar{\omega}_j = \sum_{j=1}^{N} \Gamma_j \frac{e^{-r^2/\delta^2}}{\pi\delta^2}$$

(10)

The point vortices can be interpolated to a Cartesian grid in order to plot the vorticity contours in a given field. The final manuscript will also cover re-meshing techniques to overcome nonphysical particle concentrations in high velocity gradients.

The 2D algorithm was extended to a three-dimensional framework using a free-wake, filament based method. The free-wake method takes boundary layer flow into account while the filaments are free to deform, thus having the ability to capture the expansion of the wake and the conditions leading up to an instability. Stretching effects of the vortices can be observed along with the viscous diffusion. The vector representation for the velocity induced by a unit filament (AB) at a given point(C) is elucidated in Figure 3.
Velocity induced at the given point from the unit filament is given by,

$$V_{ind} = \frac{\Gamma}{4\pi} \frac{(r_1 + r_2)(\vec{r}_1 \times \vec{r}_2)}{\vec{r}_1 \cdot \vec{r}_2 + \delta^2}$$  \hspace{1cm} (11)$$

where, $\delta$ represents a smoothing factor or a cut-off radius. This expression can be further expanded to add the viscous effects to the vortex core using a simple diffusion model as$^{10,11}$,

$$V_{ind} = K_v \frac{\Gamma}{4\pi} \frac{(r_1 + r_2)(\vec{r}_1 \times \vec{r}_2)}{\vec{r}_1 \cdot \vec{r}_2}$$  \hspace{1cm} (12)$$

where,

$$K_v = \frac{h^n}{(r_c^{2n} + h^{2n})^{1/n}}$$  \hspace{1cm} (13)$$

The $h$ stands for the perpendicular distance from the filament to the point of interest given by,

$$h = \frac{|\vec{r}_1 \times \vec{r}_2|}{|\vec{L}|}$$  \hspace{1cm} (14)$$

The factor $n = 2$ is recommended for the tip vortices of a rotor and the core radius can be found using a growth model $r_c = \sqrt{4\alpha\nu t}$ where $t$ represents the current time and $\alpha = 1.25643$. Kinematic viscosity is denoted as $\nu$. A flow chart has been presented in Figure 4 depicting the flow of information and the time integration loop. For the purpose of this manuscript, we will be using an unbounded filament with known circulation.
Fig 4: Construction of a simple algorithm

III. Results and Discussion

Preliminary results for these simulations were obtained from the stream function representations discussed previously using modified Bessel functions. An iso-surface tangential to the propagating filament (+Z direction) is shown in Figure 5, along with its stream-function. In the stream function representation, the filament is intersecting normal to the XY plane at (1,0) location. At the point of intersection, a high concentration of streamlines is present suggesting the presence of a sharp velocity gradient in which complex flow features are present.

Fig 5: Velocity iso-surface (left) and stream-function (right)
A stream-function representation for a tip vortex filament alongside a mid-span vortex filament was developed using the superposition principle. Theoretical velocities were compared to the experimental solutions obtained from ensemble-averaged PIV measurements as shown in Figure 6.

Fig 6: stream-function (left) and velocity comparison (right)

The complexities of the flow is revealed at $X = 0.6$ and $X = 1.0$ where the filaments intersect orthogonally with the plane of interest. The stream function solution is asymmetric in the radial direction with streamlines converging and diverging near singularities suggesting significant gradients in velocity fluctuations. The velocity distributions between filament intersections show a rapid decrease in velocity magnitude and an immediate increase as the observer gets closer to the tip vortex. The analytical solution manages to capture the decreasing trend in velocity magnitude aft of the mid-span filament but doesn’t capture the gradual increase towards the tip vortex. The current model doesn’t evaluate velocity induced by the consecutive turns in $z$ direction or the diameter of the filament (core size). The effects of the ambient turbulence also do not factor in to the model.

1. Kelvin Helmholtz Instability (Validation 1)

Kelvin-Helmholtz instability occurs due to a velocity shear in a single continuous fluid. This is a short-wave instability which can be simulated with a de-singularized kernel functions given as\(^{20}\):

\[
\frac{dx_i}{dt} = -\frac{1}{2N} \sum_{j=1, i \neq j}^{N} \frac{\sinh(2\pi(y_i - y_j))}{\cosh(2\pi(y_i - y_j)) - \cos(2\pi(x_i - x_j)) + \delta^2}
\]

\[
\frac{dy_i}{dt} = -\frac{1}{2N} \sum_{j=1, i \neq j}^{N} \frac{\sinh(2\pi(x_i - x_j))}{\cosh(2\pi(y_i - y_j)) - \cos(2\pi(x_i - x_j)) + \delta^2}
\]

where the kernels are de-singularized using by declaring a blob width of $\delta^{12}$. The initial Lagrange grid was generated using with the following circulation strengths and spatial perturbations.

\[x_i = \Gamma_i + 0.01 \sin(2\pi f_i)\]
\[y_i = -0.01 \sin(2\pi f_i)\]

The circulation strengths are distributed among the points as $\Gamma_i = (i - 1)\Delta \Gamma$. The evolution of this instability in the time is shown in Figure 3. The simulation was run using 400 blobs and a blob width of 0.5. As the time progresses, the vortex filament starts to deform with a counter-clockwise moment. However, the simulation...
doesn’t take surface tension of the material into account. This reproduction of KH instability phenomenon paved the way to the development of a generalized solver based on Biot-Savart law and a better understanding of vortex blob methods.

2. Elliptic Wing Tip Vortices Simulation (Validation 2)

Based on the work by Krasney\textsuperscript{13}, we also reproduced the case of tip vortices shed by an elliptically loaded wing. A vortex filament with equally spaced blobs was declared as an initial grid. The circulation strength changed from positive to negative along the filament with the particles in the middle having the lowest circulations. Minima and maxima occurred at the end of the filaments. The simulation was run for 4 seconds with 1000 vortex blobs and the final results are shown in Figure 4.

![Fig 7: Evolution of the KH Instability](image-url)
The Lagrange grid and the circulation strengths were interpolated onto a regular grid in order to construct a vorticity plot as shown in Figure 5 using an interpolation kernel.

The solution was symmetric and produced tip vorticity patches of opposite signs as intended. A qualitative observation with the literature confirmed the validity and the accuracy of the algorithm. This initial simulation with an inviscid flow assumption was configured to include viscous effects via core-spreading (CS) and comparison is shown in Figure 10.
The solutions obtained from core-spreading viscous diffusion scheme shows variations in vortex roll-up and the expansion of the filament compared to the inviscid case. Introduction of viscosity has adversely affected the rate of roll-up on the tip vortices as well as the stretching of the filament. A high velocity gradient can be seen at the core of each vortex which may lead to numerical instabilities if neglected.

3. **Formation of self-similar vortex sheets (Validation)**

The formation of the self-similar vortex sheets can also be studied using the de-singularized Biot-Savart kernel. Position of the Lagrange particles is advanced in time via fourth order RK4 and the initial grid location and the corresponding circulation is given by,

\[
x(\alpha, 0) = \cos(\alpha) , \quad y(\alpha, 0) = \sin(\alpha) , \quad \text{and} \quad \Gamma(\alpha) = (1 - \cos(\alpha))/2.0
\]

where, \(0 \leq \alpha \leq \pi\). The vortex ring has an initial velocity and the initial and the final results are shown in Figure 11.

The final solution depicts a roll-up in identical vortex rings with an extending tail in the axis of symmetry. Further simulations require a smoothing scheme where Lagrange points can be added between rapidly expanding filaments to prevent non-physical artifacts.
4. **Wind Turbine Rotor Simulation**

A simplified rotor model based on the actuator disk model was implemented to model the tip vortices emanating from a rotor\(^{15}\). A vortex blob is generated at each tip location, at every time step, with opposite circulation strengths and an initial unit velocity. The circulation strength magnitude for each tip vortex is given by,

\[
\Gamma = \frac{\Delta P}{\rho} \Delta t \quad (17)
\]

where \(\Delta P = \frac{1}{2} \rho U^2 C_T\). A thrust coefficient \(C_T\), was determined assuming the optimum axial induction \((\alpha = 1/3)\). The case was run for 5.0 seconds and the results are shown in Figure 12.

![Image of rotor and interpolated vorticity solution](image)

**Fig 12**: Particle locations (Right) and Interpolated vorticity solution (Right)

The solution shows symmetry along the axis \(Y = 0.0\) as the wake propagates downstream. Vorticity was calculated for individual points and interpolated on to a regular grid. The positive and negative circulation strengths had resulted in two tip vortex filaments with equal and opposite vortex strengths. The wake has undergone expansion as the vortex blobs propagate downstream.

This 2D method can be further extended to simulate the interactions between a mid-span and tip vortex, where a secondary stream of vortices is shed from the \(R = 0.3\) location analogous to what has been seen in the PIV measurements. The circulation of the mid-span vortex has been set to a value twice as high as that of the tip vortex filament. The aggregate behavior of the discreet vortices in XY plane for a normalized hub velocity of 1.0 is provided in Figure 13. The mid-span vortex blobs show roll up in the near-wake regions, while the tip vortices roll-up in the far-wake. The wake shows an expansion in the near-wake region and a decrease in proximity between the mid-span and the tip vortex blobs at \(X = 0.5D\), as highlighted in red, paving the way for a short wave instability. Short wave instability will lead to merging of vortical cores thus causing break-up and diffusion into the far wake.
Short wave instability can also be simulated using the vortex blobs. Sample solutions for two vortices with similar and different circulation distributions are shown in Figure 14. Lagrange grid based solution was interpolated into a 2D Cartesian grid where the potential function was obtained.

Short-wave instabilities or elliptical instabilities which take place between vortices with similar circulations yield a symmetric elliptical vortex after they merge\(^{16}\). An asymmetric solution is obtained for the merging between two vortices with different circulations, similar to the situation observed between mid-span and tip vortices.

The three-dimensional filament based algorithm was utilized to run inviscid, incompressible simulations of the vortex behavior. A Kelvin wave assumption was used to determine the initial geometry of an un-bounded vortex filament. Normalized input including wave radius and the wave length has been used for the initial conditions. No boundary conditions or a specific rotor model has been used. The Kelvin wave equations are given by\(^{17}\),

\[
\begin{align*}
y &= a \cdot \cos(kx - \omega t) \\
z &= a \cdot \sin(kx - \omega t)
\end{align*}
\]

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**Fig 13:** Vortex interactions in 2D

**Fig 14:** Stages of a core instability potential function
where \( k \) stands for wave number and \( a \) stands for the amplitude of the wave. Results obtained from the filament based algorithm are shown in Figure 15.

![Wave Positions (3D)](image1)

(a) Initial solution

![Wave Positions (3D)](image2)

(b) Final Solution (isometric)

![Wave Positions (3D)](image3)

(c) Final Solution (front)

![Wave Positions (3D)](image4)

(d) Final Solution (top)

**Fig 15:** Deformation of the filament due to induced velocities

As the time progresses both filaments showed expansion. The axial propagation was neglected for the moment to focus on the vortex stretching phenomenon. Vortex stretching term doesn’t have to be explicitly solved as each point is part of a continuous filament. The mid-span vortex stretches surpassing the tip vortex filament. The mid-span vortex gets closer to the tip vortex, which could lead up to break-up instability. Some of the instances where both filaments have converged on to each other are highlighted in red boxes as potential candidates for break-up phenomenon. Such qualitative trends obtained from the filament based algorithm agreed with the trends observed in the 2D vortex-blob algorithm as shown in Figure 13.
IV. Conclusion

Vortex blob methods (2D) and filament based methods (3D) were employed to understand the behavior of flow structures observed in the near wake regions of a HAWT. PIV wake measurements of a scaled (1:350) turbine revealed the presence of a tip vortex, a mid-span vortex and a hub vortex that dissipates aft of the nacelle. The tip and the mid-span vortex advect downstream until $X/D = 0.5$, where they start to break-up due to the presence of an external strain field. As shown in Figure 1(b) both filaments show a converging behavior downstream. The vortex blobs break-up into turbulent eddies creating a shear layer where momentum is transferred from high velocity fluid outside wake to the low velocity fluid elements. This is also known as wake recharging which plays a crucial role in determining the distance between each turbine in a wind-farm setting. Absence of volumetric measurements presented a challenge in identifying a primary mechanism behind vortex break-up experimentally. Unsteady vortex methods allowed us to perform a qualitative investigation into the near-wake region, with a relatively lower computational cost and a smaller implementation time line.

Implementation of the vortex blob based method showed its ability to perform a variety of validations, including vortex roll-up in an elliptically loaded wing, KH instability and the deformation of a self-similar vortex ring. The vortex blob method coupled with an actuator disk model was used to simulate the aggregate behavior of tip vortices and mid-span vortices. The mid-span vortex, with twice the circulation of tip vortex blobs, showed roll-up behavior in the near-wake region. A convergence between tip and mid-span vortices was observed prior to $X/D = 0.5$. Such convergence leads into short-wave instabilities in the vortex cores, which could result in the final-break up into smaller eddies as the vortices merge to create asymmetric vortex structures. Simulation of unbounded tip and mid-span vortex filaments via a three-dimensional vortex filament algorithm also confirmed the converging behavior previously observed as a precursor to the short-wave instability, thus validating the behavior of seen in ensemble-averaged PIV measurements.

As for the future work, a rotor model will be introduced to the filament based solution with a criterion to handle instability. Lifting line based rotor models are currently being investigated for this purpose. This will place proper boundary and initial conditions to ensure the accuracy of the filament based simulations. Using a tree code or a Fast Multipole Method (FMM) may improve the computational costs that arise due to $N^2$ calculations that has to be performed at every given time step. Simulations resulting in severe vortex expansion or stretching may lead to non-physical numerical artifacts that can be mitigated via a smoothing criterion which introduces grid points into expanding filaments. A vortex-in-cell approach may also alleviate the non-physical numerical effects due to grid points congregating at regions where high velocity gradients are present. However, this approach also introduces numerical diffusion and interpolation errors which tend to grow as the time progresses.
References


