Chapter 1   Review of Vector Algebra

1.1.  Vector

1.1.1  Definition of a Vector

Definition: A vector is a quantity that possesses both magnitude and direction, and obeys the parallelogram law of addition.

Commutative: \( \vec{A} + \vec{B} = \vec{B} + \vec{A} \)
\( \vec{C} = \vec{A} + \vec{B} \)
\( \vec{D} = \vec{A} + \vec{B} \)

Unit vector: \( \hat{e}_A = \frac{\vec{A}}{|\vec{A}|} \)

1.1.2  Scalar Product (Dot Product)

\( \vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta \)

Where \( |\vec{A}| \) and \( |\vec{B}| \) are the magnitude of the vectors \( \vec{A} \) and \( \vec{B} \).

\( \theta \) (\( 0 \leq \theta \leq \pi \)) is the angle between the vectors \( \vec{A} \) and \( \vec{B} \) when they are arranged “tail to tail”.

- \( |\vec{B}| \cos \theta \) is the projection of vector \( \vec{B} \) to vector \( \vec{A} \).
- If \( \theta = \pi / 2 \), \( \vec{A} \) and \( \vec{B} \) are orthogonal to each other, and \( \vec{A} \cdot \vec{B} = 0 \)
- Commutative: \( \vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A} \)

Example:

Work done by a force \( \vec{F} \) during an infinitesimal displacement \( \vec{S} \)
1.1.3. Vector Product (Cross Product)

\[ \vec{A} \times \vec{B} = |\vec{A}| |\vec{B}| \sin \theta \hat{e}_n \]

Where \( \hat{e}_n \) is the unit vector normal to the plane containing \( \vec{A} \) and \( \vec{B} \). Direction is determined according to the “right-hand” rule. \( 0 \leq \theta \leq \pi \)

\[ |\vec{A} \times \vec{B}| = \text{Area of the parallelogram} \]

If the two vectors are parallel, that is if \( \theta = 0 \) or \( \theta = \pi \), then \( \vec{A} \times \vec{B} = \vec{0} \).

- Vector product is not commutative. i.e., \( \vec{A} \times \vec{B} \neq \vec{B} \times \vec{A} \). However, \( \vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \)

Application example:

Moment about \( O \): \( \vec{M}_O = \vec{R} \times \vec{F} \)
1.1.4. **Scalar Triple Product:**

\[ \vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{C} \times \vec{A}) \]

is the volume of the parallelepiped formed by the non-coplanar vectors \( \vec{A}, \vec{B} \) and \( \vec{C} \).

![Parallelepiped Diagram](image)

1.1.5. **Vector Triple Product:**

\[ \vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C} \]

\[ = m\vec{B} - n\vec{C} \]

Where \( m, n \) are scalar parameters.

- \( \vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C} \)

**Proof:**

\[
(\vec{A} \times \vec{B}) \times \vec{C} = -\vec{C} \times (\vec{A} \times \vec{B}) \\
= -(\vec{C} \cdot \vec{B})\vec{A} - (\vec{C} \cdot \vec{A})\vec{B} \\
= (\vec{C} \cdot \vec{A})\vec{B} - (\vec{C} \cdot \vec{B})\vec{A}
\]

Thus, vector \( \vec{A} \times (\vec{B} \times \vec{C}) \) is inside the plane of vectors \( \vec{A} \) and \( \vec{B} \), while the vector \( \vec{A} \times (\vec{B} \times \vec{C}) \) is inside the plane of vectors \( \vec{B} \) and \( \vec{C} \).

Therefore: \( \vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C} \)
1.2 Vector Calculus

If $\vec{A}$ and $\vec{B}$ are differentiable vector functions of a scalar $q$, and $\vec{U} = \vec{A} + \vec{B}$, then,

\[
\frac{d\vec{U}}{dq} = \frac{d\vec{A}}{dq} + \frac{d\vec{B}}{dq}
\]

\[
\frac{d(\alpha \vec{U})}{dq} = \frac{d\alpha}{dq} \vec{U} + \alpha \frac{d\vec{U}}{dq} ; \quad \alpha, q \text{ are scalars.}
\]

\[
\frac{d(\vec{A} \cdot \vec{B})}{dq} = \frac{d\vec{A}}{dq} \cdot \vec{B} + \vec{A} \cdot \frac{d\vec{B}}{dq}
\]

\[
\frac{d(\vec{A} \times \vec{B})}{dq} = \frac{d\vec{A}}{dq} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dq}
\]

\[
\frac{d[\vec{A} \cdot (\vec{B} \times \vec{C})]}{dq} = \frac{d\vec{A}}{dq} \cdot (\vec{B} \times \vec{C}) + \vec{A} \cdot [\frac{d\vec{B}}{dq} \times \vec{C} + \vec{B} \times \frac{d\vec{C}}{dq}]
\]

\[
\frac{d[\vec{A} \times (\vec{B} \times \vec{C})]}{dq} = \frac{d\vec{A}}{dq} \times (\vec{B} \times \vec{C}) + \vec{A} \times [\frac{d\vec{B}}{dq} \times \vec{C} + \vec{B} \times \frac{d\vec{C}}{dq}]
\]
1.3 Partial Derivatives of Vectors

If \( \vec{A} \) and \( \vec{B} \) are two vectors depending on more than one scalar variable, say \( q_1, q_2 \) and \( q_3 \) for example, then,

\[
\frac{\partial \vec{A}}{\partial q_1} = \lim_{\Delta q_1 \to 0} \frac{A(q_1 + \Delta q_1, q_2, q_3) - A(q_1, q_2, q_3)}{\Delta q_1}
\]

\[
\frac{\partial (\vec{A} \cdot \vec{B})}{\partial q_1} = \frac{\partial \vec{A}}{\partial q_1} \cdot \vec{B} + \vec{A} \cdot \frac{\partial \vec{B}}{\partial q_1}
\]

\[
\frac{\partial (\vec{A} \times \vec{B})}{\partial q_1} = \frac{\partial \vec{A}}{\partial q_1} \times \vec{B} + \vec{A} \times \frac{\partial \vec{B}}{\partial q_1}
\]

\[
\frac{\partial^2 (\vec{A} \cdot \vec{B})}{\partial q_1 \partial q_2} = \frac{\partial}{\partial q_1} \left( \frac{\partial (\vec{A} \cdot \vec{B})}{\partial q_2} \right)
\]

\[
= \frac{\partial}{\partial q_1} \left( \frac{\partial \vec{A}}{\partial q_2} \cdot \vec{B} + \vec{A} \cdot \frac{\partial \vec{B}}{\partial q_2} \right)
\]

\[
= \frac{\partial^2 \vec{A}}{\partial q_1 \partial q_2} \cdot \vec{B} + \frac{\partial \vec{A}}{\partial q_2} \cdot \frac{\partial \vec{B}}{\partial q_1} + \frac{\partial \vec{A}}{\partial q_1} \cdot \frac{\partial \vec{B}}{\partial q_2} + \vec{A} \cdot \frac{\partial^2 \vec{B}}{\partial q_1 \partial q_2}
\]

\[
\frac{\partial^2 \vec{A}}{\partial q_1 \partial q_2} = \frac{\partial^2 \vec{A}}{\partial q_2 \partial q_1} \quad \text{if} \quad \vec{A} \quad \text{has continuous partial derivatives of the second order at least.}
\]
1.4 Vector Space and Bases

A set of $S^n$ of all n-tuples $[A_1, A_2, \cdots A_n]$ of real/complex numbers is called a linear vector space, and its elements are called vectors.

We denote $[A_1, A_2, \cdots A_n]$ by the symbol $\vec{A}$ and the numbers of $A_1, A_2, \cdots A_n$ are called components of $\vec{A}$.

The vector spaces $S^1$, $S^2$ and $S^3$ have simple geometric interpretations.

To picture $S^3$, for instance, we represent the vector $\vec{A}$ by $[A_1, A_2, A_3]$ by the line segment in space having its initial point at the origin and its end point at the point with coordinate $(q_1, q_2, q_3)$. $(q_1, q_2, q_3)$ needs not necessarily possess the dimension of length.

There is a distinction between the triple $[A_1, A_2, A_3]$, which we call a vector, and the triple $(q_1, q_2, q_3)$, which represents a point. We do different things with them.

- We can add two triples $[A_1, A_2, A_3]$ and $[B_1, B_2, B_3]$ together for instance, but we certainly do not add two points together.

- On the other hand, we speak of the distance between two points, but not the distance between two vectors.

1.4.1. Linearly Independence, Bases and Dimension

A set of vectors $\vec{A}_1, \vec{A}_2, \cdots \vec{A}_p$ in $S^n$ is linearly independent if and only if the only linear combination of

$$C_1\vec{A}_1 + C_2\vec{A}_2 + \cdots + C_p\vec{A}_p = \vec{0}$$

is true when $C_1 = C_2 = \cdots = C_p = 0$. 
1.4.2. Linear Dependence

A set of vectors $\vec{A}_1, \vec{A}_2, \ldots, \vec{A}_p$ in $S^n$ is linearly dependent if there exist scalars $C_j$, not all zero, to satisfy the equation of

$$C_1 \vec{A}_1 + C_2 \vec{A}_2 + \cdots + C_p \vec{A}_p = \vec{0}.$$ 

If they are linearly dependent, then, at least one of the $\vec{A}_j$s can be expressed as a linear combination of the others. For instance, Suppose $C_2 \neq 0$, then, it follows that

$$\vec{A}_2 = \left[ \left( \frac{C_1}{C_2} \right) \vec{A}_1 + \left( \frac{C_3}{C_2} \right) \vec{A}_3 + \cdots + \left( \frac{C_p}{C_2} \right) \vec{A}_p \right]$$

We say that a vector space $S$ is $n$-dimensional if it contains a set of $n$ linearly independent vectors, but not $n+1$ linearly independent vectors.

1.4.3. Basis

A basis for a given vector space $S^n$ is a set of linearly independent vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$. Any vector $\vec{A}$ in $S^n$ can be “expanded in terms of them” – expressed as a linear combination of them.

$$\vec{A} = C_1 \vec{e}_1 + C_2 \vec{e}_2 + \cdots + C_n \vec{e}_n \quad (1)$$

or

$$\vec{A} = [C_1, C_2 \ldots C_n]$$

1.4.4. Uniqueness

Question: Is the expression of $\vec{A} = [C_1, C_2 \ldots C_n]$ unique?

Proof: If $\vec{A}$ also have another representation in the form of

$$\vec{A} = d_1 \vec{e}_1 + d_2 \vec{e}_2 + \cdots + d_n \vec{e}_n \quad (2)$$

Subtracting (2) from (1), we have
(c_1 - d_1) \vec{e}_1 + (c_2 - d_2) \vec{e}_2 + \cdots + (c_n - d_n) \vec{e}_n = \vec{0}.

Since \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n are linear independent, we can say,

(c_1 - d_1) = (c_2 - d_2) = \cdots (c_n - d_n) = 0

Therefore, the expression is unique.

1.4.5. \textbf{Orthogonal and orthonormal}

\textbf{Definition:} The bases \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n are orthogonal if \vec{e}_i \cdot \vec{e}_j = 0 for \ i \neq j.

\textbf{Definition:} The bases \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n are orthonormal if \vec{e}_i \cdot \vec{e}_j = 0 for \ i \neq j \ \text{and} \ \vec{e}_i \cdot \vec{e}_i = 1 \ \text{for} \ i = j.

\textbf{Example:}

\vec{e}_1 = (1, 0, 0), \quad \vec{e}_2 = (1, 1, 0), \quad \vec{e}_3 = (1, 1, 1),

Consider the vector sets \vec{e}_1, \vec{e}_2, \vec{e}_3.

\alpha_1 (1, 0, 0) + \alpha_2 (1, 1, 0) + \alpha_3 (1, 1, 1) = (0, 0, 0)

Then

\alpha_1 + \alpha_2 + \alpha_3 = 0; \quad \alpha_1 + \alpha_2 = 0; \quad \alpha_3 = 0

Therefore, vector set \vec{e}_1, \vec{e}_2, \vec{e}_3 is linear independent, are they can be used as a set of base vectors.

However, since \vec{e}_1 \cdot \vec{e}_2 \neq 0, \quad \vec{e}_1 \cdot \vec{e}_3 \neq 0, \quad \vec{e}_2 \cdot \vec{e}_3 \neq 0, \ \text{thus, they are not orthogonal}.

Normalized the vector: unit vector \vec{e}_2'' = \frac{\vec{e}_2}{\|\vec{e}_2\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \ \text{in the direction of} \ \vec{e}_2
1.5. Curvilinear Coordinates

1.5.1. Cartesian Coordinate System

Rectangular coordinate system X, Y, Z coordinates and the corresponding unit base vector \( \hat{i}, \hat{j}, \hat{k} \) which are orthonormal.

\[
\hat{i} \cdot \hat{j} = \hat{i} \cdot \hat{k} = \hat{j} \cdot \hat{k} = 0
\]
\[
\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1
\]
\[
\hat{i} \times \hat{j} = \hat{k}; \quad \hat{j} \times \hat{k} = \hat{i}; \quad \hat{k} \times \hat{i} = \hat{j}
\]
\[
\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}
\]

Where \( A_x = \vec{A} \cdot \hat{i}; \quad A_y = \vec{A} \cdot \hat{j}; \quad A_z = \vec{A} \cdot \hat{k}. \)

In other words, \( A_x, A_y, A_z \) are the components of vector \( \vec{A} \), and there are the projections of \( \vec{A} \) on X, Y, Z axes respectively.
1.5.2 Curvilinear Coordinates

In vector space $S^3$ (three-dimensional space), we defined a vector, analytically, as an ordered set of three numbers that are unique respect to a chosen base, i.e., $\vec{A} = [A_1, A_2, A_3]$.

Let coordinate $q_1, q_2, q_3$ are known as the general coordinates of a point (they need not necessarily to posses the dimension of length). In other words, they are not necessary the component of the position vector describing the point.

Consider a Cartesian coordinate space $X, Y, Z$, consider three function defined by

$$
q_1 = q_1(x, y, z) = \text{const} = C_1 \\
q_2 = q_2(x, y, z) = \text{const} = C_2 \\
q_3 = q_3(x, y, z) = \text{const} = C_3
$$

If these functions are single valued and can be solved uniquely for $x, y, z$ by relations

$$x = x(q_1, q_2, q_3); \quad y = y(q_1, q_2, q_3); \quad z = z(q_1, q_2, q_3),$$

and also if these functions have continuous derivatives,

then, $q_1, q_2, q_3$ can be a curvilinear coordinate of $P = P(x, y, z)$.

The surfaces:

$$
q_1 = q_1(x, y, z) = C_1 \\
q_2 = q_2(x, y, z) = C_2 \\
q_3 = q_3(x, y, z) = C_3
$$

are coordinate surfaces, and each pair of these surfaces interact in curves called coordinate curves or lines.

If the coordinate surfaces interact at right angles, then the curvilinear coordinate system is orthogonal.
1.5.3 Space Curve

If $\vec{R} = \vec{R}(q)$ is the position vector joining origin $O$ of a coordinate system and any point $P(x, y, z)$, then $\vec{R} = \vec{R}(q)$ is given by:

$$\vec{R}(q) = x(q)\hat{i} + y(q)\hat{j} + z(q)\hat{k}.$$ 

The space curve $C$ is defined by $\vec{R}(q)$.

$$\frac{d\vec{R}}{dq}$$ is a vector in the direction of the tangent to $C$.

If $q$ is taken as the arc length $S$ measured from some fixed points on $C$, then $\frac{d\vec{R}}{dS}$ is a unit tangent vector, is denoted by $\hat{e}_T$.

The rate at which $\hat{e}_T$ changes with respect to $S$ is a measure of the curvature of $C$ and is given by $\frac{d\hat{e}_T}{ds}$, a normal to the curve at the point. If $\hat{e}_N$ is the unit vector in the direction of this normal, then $\hat{e}_N = K\frac{d\hat{e}_T}{dS}$, where $K$ is called curvature of $C$ at the specified point and $\rho = \frac{1}{K}$ is the radius of curvature at that point.

A unit vector $\hat{e}_B$ perpendicular to $\hat{e}_T$ and $\hat{e}_N$ such as $\hat{e}_B = \hat{e}_T \times \hat{e}_N$ is called the binormal to the curve.

The directions $T$, $N$ and $B$ form a localized right-handed rectangular coordinate system at any specified points of $C$. 

![Diagram of space curve with tangent, normal, and binormal vectors]
1.5.4 Definition of Scale factors and Unit Vectors

Let \( \mathbf{R} = x \hat{i} + y \hat{j} + z \hat{k} \) be the position vector of a point \( P(q_1, q_2, q_3) \) in a curvilinear coordinate system, i.e., \( \mathbf{R} = \mathbf{R}(q_1, q_2, q_3) \). A tangent to \( q_1 \)-curve (axis) at \( P \) (at which \( q_2 \) and \( q_3 \) are constant) is \( \frac{\partial \mathbf{R}}{\partial q_1} \). Then, a unit vector is given by:

\[
\hat{e}_1 = \left| \frac{\partial \mathbf{R}}{\partial q_1} \right|, \quad h_1 = \frac{\partial \mathbf{R}}{\partial q_1}.
\]

Similarly

\[
\hat{e}_2 = \left| \frac{\partial \mathbf{R}}{\partial q_2} \right|, \quad h_2 = \frac{\partial \mathbf{R}}{\partial q_2},
\]

\[
\hat{e}_3 = \left| \frac{\partial \mathbf{R}}{\partial q_3} \right|, \quad h_3 = \frac{\partial \mathbf{R}}{\partial q_3}.
\]

\( h_1, h_2, h_3 \) are called the scale factors and \( \hat{e}_1, \hat{e}_2, \hat{e}_3 \) are called unit vectors in the increasing direction of \( q_1, q_2 \) and \( q_3 \).

Note: The scale factors relate the differential distance to the differential of the coordinates. These scale factors varying from point to point, and thus are, in general, function of position.

Since \( \mathbf{R} = \mathbf{R}(q_1, q_2, q_3) \), we have

\[
d\mathbf{R} = \frac{\partial \mathbf{R}}{\partial q_1} dq_1 + \frac{\partial \mathbf{R}}{\partial q_2} dq_2 + \frac{\partial \mathbf{R}}{\partial q_3} dq_3 = h_1 \hat{e}_1 dq_1 + h_2 \hat{e}_2 dq_2 + h_3 \hat{e}_3 dq_3
\]

\[
= h_1 dq_1 \hat{e}_1 + h_2 dq_2 \hat{e}_2 + h_3 dq_3 \hat{e}_3 = ds_1 \hat{e}_1 + ds_2 \hat{e}_2 + ds_3 \hat{e}_3
\]

\[
= d\mathbf{S}
\]

Where \( ds_1, ds_2, ds_3 \) denote elemental distance along \( q_1, q_2 \) and \( q_3 \) axes respectively.

The differential of arc length \( d\mathbf{S} \) can be determined from \( (d\mathbf{S})^2 = d\mathbf{R} \cdot d\mathbf{R} \)

In general \( (d\mathbf{S})^2 = \sum_{m=1}^{M} \sum_{n=1}^{N} (h_{m} \hat{e}_m \cdot h_{n} \hat{e}_n) dq_m dq_n \)

[Along \( q_1 \)-curve (axis), \( q_2 \) and \( q_3 \) are constants, thus, \( d\mathbf{R} = h_1 dq_1 \hat{e}_1 \), therefore, arc length along \( q_1 \) at point \( P \) is \( ds_1 = h_1 dq_1 \). Similarly, \( ds_2 = h_2 dq_2 \) and \( ds_3 = h_3 dq_3 \).]
CHAPTER 1  REVIEW OF VECTOR ALGEBRA

For orthogonal systems: \( \hat{e}_1 \cdot \hat{e}_2 = \hat{e}_1 \cdot \hat{e}_3 = \hat{e}_2 \cdot \hat{e}_3 = 0 \),

Therefore, \( (dS)^2 = h_1^2 dq_1^2 + h_2^2 dq_2^2 + h_3^2 dq_3^2 \)

A space curve can be represented by parametric equations such as \( x = x(q) \), \( y = y(q) \) and \( z = z(q) \).

A space surface can be represented by a two parameter function family, \( x = x(q_1, q_2) \), \( y = y(q_1, q_2) \) and \( z = z(q_1, q_2) \),

A volume in space can be represented by a three-parametric family, \( x = x(q_1, q_2, q_3) \), \( y = y(q_1, q_2, q_3) \) and \( z = z(q_1, q_2, q_3) \). Keeping any one of the \( q \)s in constant generates a surface. Keeping any two of the \( q \)'s in constant generates a space curve.

Example:
- Cartesian system (X,Y,Z): X constant \( \Rightarrow \) a surface parallel to Y-Z plane
- Cylindrical system(R,\( \theta \),z): R constant \( \Rightarrow \) a surface of a cylinder
- Spherical system: R constant \( \Rightarrow \) a surface of a sphere

Review:
- \( x = x(q) \), \( y = y(q) \), \( z = z(q) \) \( \Rightarrow \) Curve in space
- \( x = x(q_1, q_2) \), \( y = y(q_1, q_2) \), \( z = z(q_1, q_2) \) \( \Rightarrow \) Surface in space
- \( x = x(q_1, q_2, q_3) \), \( y = y(q_1, q_2, q_3) \), \( z = z(q_1, q_2, q_3) \) \( \Rightarrow \) Volume in space
1.6 Curvilinear Surfaces

Let $\vec{R} = \vec{R}(q_1, q_2)$ represent a space surface, let $(u_0, v_0)$ be a given point on the surface. $\vec{R}(u_0, q_2)$ represents a space curve given by $q_1 = u_0$ (i.e., a line parallel to the $q_2$-axis). $\vec{R}(q_1, v_0)$ represents another space curve given by $q_2 = v_0$ (i.e., a line parallel to the $q_1$-axis).

\[
\frac{\partial \vec{R}}{\partial q_2} \bigg|_{(u_0, v_0)} \text{ is a tangent to the curve } q_1 = u_0
\]

\[
\frac{\partial \vec{R}}{\partial q_1} \bigg|_{(u_0, v_0)} \text{ is a tangent to the curve } q_2 = v_0
\]

\[
\frac{\partial \vec{R}}{\partial q_1} \times \frac{\partial \vec{R}}{\partial q_2} \text{ is a vector normal to the plane containing the two tangents.}
\]

\[
\frac{\partial \vec{R}}{\partial q_1} \times \frac{\partial \vec{R}}{\partial q_2} \text{ is the unit vector normal to the given surface at } (u_0, v_0)
\]

Elemental/differential distance along the curve $q_2 = v_0$ is $d\vec{s}_1 = h_1 dq_1 \hat{e}_1 = \left| \frac{\partial \vec{R}}{\partial q_1} \right| dq_1 \hat{e}_1$. It becomes

\[
d\vec{R} = d\vec{s}_1 = \frac{\partial \vec{R}}{\partial q_1} dq_1 + \frac{\partial \vec{R}}{\partial q_2} dq_2 + \frac{\partial \vec{R}}{\partial q_3} dq_3 = \frac{\partial \vec{R}}{\partial q_1} dq_1 \left\{ \begin{array}{l}
\text{along the curve } q_2 = v_0 \\
\text{on the surface } q_3 = C
\end{array} \right.
\]

Similarly, elemental/differential distance along the curve $q_1 = u_0$ is $d\vec{s}_2 = h_2 dq_2 \hat{e}_2 = \left| \frac{\partial \vec{R}}{\partial q_2} \right| dq_2 \hat{e}_2$. It became

\[
d\vec{R} = d\vec{s}_2 = \frac{\partial \vec{R}}{\partial q_1} dq_1 + \frac{\partial \vec{R}}{\partial q_2} dq_2 + \frac{\partial \vec{R}}{\partial q_3} dq_3 = \frac{\partial \vec{R}}{\partial q_2} dq_2 \left\{ \begin{array}{l}
\text{along the curve } q_2 = v_0 \\
\text{on the surface } q_3 = C
\end{array} \right.
\]
Elemental/differential area at point \((u_0, v_0)\) is given by
\[
d\vec{S}_1 \times d\vec{S}_2 = h_1 h_2 dq_1 dq_2 \hat{e}_1 \times \hat{e}_2 = d\vec{A}_3,
\]
where \(\hat{e}_1\) and \(\hat{e}_2\) are unit vectors.

\(d\vec{A}_3\) is directed out from the surface. \(d\vec{A}_3 = h_1 h_2 dq_1 dq_2\)

Similarly,
\[
\begin{align*}
    d\vec{A}_1 &= h_2 h_3 dq_2 dq_3 \hat{e}_1 \quad \text{and} \quad d\vec{A}_1 = h_2 h_3 dq_2 dq_3, \\
    d\vec{A}_2 &= h_3 h_1 dq_3 dq_1 \hat{e}_2 \quad \text{and} \quad d\vec{A}_2 = h_3 h_1 dq_3 dq_1.
\end{align*}
\]
1.7 Determination of Unit Vectors and Scale Factors

\[ \frac{\partial \hat{R}}{\partial q_1} = h_1 \hat{e}_1; \quad \frac{\partial \hat{R}}{\partial q_2} = h_2 \hat{e}_2; \quad \frac{\partial \hat{R}}{\partial q_3} = h_3 \hat{e}_3 \]

i.e., \[ \frac{\partial R}{\partial q_i} = h_i \hat{e}_i \]

\[ d\hat{R} = \frac{\partial \hat{R}}{\partial q_1} dq_1 + \frac{\partial \hat{R}}{\partial q_2} dq_2 + \frac{\partial \hat{R}}{\partial q_3} dq_3 \]

\[ \hat{R} = \hat{R}(q_1, q_2, q_3) = x \hat{i} + y \hat{j} + z \hat{k} = h_i \hat{e}_i \]

\[ = x(q_1, q_2, q_3) \hat{i} + y(q_1, q_2, q_3) \hat{j} + z(q_1, q_2, q_3) \hat{k} \]

Therefore:

\[ \frac{\partial \hat{R}}{\partial q_i} = \frac{\partial x}{\partial q_i} \hat{i} + \frac{\partial y}{\partial q_i} \hat{j} + \frac{\partial z}{\partial q_i} \hat{k} = h_i \hat{e}_i \]

\[ (h_i \hat{e}_i) \cdot (h_i \hat{e}_i) = \left( \frac{\partial x}{\partial q_i} \right)^2 + \left( \frac{\partial y}{\partial q_i} \right)^2 + \left( \frac{\partial z}{\partial q_i} \right)^2 = h_i^2 \]

\[ \Rightarrow \begin{cases} h_i = \sqrt{\left( \frac{\partial x}{\partial q_i} \right)^2 + \left( \frac{\partial y}{\partial q_i} \right)^2 + \left( \frac{\partial z}{\partial q_i} \right)^2} \\ \hat{e}_i = \frac{1}{h_i} \left( \frac{\partial x}{\partial q_i} \hat{i} + \frac{\partial y}{\partial q_i} \hat{j} + \frac{\partial z}{\partial q_i} \hat{k} \right) \end{cases} \]
**Cylindrical Coordinate System**

A point P in space is given by \( p(q_1, q_2, q_3) \) or \( p(r, \theta, z) \) with base vector \((\hat{e}_1, \hat{e}_2, \hat{e}_3)\) given by \((\hat{e}_r, \hat{e}_\theta, \hat{e}_z)\).

\[
\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta \\
z &= Z
\end{align*}
\]

\( r \geq 0, \quad 0 \leq \theta \leq 2\pi; \quad -\infty \leq Z \leq \infty \)

\[
\begin{align*}
dx &= dr \cos \theta - r \sin \theta \, d\theta \\
y &= dr \sin \theta + r \cos \theta \, d\theta \\
dz &= dz
\end{align*}
\]

\[
(ds)^2 = h_1^2 (dq_1)^2 + h_2^2 (dq_2)^2 + h_3^2 (dq_3)^2
\]

\[
= (dx)^2 + (dy)^2 + (dz)^2
\]

\[
= (\cos \theta \, dr)^2 - 2r \sin \theta \, d\theta \cos \theta \, dr + (r \sin \theta \, d\theta)^2 \\
+ (\sin \theta \, dr)^2 + 2r \sin \theta \cos \theta \, dr \, d\theta + (r \cos \theta \, d\theta)^2 + (dz)^2
\]

\[
= (dr)^2 + r^2 (d\theta)^2 + (dz)^2
\]

Therefore: \( h_1 = 1; \quad h_2 = r; \quad h_3 = 1 \)

\[
\frac{\partial \vec{R}}{\partial q_i} = h_i \, \hat{e}_i
\]

\[
\frac{\partial \vec{R}}{\partial q_i} = \frac{\partial x}{\partial q_i} \hat{i} + \frac{\partial y}{\partial q_i} \hat{j} + \frac{\partial z}{\partial q_i} \hat{k} = h_i \, \hat{e}_i
\]

\(r\) - Direction:

\[
h_r \, \hat{e}_r = \frac{\partial \vec{R}}{\partial r} = \frac{\partial x}{\partial r} \hat{i} + \frac{\partial y}{\partial r} \hat{j} + \frac{\partial z}{\partial r} \hat{k} = \cos \theta \, \hat{i} + \sin \theta \, \hat{j} + 0 \, \hat{k}
\]

\[
(h_r \, \hat{e}_r) \cdot (h_r \, \hat{e}_r) = (h_r)^2 = \cos^2 \theta + \sin^2 \theta = 1
\] \(\Rightarrow\) \[
\begin{align*}
h_r &= 1 \\
\hat{e}_r &= \cos \theta \, \hat{i} + \sin \theta \, \hat{j}
\end{align*}
\]

\(\theta\) - Direction:
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\[ h_\theta \hat{e}_\theta = \frac{\partial \hat{R}}{\partial \theta} = \frac{\partial x}{\partial \theta} \hat{i} + \frac{\partial y}{\partial \theta} \hat{j} + \frac{\partial z}{\partial \theta} \hat{k} = -r \sin \theta \hat{i} + r \cos \theta \hat{j} + 0 \hat{k} \]

\[ (h_\theta \hat{e}_\theta) \cdot (h_\theta \hat{e}_\theta) = (h_\theta)^2 = r^2 (\sin^2 \theta + \cos^2 \theta) = r^2 \quad \Rightarrow \quad \begin{cases} h_\theta = r \\ \hat{e}_\theta = -\sin \theta \hat{i} + \cos \theta \hat{j} \end{cases} \]

**Z - Direction:**

\[ h_z \hat{e}_z = \frac{\partial \hat{R}}{\partial Z} = \frac{\partial x}{\partial Z} \hat{i} + \frac{\partial y}{\partial Z} \hat{j} + \frac{\partial z}{\partial Z} \hat{k} = 0 \hat{i} + 0 \hat{j} + 1 \hat{k} \]

\[ (h_z \hat{e}_z) \cdot (h_z \hat{e}_z) = (h_z)^2 = 1^2 \quad \Rightarrow \quad \begin{cases} h_z = 1 \\ \hat{e}_z = \hat{k} \end{cases} \]

**Summarize:**

\[ \hat{e}_r = \cos \theta \hat{i} + \sin \theta \hat{j} \quad h_r = 1 \]

\[ \hat{e}_\theta = -\sin \theta \hat{i} + \cos \theta \hat{j} \quad h_\theta = r \]

\[ \hat{e}_z = \hat{k} \quad h_z = 1 \]

**Transformation relationship**

\[
\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_z \end{bmatrix} \quad ; \quad \text{or} \quad \begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_z \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}
\]

**Derivatives of the unit vectors:**

\[
\frac{\partial \hat{e}_r}{\partial r} = 0 \quad \frac{\partial \hat{e}_\theta}{\partial r} = 0 \quad \frac{\partial \hat{e}_z}{\partial r} = 0
\]

\[
\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta \quad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r \quad \frac{\partial \hat{e}_z}{\partial \theta} = 0
\]

\[
\frac{\partial \hat{e}_r}{\partial Z} = 0 \quad \frac{\partial \hat{e}_\theta}{\partial Z} = 0 \quad \frac{\partial \hat{e}_z}{\partial Z} = 0
\]
Example:

If \( \vec{R} = \vec{R}(t) = r\hat{e}_r + z\hat{e}_z \) is the position vector of a particle in cylindrical coordinates, obtain expression for velocity vector, \( \vec{V} \), and acceleration vector, \( \vec{a} \), at that point.

Since \( \hat{e}_r = \hat{e}_r(r, \theta, z) \), then, \( \frac{d\hat{e}_r}{dt} = \frac{\partial \hat{e}_r}{\partial \theta} d\theta + \frac{\partial \hat{e}_r}{\partial r} dr + \frac{\partial \hat{e}_r}{\partial z} dz \)

Therefore, \( \frac{d\hat{e}_r}{dt} = \frac{\partial \hat{e}_r}{\partial \theta} d\theta + \frac{\partial \hat{e}_r}{\partial r} dr + \frac{\partial \hat{e}_r}{\partial z} dz \)

Similarly, \( \frac{d\hat{e}_\theta}{dt} = \frac{\partial \hat{e}_\theta}{\partial \theta} d\theta + \frac{\partial \hat{e}_\theta}{\partial r} dr + \frac{\partial \hat{e}_\theta}{\partial z} dz \)

\( \frac{d\hat{e}_z}{dt} = \frac{\partial \hat{e}_z}{\partial \theta} d\theta + \frac{\partial \hat{e}_z}{\partial r} dr + \frac{\partial \hat{e}_z}{\partial z} dz \)

\( \vec{V} = \frac{d\vec{R}}{dt} = r \frac{d\hat{e}_r}{dt} + \frac{dr}{dt} \hat{e}_r + \frac{dz}{dt} \hat{e}_z + z \frac{d\hat{e}_z}{dt} \)

\( = r(\frac{\partial \hat{e}_r}{\partial \theta} d\theta + \frac{\partial \hat{e}_r}{\partial r} dr + \frac{\partial \hat{e}_r}{\partial z} dz) + \frac{dr}{dt} \hat{e}_r + \frac{dz}{dt} \hat{e}_z + z(\frac{\partial \hat{e}_z}{\partial \theta} d\theta + \frac{\partial \hat{e}_z}{\partial r} dr + \frac{\partial \hat{e}_z}{\partial z} dz) \)

\( = \hat{e}_\theta \frac{d\theta}{dt} + \frac{dr}{dt} \hat{e}_r + \frac{dz}{dt} \hat{e}_z \)

\( \vec{a} = \frac{d\vec{V}}{dt} = \frac{d(r\hat{e}_\theta \frac{d\theta}{dt} + \frac{dr}{dt} \hat{e}_r + \frac{dz}{dt} \hat{e}_z)}{dt} \)

\( = \frac{dr}{dt} \hat{e}_\theta \frac{d\theta}{dt} + r \frac{d\hat{e}_\theta}{dt} \frac{d\theta}{dt} + \hat{e}_\theta \frac{d^2\theta}{dt^2} + \frac{dr}{dt} \hat{e}_r + \frac{d^2r}{dt^2} \hat{e}_r + \frac{dz}{dt} \hat{e}_z + \frac{d^2z}{dt^2} \hat{e}_z + z \frac{d\hat{e}_z}{dt} \frac{d\theta}{dt} + \frac{\partial \hat{e}_z}{\partial r} \frac{d\theta}{dt} dr + \frac{\partial \hat{e}_z}{\partial z} dz + \frac{dz}{dt} \hat{e}_z + \frac{d^2z}{dt^2} \hat{e}_z \)

\( = \frac{dr}{dt} \hat{e}_\theta + \hat{e}_\theta \frac{d^2\theta}{dt^2} + \frac{d^2r}{dt^2} \hat{e}_r + \frac{d^2z}{dt^2} \hat{e}_z \)

\( = \left[ \frac{d^2r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 \right] \hat{e}_r + \left[ \frac{dr}{dt} \hat{e}_\theta + \frac{d^2\theta}{dt^2} \right] \hat{e}_r + \frac{d^2z}{dt^2} \hat{e}_z \)
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Scale factors and unit vectors in Spherical coordinate system \((R, \varphi, \theta)\)

\[ \vec{OB} = R \hat{e}_R \]
\[ \vec{OA} = R \sin \varphi \]
\[ x = R \sin \varphi \cos \theta \]
\[ z = r \cos \varphi \]

\[ h_R = 1 \]
\[ \hat{e}_R = \sin \varphi \cos \theta \hat{i} + \sin \varphi \sin \theta \hat{j} + \cos \varphi \hat{k} \]

\[ h_\varphi = R \]
\[ \hat{e}_\varphi = \cos \varphi \cos \theta \hat{i} + \cos \varphi \sin \theta \hat{j} - \sin \varphi \hat{k} \]

\[ h_\theta = R \sin \varphi \]
\[ \hat{e}_\theta = -\sin \theta \hat{i} + \cos \theta \hat{j} \]
1.8 Functions of Vector to Describe Physical Problems

Type of functions

- A scalar as a function of a scalar, for example: $\mu = \mu(T)$
- A vector as a function of a scalar, for example: $\vec{R} = \vec{R}(t)$
- A scalar as a function of a vector, for example: $T = T(\vec{R})$
- A vector as a function of a vector, for example: $\vec{V} = \vec{V}(\vec{R})$

General description: $\phi = \phi(\vec{R}, t)$ and $\vec{A} = \vec{A}(\vec{R}, t)$

**Scalar field:** A scalar quantity given as a function of coordinate space and time, $t$, is called scalar field.

For examples: $p = p(x, y, z, t)$ and $T = T(x, y, z, t)$

$= p(\vec{R}, t)$ and $= T(\vec{R}, t)$

**Vector field:** A vector quantity given as a function of coordinate space and time, $t$, is called vector field.

For examples: $\vec{V} = \vec{V}(x, y, z, t) = \vec{V}(\vec{R}, t)$ and $\vec{M} = \vec{M}(x, y, z, t) = \vec{M}(\vec{R}, t)$

- In general, a field denotes a region throughout which a quantity is defined as a function of location within the region and time.
- If the quantity is independent of time, the field is steady or stationary.
1.9 Gradient

Gradient is a vector generated by the differentiation of a scalar function

Let $\phi = \phi(\vec{R}) = \phi(q_1, q_2, q_3)$

We refer to the spatial variation of $\phi$ in a particulate direction as a directional derivative, and in general this derivative is different in different directions.

A simple but useful representation of directional derivative is introduced through grouping together the partial derivatives of $\phi$ along the coordinate axes (bases) as the components of a vector called the gradient of $\phi$.

Consider the change of $\phi$ over the directed distance $d\vec{R}$ (i.e., $\vec{R} \rightarrow \vec{R} + \Delta\vec{R}$), find $d\phi = \lim_{\Delta\vec{R} \to 0} [\phi(\vec{R} + \Delta\vec{R}) - \phi(\vec{R})] = ?$

From the total differential formula of the calculus, the first order differential in $\phi$ will be

$$d\phi = \frac{\partial \phi}{\partial q_1} dq_1 + \frac{\partial \phi}{\partial q_2} dq_2 + \frac{\partial \phi}{\partial q_3} dq_3 + \text{high Orders terms}$$

$$\approx \frac{\partial \phi}{\partial q_1} dq_1 + \frac{\partial \phi}{\partial q_2} dq_2 + \frac{\partial \phi}{\partial q_3} dq_3 = \frac{1}{h_1} \frac{\partial \phi}{\partial q_1} h_1 dq_1 + \frac{1}{h_2} \frac{\partial \phi}{\partial q_2} h_2 dq_2 + \frac{1}{h_3} \frac{\partial \phi}{\partial q_3} h_3 dq_3$$

$$= \frac{1}{h_1} \frac{\partial \phi}{\partial q_1} ds_1 + \frac{1}{h_2} \frac{\partial \phi}{\partial q_2} ds_2 + \frac{1}{h_3} \frac{\partial \phi}{\partial q_3} ds_3$$

Since $d\vec{R} = d\vec{S} = ds_1 \hat{e}_1 + ds_2 \hat{e}_2 + ds_3 \hat{e}_3$

Now introduce a vector $[\frac{1}{h_1} \frac{\partial \phi}{\partial q_1}, \frac{1}{h_2} \frac{\partial \phi}{\partial q_2}, \frac{1}{h_3} \frac{\partial \phi}{\partial q_3}]$ denoted by $\nabla \phi$ in the curvilinear orthogonal coordinate system with unit vector $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$, then,

$$d\phi = [\frac{1}{h_1} \frac{\partial \phi}{\partial q_1}, \frac{1}{h_2} \frac{\partial \phi}{\partial q_2}, \frac{1}{h_3} \frac{\partial \phi}{\partial q_3}] \cdot [ds_1, ds_2, ds_3]$$

$$= \nabla \phi \cdot d\vec{R} = \nabla \phi \cdot d\vec{S}$$

Since $d\vec{S} = dS \cdot \hat{e}_s$ therefore, $\frac{d\phi}{dS} = \nabla \phi \cdot \hat{e}_s$

- $\frac{d\phi}{dS} = \nabla \phi \cdot \hat{e}_s$ is a maximum when $\nabla \phi \cdot \hat{e}_s$ is a maximum. i.e., when $\nabla \phi$ and $\hat{e}_s$ are in the same direction. In other words, $\nabla \phi$ is the direction of maximum changes of $\phi$ and $|\nabla \phi|$ is the magnitude of the change.

- The direction of the gradient to a level of a scalar field is normal to the surface at a given point, i.e., $\nabla \phi$ is a vector normal to $\phi(q_1, q_2, q_3) = C$. In the
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\[ \phi(q_1, q_2, q_3) = C \text{ plane, } d\phi = \nabla \phi \cdot d\vec{S} = 0 \text{ because } \phi \text{ is a constant. } \nabla \phi \cdot d\vec{S} = 0 \text{ means } \nabla \phi \text{ and } d\vec{S} \text{ are orthogonal.} \]

• Gradient is a vector composed of partial derivatives of a scalar \( \phi \) in the coordinate directions or along the orthogonal basis of a vector space.

General notion of gradient:

In a scalar field of \( \phi = \phi(\vec{R}) \), the spatial variation of \( \phi \) can be calculated by
\[ d\phi = \nabla \phi \cdot d\vec{R} = \nabla \phi \cdot d\vec{S}. \]

Suppose a scalar field \( \phi \) is a function of a vector, i.e., \( \phi = \phi(\vec{V}) \) where \( \vec{V} \) is a vector.

Then:
\[ d\phi = \frac{\partial \phi}{\partial V_1} dV_1 + \frac{\partial \phi}{\partial V_2} dV_2 + \frac{\partial \phi}{\partial V_3} dV_3 \text{ or } d\phi = d\vec{V} \cdot \nabla \phi. \]

Where \( V_1, V_2, V_3 \) are the components of vector \( \vec{V} \).

If \( \phi \) is a function of more than one vectors or is a function of several sets of independent variables, i.e.,
\[ \phi = \phi(q_1, q_2, q_3, V_1, V_2, V_3, t) = \phi(\vec{R}, \vec{V}, t) \]

Then,
\[ d\phi = \frac{\partial \phi}{\partial q_1} dq_1 + \frac{\partial \phi}{\partial q_2} dq_2 + \frac{\partial \phi}{\partial q_3} dq_3 + \frac{\partial \phi}{\partial V_1} dV_1 + \frac{\partial \phi}{\partial V_2} dV_2 + \frac{\partial \phi}{\partial V_3} dV_3 + \frac{\partial \phi}{\partial t} dt \]
\[ = d\vec{R} \cdot \nabla \phi + d\vec{V} \cdot \nabla \phi + \frac{\partial \phi}{\partial t} dt \]

**Integral definition of gradient:**

\[ \nabla \phi = \lim_{\Delta V \to 0} \left[ \int_{\Delta S} \phi \, d\vec{A} / \Delta V \right] = \lim_{\Delta V \to 0} \left[ \int_{\Delta S} \phi \, \hat{e}_n \, dA / \Delta V \right] \]

where \( \Delta V \) is a infinitesimal arbitrary volume and \( \Delta S \) is the surface of the volume considered. \( d\vec{A} \) is elemental area on the surface. \( d\vec{A} = \hat{e}_n dA \) and \( \hat{e}_n \) is a unit vector pointing outward normal to the surface.
1.10 Contra-variant Vector and Covariant Vector

Contravariant components of a vector

For a position vector \( \vec{R} = \vec{R}(q_1, q_2, q_3) \)

\[
d\vec{R} = \frac{\partial \vec{R}}{\partial q_1} dq_1 + \frac{\partial \vec{R}}{\partial q_2} dq_2 + \frac{\partial \vec{R}}{\partial q_3} dq_3
\]

\[
= h_1 \hat{e}_1 dq_1 + h_2 \hat{e}_2 dq_2 + h_3 \hat{e}_3 dq_3
\]

Consider the basis \( (h_1 \hat{e}_1, h_2 \hat{e}_2, h_3 \hat{e}_3) \), and expanding a vector \( \vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3 \) in terms of this basis and obtained.

\[
\vec{A} = \vec{A}_1 (h_1 \hat{e}_1) + \vec{A}_2 (h_2 \hat{e}_2) + \vec{A}_3 (h_3 \hat{e}_3) \quad \text{or} \quad \vec{A} = \vec{A}_1 \frac{\partial \vec{R}}{\partial q_1} + \vec{A}_2 \frac{\partial \vec{R}}{\partial q_2} + \vec{A}_3 \frac{\partial \vec{R}}{\partial q_3}
\]

\( (\vec{A}_1, \vec{A}_2, \vec{A}_3) \) are called the contra-variant component of \( \vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3 \).

Since \( \frac{\partial \vec{R}}{\partial q_i} \) is a tangent to the \( i \)-th coordinate axes, the bases are in the tangent space.

\[
A_i = \frac{\vec{A}_1}{h_i}; \quad A_2 = \frac{\vec{A}_2}{h_2}; \quad A_3 = \frac{\vec{A}_3}{h_3}.
\]

Covariant components of vector \( \vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3 \)

\( \nabla q_i \) is a gradient to the \( i \)-th coordinate surface. In other words, \( \nabla q_i \) is a gradient to the surface \( q_i = \text{const} \).

\[
\nabla = \frac{\hat{e}_1}{h_1} \frac{\partial}{\partial q_1} + \frac{\hat{e}_2}{h_2} \frac{\partial}{\partial q_2} + \frac{\hat{e}_3}{h_3} \frac{\partial}{\partial q_3}
\]

\[
\nabla q_i = \frac{\hat{e}_1}{h_1} \frac{\partial q_1}{\partial q_1} + \frac{\hat{e}_2}{h_2} \frac{\partial q_1}{\partial q_2} + \frac{\hat{e}_3}{h_3} \frac{\partial q_1}{\partial q_3} = \frac{\hat{e}_1}{h_1} \quad \text{where} \quad \hat{e}_1 \text{ is a vector normal to the} \ q_i = \text{const} \ \text{surface}.
\]

Similarly, \( \nabla q_2 = \frac{\hat{e}_2}{h_2} \) and \( \nabla q_3 = \frac{\hat{e}_3}{h_3} \).
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If we consider the basis \((\nabla q_1, \nabla q_2, \nabla q_3)\), a vector \( \vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3 \) can be expanded in terms of this basis as \( \vec{A} = \vec{A}_1 \nabla q_1 + \vec{A}_2 \nabla q_2 + \vec{A}_3 \nabla q_3 \) or \( \vec{A} = \hat{A}_1 \frac{\hat{e}_1}{h_1} + \hat{A}_2 \frac{\hat{e}_2}{h_2} + \hat{A}_3 \frac{\hat{e}_3}{h_3} \).

Where \((\vec{A}_1; \vec{A}_2; \vec{A}_3)\) are called the covariant component of vector \( \vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3 \)

Where \( A_1 = \frac{\vec{A}_1}{h_1}; \quad A_2 = \frac{\vec{A}_2}{h_2}; \quad A_3 = \frac{\vec{A}_3}{h_3} \).
1.11 Divergence of a Vector Field

Definition: The divergence of a vector \( (\nabla \bullet \vec{B}) \) at a point is the net outflow (efflux) of the vector field per unit volume enclosing the point.

Let \( \Delta \mathbf{V} \) be an elemental volume with a surface \( \Delta S \). An element area on the surface \( \Delta S \) is \( \Delta \mathbf{A} \). If \( \vec{B} \) is a vector at a point in this vector field, then:

\[
\nabla \bullet \vec{B} = \text{Div} \vec{B} = \lim_{\Delta \mathbf{V} \to 0} \left[ \frac{\iiint_{B} \vec{B} \cdot d\mathbf{A}}{\Delta \mathbf{V}} \right]
\]

Where \( \vec{B} \cdot d\mathbf{A} = \vec{B} \cdot \hat{\mathbf{e}}_n dA \) is the outflow of \( \vec{B} \) through \( dA \), and \( \hat{\mathbf{e}}_n \) is a unit vector pointing outward and normal to the surface. \( \iiint_{\Delta S} \vec{B} \cdot d\mathbf{A} \) is the net outflow (efflux) from the surface.

- If \( \vec{B} = \vec{V} \) the velocity vector, then, \( \nabla \bullet \vec{V} \) is the volume flux from the point, i.e., the rate at which fluid volume is leaving a point per unit volume.
- If \( \vec{B} = \rho \vec{V} \) the vector composed of density times the velocity vector, then, \( \nabla \bullet \rho \vec{V} \) is the mass outflow (efflux) from the surface.

\[ \text{Div} \vec{V} = \nabla \bullet \vec{V} ; \]

\[ \vec{V} = V_1 \hat{\mathbf{e}}_1 + V_2 \hat{\mathbf{e}}_2 + V_3 \hat{\mathbf{e}}_3 \]

\[
\nabla = \hat{\mathbf{e}}_1 \frac{\partial}{\partial q_1} + \hat{\mathbf{e}}_2 \frac{\partial}{\partial q_2} + \hat{\mathbf{e}}_3 \frac{\partial}{\partial q_3}
\]

\[
\nabla \bullet \vec{V} = \left( \hat{\mathbf{e}}_1 \frac{\partial}{\partial q_1} + \hat{\mathbf{e}}_2 \frac{\partial}{\partial q_2} + \hat{\mathbf{e}}_3 \frac{\partial}{\partial q_3} \right) \bullet (V_1 \hat{\mathbf{e}}_1 + V_2 \hat{\mathbf{e}}_2 + V_3 \hat{\mathbf{e}}_3)
\]

Cartesian system:

\[
\nabla \bullet \vec{V} = \left( \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \bullet (i V_x + j V_y + k V_z)
\]

\[
= \left( i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) (i V_x + j V_y + k V_z)
\]

\[
= i \left( \frac{\partial}{\partial x} V_x + \frac{\partial V_x}{\partial x} + j \frac{\partial}{\partial y} V_y + \frac{\partial V_y}{\partial y} + k \frac{\partial}{\partial z} V_z + \frac{\partial V_z}{\partial z} \right)
\]

- \[ \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \]
Cylindrical system:
\[
\nabla \cdot \vec{V} = \left( \hat{e}_r \frac{\partial}{\partial r} + \frac{\hat{e}_\theta}{r} \frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z} \right) \cdot (V_r \hat{e}_r + V_\theta \hat{e}_\theta + V_z \hat{e}_z)
\]
\[
= \hat{e}_r \left( \frac{\partial}{\partial r} (V_r \hat{e}_r + V_\theta \hat{e}_\theta + V_z \hat{e}_z) \right) + \frac{\hat{e}_\theta}{r} \left( \frac{\partial}{\partial \theta} (V_r \hat{e}_r + V_\theta \hat{e}_\theta + V_z \hat{e}_z) \right) + \hat{e}_z \left( \frac{\partial}{\partial z} (V_r \hat{e}_r + V_\theta \hat{e}_\theta + V_z \hat{e}_z) \right)
\]
\[
\text{Term 1} = \hat{e}_r \left( \frac{\partial}{\partial r} (V_r \hat{e}_r + V_\theta \hat{e}_\theta + V_z \hat{e}_z) \right)
\]
\[
= \hat{e}_r \left( \frac{\partial V_r}{\partial r} \hat{e}_r + \frac{V_r}{r} \frac{\partial \hat{e}_r}{\partial r} + \frac{\partial V_\theta}{\partial r} \hat{e}_\theta + \frac{V_\theta}{r} \frac{\partial \hat{e}_\theta}{\partial r} + \frac{\partial V_z}{\partial r} \hat{e}_z + \frac{V_z}{r} \frac{\partial \hat{e}_z}{\partial r} \right)
\]
\[
= \frac{\partial V_r}{\partial r}
\]
\[
\text{Term 2} = \hat{e}_\theta \left( \frac{\partial}{\partial \theta} (V_r \hat{e}_r + V_\theta \hat{e}_\theta + V_z \hat{e}_z) \right)
\]
\[
= \hat{e}_\theta \left( \frac{\partial V_r}{\partial \theta} \hat{e}_r + \frac{V_r}{r} \frac{\partial \hat{e}_r}{\partial \theta} + \frac{\partial V_\theta}{\partial \theta} \hat{e}_\theta + \frac{V_\theta}{r} \frac{\partial \hat{e}_\theta}{\partial \theta} + \frac{\partial V_z}{\partial \theta} \hat{e}_z + \frac{V_z}{r} \frac{\partial \hat{e}_z}{\partial \theta} \right)
\]
\[
= \hat{e}_\theta \left( \frac{\partial V_r}{\partial \theta} \hat{e}_r + \frac{V_r}{r} \frac{\partial \hat{e}_r}{\partial \theta} + \frac{\partial V_\theta}{\partial \theta} \hat{e}_\theta + \frac{V_\theta}{r} \frac{\partial \hat{e}_\theta}{\partial \theta} + \frac{\partial V_z}{\partial \theta} \hat{e}_z + \frac{V_z}{r} \frac{\partial \hat{e}_z}{\partial \theta} \right)
\]
\[
= \frac{V_r}{r} + \frac{1}{r} \frac{\partial V_r}{\partial \theta}
\]
\[
\text{Term 3} = \hat{e}_z \left( \frac{\partial}{\partial z} (V_r \hat{e}_r + V_\theta \hat{e}_\theta + V_z \hat{e}_z) \right)
\]
\[
= \hat{e}_z \left( \frac{\partial V_r}{\partial z} \hat{e}_r + \frac{V_r}{r} \frac{\partial \hat{e}_r}{\partial z} + \frac{\partial V_\theta}{\partial z} \hat{e}_\theta + \frac{V_\theta}{r} \frac{\partial \hat{e}_\theta}{\partial z} + \frac{\partial V_z}{\partial z} \hat{e}_z + \frac{V_z}{r} \frac{\partial \hat{e}_z}{\partial z} \right)
\]
\[
= \frac{\partial V_z}{\partial z}
\]
Therefore,
\[
\nabla \cdot \vec{V} = \left( \hat{e}_r \frac{\partial}{\partial r} + \frac{\hat{e}_\theta}{r} \frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z} \right) \cdot (V_r \hat{e}_r + V_\theta \hat{e}_\theta + V_z \hat{e}_z)
\]
\[
= \frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{1}{r} \frac{\partial V_r}{\partial \theta} + \frac{\partial V_z}{\partial z}
\]
\[
= \frac{1}{r} \left[ \frac{\partial (r V_r)}{\partial r} + \frac{\partial V_\theta}{\partial \theta} + \frac{\partial (r V_z)}{\partial z} \right]
\]
\[
\text{In general form:}
\]
\[
\nabla \cdot \vec{V} = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial (h_2 h_3 V_r)}{\partial q_1} + \frac{\partial (h_1 h_3 V_z)}{\partial q_2} + \frac{\partial (h_1 h_2 V_z)}{\partial q_3} \right]
\]
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Proof 1:

Generalized definition of divergence (from vector algebra)

\[ \nabla = \frac{\hat{e}_1}{h_1} \frac{\partial}{\partial q_1} + \frac{\hat{e}_2}{h_2} \frac{\partial}{\partial q_2} + \frac{\hat{e}_3}{h_3} \frac{\partial}{\partial q_3} \]

From the definition of covariant vector

\[ \nabla q_1 = \frac{\hat{e}_1}{h_1}; \quad \nabla q_2 = \frac{\hat{e}_2}{h_2}; \quad \nabla q_3 = \frac{\hat{e}_3}{h_3} \]

\[ \nabla \cdot \vec{A} = \nabla \cdot (A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3) \]

Consider the first term: \( \nabla \cdot (A_1 \hat{e}_1) = \nabla \cdot (A_1 (h_1 h_2 \nabla q_2 \times \nabla q_3)) \)

Since \( \hat{e}_1, \hat{e}_2, \hat{e}_3 \) is an orthogonal base vector, \( \hat{e}_1 = \hat{e}_2 \times \hat{e}_3 = h_2 \nabla q_2 \times h_3 \nabla q_3 = h_2 h_3 \nabla q_2 \times \nabla q_3 \), then

\[ \nabla \cdot (A_1 \hat{e}_1) = \nabla \cdot (A_1 h_2 h_3) \cdot (\nabla q_2 \times \nabla q_3) + (A_1 h_2 h_3) \nabla \cdot (\nabla q_2 \times \nabla q_3) \]

\[ = \nabla \cdot (A_1 h_2 h_3) \cdot (\frac{\hat{e}_2}{h_2} \times \frac{\hat{e}_3}{h_3}) + (A_1 h_2 h_3) \nabla \cdot (\nabla q_2 \times \nabla q_3) \]

Since \( \nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}) \)

\[ \nabla \cdot (A_1 \hat{e}_1) = \nabla \cdot (A_1 h_2 h_3) \cdot (\frac{\hat{e}_2}{h_2} \times \frac{\hat{e}_3}{h_3}) + (A_1 h_2 h_3) \nabla \cdot (\nabla q_2 \times \nabla q_3) \]

\[ = \nabla \cdot (A_1 h_2 h_3) \cdot (\frac{\hat{e}_1}{h_1}) \]

\[ = \left( \frac{\hat{e}_1}{h_1} \frac{\partial}{\partial q_1} + \frac{\hat{e}_2}{h_2} \frac{\partial}{\partial q_2} + \frac{\hat{e}_3}{h_3} \frac{\partial}{\partial q_3} \right) \cdot \left( \frac{\hat{e}_1}{h_1 h_2 h_3} \right) \]

\[ = \frac{1}{h_1 h_2 h_3} \frac{\partial (A_1 h_2 h_3)}{\partial q_1} \]

Similarly, term 2 and term 3 can be calculated.

Therefore:

\[ \nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial (A_1 h_2 h_3)}{\partial q_1} + \frac{\partial (h_1 h_3 A_2)}{\partial q_2} + \frac{\partial (h_2 h_3 A_3)}{\partial q_3} \right\} \]
Proof 2:

Generalized definition of Divergence of a vector field using integral form

Let \( \vec{B} = B_1 \hat{e}_1 + B_2 \hat{e}_2 + B_3 \hat{e}_3 \), by definition

\[
\nabla \cdot \vec{B} = \text{Div} \vec{B} = \lim_{\Delta V \to 0} \left[ \frac{\Delta}{\Delta V} \right] = \iint \vec{B} \cdot d\bar{A}
\]

Consider a volume element in a curvilinear space around the point \( P(q_1, q_2, q_3) \) with the \( h_1 \Delta q_1, h_2 \Delta q_2, h_3 \Delta q_3 \) as the edge of the volume.

Outflow in the DIV of \( q_1 \) at point \( P(q_1, q_2, q_3) \) is given by

\[
(B_1 \hat{e}_1 + B_2 \hat{e}_2 + B_3 \hat{e}_3) \cdot (\hat{e}_1 h_2 h_3 \Delta q_2 \Delta q_3) = B_1 h_2 h_3 \Delta q_2 \Delta q_3
\]

Net outflow from the surface 1, which is normal to \( q_1 \)-axis with a distance \( \Delta q_1 \) from the point \( P(q_1, q_2, q_3) \) is given by

\[
\text{Surface 1} = B_1 h_2 h_3 \Delta q_2 \Delta q_3 + \frac{\partial (B_1 h_2 h_3 \Delta q_2 \Delta q_3)}{\partial q_1} \frac{\Delta q_1}{2}.
\]

Similarly, inflow from the surface 2, which is normal to \( q_1 \)-axis with a distance \( -\frac{\Delta q_1}{2} \) from the point \( P(q_1, q_2, q_3) \) is given by

\[
\text{Surface 2} = B_1 h_2 h_3 \Delta q_2 \Delta q_3 - \frac{\partial (B_1 h_2 h_3 \Delta q_2 \Delta q_3)}{\partial q_1} \frac{\Delta q_1}{2}.
\]

The net outflow (efflux) along the \( q_1 \) direction will be:

\[
\text{Surface 1 - Surface 2} = \frac{\partial (B_1 h_2 h_3 \Delta q_2 \Delta q_3)}{\partial q_1} \frac{\Delta q_1}{2} = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial (B_1 h_2 h_3)}{\partial q_1} \right] h_1 \Delta q_1 h_2 \Delta q_2 h_3 \Delta q_3
\]

\[
= \frac{1}{h_1 h_2 h_3} \frac{\partial (B_1 h_2 h_3)}{\partial q_1} \Delta V
\]
Similarly, the net outflow (efflux) along the $q_2$ and $q_3$ direction will be:

$$\frac{1}{h_2 h_3} \frac{\partial (B_2 h_1 h_3)}{\partial q_2} \Delta V$$

and

$$\frac{1}{h_2 h_3} \frac{\partial (B_3 h_1 h_2)}{\partial q_3} \Delta V$$

Therefore

$$\nabla \cdot \vec{B} = \lim_{\Delta V \to 0} \frac{\iint \vec{B} \cdot d\vec{A}}{\Delta V}$$

$$= \lim_{\Delta V \to 0} \left[ \frac{1}{h_2 h_3} \frac{\partial (B_1 h_2 h_3)}{\partial q_1} \Delta V + \frac{1}{h_2 h_3} \frac{\partial (B_2 h_1 h_3)}{\partial q_2} \Delta V + \frac{1}{h_2 h_3} \frac{\partial (B_3 h_1 h_2)}{\partial q_3} \Delta V \right]$$

$$= \lim_{\Delta V \to 0} \frac{1}{h_2 h_3} \left[ \frac{\partial (B_1 h_2 h_3)}{\partial q_1} + \frac{\partial (B_2 h_1 h_3)}{\partial q_2} + \frac{\partial (B_3 h_1 h_2)}{\partial q_3} \right]$$

$$= \frac{1}{h_2 h_3} \left[ \frac{\partial (B_1 h_2 h_3)}{\partial q_1} + \frac{\partial (B_2 h_1 h_3)}{\partial q_2} + \frac{\partial (B_3 h_1 h_2)}{\partial q_3} \right]$$
1.12 The Curl of a Vector Field

\[ \nabla \times \vec{B} = \text{Curl } \vec{B} \]

\[ \nabla q_1 = \frac{\hat{e}_1}{h_1}; \quad \nabla q_2 = \frac{\hat{e}_2}{h_2}; \quad \nabla q_3 = \frac{\hat{e}_3}{h_3} \]

\[ \vec{B} = B_1 \hat{e}_1 + B_2 \hat{e}_2 + B_3 \hat{e}_3 \]

\[ \nabla \times \vec{B} = \nabla \times (B_1 \hat{e}_1 + B_2 \hat{e}_2 + B_3 \hat{e}_3) \]

Consider the first term

\[ \nabla \times (B_1 \hat{e}_1) = \nabla \times (B_1 h_1 \nabla q_1) = \nabla \times (B_1 h_1 \nabla q_1) \]

Since \( \nabla \times (\phi \vec{A}) = \nabla \phi \times (\vec{A}) + \phi \nabla \times \vec{A} \)

\[ \nabla \times (B_1 \hat{e}_1) = \nabla (B_1 h_1) \times \nabla q_1 + (B_1 h_1) \nabla \times \nabla q_1 \]

\[ = \nabla (B_1 h_1) \times \nabla q_1 \]

\[ = \left[ \frac{\hat{e}_1}{h_1} \frac{\partial (B_1 h_1)}{\partial q_1} + \frac{\hat{e}_2}{h_2} \frac{\partial (B_1 h_1)}{\partial q_2} + \frac{\hat{e}_3}{h_3} \frac{\partial (B_1 h_1)}{\partial q_3} \right] \times \left( \frac{\hat{e}_1}{h_1} \right) \]

\[ = \frac{\hat{e}_2 \times \hat{e}_1}{h_2 h_1} \frac{\partial (B_1 h_1)}{\partial q_2} + \frac{\hat{e}_1 \times \hat{e}_1}{h_3 h_1} \frac{\partial (B_1 h_1)}{\partial q_3} \]

\[ = -\frac{\hat{e}_3}{h_2 h_1} \frac{\partial (B_1 h_1)}{\partial q_3} + \frac{\hat{e}_2}{h_3 h_1} \frac{\partial (B_1 h_1)}{\partial q_3} \]

\[ = \frac{1}{h_1} \left( \frac{\hat{e}_2}{h_3} \frac{\partial (B_1 h_1)}{\partial q_3} - \frac{\hat{e}_3}{h_2} \frac{\partial (B_1 h_1)}{\partial q_2} \right) \]

Similarly,

\[ \nabla \times (B_2 \hat{e}_2) = \frac{1}{h_2} \left( \frac{\hat{e}_3}{h_1} \frac{\partial (B_2 h_2)}{\partial q_1} - \frac{\hat{e}_1}{h_3} \frac{\partial (B_2 h_2)}{\partial q_3} \right) \]

\[ \nabla \times (B_3 \hat{e}_3) = \frac{1}{h_3} \left( \frac{\hat{e}_1}{h_2} \frac{\partial (B_3 h_3)}{\partial q_2} - \frac{\hat{e}_2}{h_1} \frac{\partial (B_3 h_3)}{\partial q_1} \right) \]

Therefore,
\[ \nabla \times \vec{B} = \nabla \times (B_1 \hat{e}_1 + B_2 \hat{e}_2 + B_3 \hat{e}_3) \]
\[ = \frac{1}{h_1} \left( \frac{\hat{e}_2}{h_3} \frac{\partial (B_1 h_1)}{\partial q_3} - \frac{\hat{e}_3}{h_2} \frac{\partial (B_1 h_1)}{\partial q_2} \right) \]
\[ + \frac{1}{h_2} \left( \frac{\hat{e}_3}{h_1} \frac{\partial (B_2 h_2)}{\partial q_1} - \frac{\hat{e}_2}{h_3} \frac{\partial (B_2 h_2)}{\partial q_3} \right) \]
\[ + \frac{1}{h_3} \left( \frac{\hat{e}_1}{h_2} \frac{\partial (B_3 h_3)}{\partial q_2} - \frac{\hat{e}_2}{h_1} \frac{\partial (B_3 h_3)}{\partial q_1} \right) \]

Or
\[ \nabla \times \vec{B} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix}
    h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\
    \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\
    h_1 B_1 & h_2 B_2 & h_3 B_3 
\end{vmatrix} \]
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1.13 Some Relations Involving the Vector Operator \( \nabla \)

\[
\nabla \equiv \hat{e}_1 \frac{\partial}{\partial q_1} + \hat{e}_2 \frac{\partial}{\partial q_2} + \hat{e}_3 \frac{\partial}{\partial q_3}
\]
is a vector operator and not a vector. Thus, it is necessary to present the orders in which \( \nabla \) appears with respect to the other terms.

For example: \( \nabla \cdot \vec{A} \neq \vec{A} \cdot \nabla \)

Some identities of interest: \( \phi, \psi \) are scalar variables and \( \vec{A}, \vec{B} \) are vector variables:

- \( \nabla (\phi \psi) = \phi \nabla \psi + \psi \nabla \phi \)
- \( \nabla \cdot (\phi \vec{A}) = \nabla \phi \cdot \vec{A} + \phi \nabla \cdot \vec{A} \)
- \( \nabla \times (\phi \vec{A}) = \nabla \phi \times \vec{A} + \phi \nabla \times \vec{A} \)
- \( \nabla \cdot (\vec{A} \cdot \vec{B}) = (\vec{A} \cdot \nabla) \vec{B} + (\vec{B} \cdot \nabla) \vec{A} + \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}) \)
- \( \nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}) \)
- \( \nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) - (\vec{B} \cdot \nabla) \vec{A} - \vec{B}(\nabla \cdot \vec{A}) - (\vec{A} \cdot \nabla) \vec{B} \)
- \( \nabla \cdot (\nabla \times \vec{A}) = 0 \)
- \( \nabla \times (\nabla \phi) = 0 \)
- \( \nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla \cdot \nabla \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \)

Proof:

By simple expansion:

\[
\nabla \cdot (\phi \vec{A}) = (i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}) (i \phi A_x + j \phi A_y + k \phi A_z)
\]

\[
= \frac{\partial \phi A_x}{\partial x} + \frac{\partial \phi A_y}{\partial y} + \frac{\partial \phi A_z}{\partial z}
\]

\[
= A_x \frac{\partial \phi}{\partial x} + \phi \frac{\partial A_x}{\partial x} + \frac{\partial \phi}{\partial y} A_y + \phi \frac{\partial A_y}{\partial y} + \frac{\partial \phi}{\partial z} A_z + \phi \frac{\partial A_z}{\partial z}
\]

\[
= A_x \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} A_y + \frac{\partial \phi}{\partial z} A_z + \phi [\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}]
\]

\[
= \nabla \phi \cdot \vec{A} + \phi \nabla \cdot \vec{A}
\]

The vector and scalar in the identities are defined intrinsically - that is without reference to any special coordinate system. Verification of the above equations in any one coordinate system (e.g, Cartesian) is equivalent to verification of all coordinate system.
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Determination of Laplacian equation $\nabla \nabla \psi = \nabla^2 \psi$

Consider a scalar variable $\psi$

$$\nabla \psi = \hat{e}_1 \frac{\partial \psi}{\partial q_1} + \hat{e}_2 \frac{\partial \psi}{\partial q_2} + \hat{e}_3 \frac{\partial \psi}{\partial q_3} = \vec{B}$$

i.e., $\vec{B} = \left[ \frac{1}{h_1} \frac{\partial \psi}{\partial q_1}; \frac{1}{h_2} \frac{\partial \psi}{\partial q_2}; \frac{1}{h_3} \frac{\partial \psi}{\partial q_3} \right] = (B_1, B_2, B_3)$

$$\nabla \cdot \vec{B} = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial (B_1 h_2 h_3)}{\partial q_1} + \frac{\partial (h_1 h_3 B_2)}{\partial q_2} + \frac{\partial (h_1 h_2 B_3)}{\partial q_3} \right\}$$

$$= \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial q_1} \left( h_2 h_3 \frac{\partial \psi}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left( h_1 h_3 \frac{\partial \psi}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left( h_1 h_2 \frac{\partial \psi}{\partial q_3} \right) \right\}$$

$$= \nabla^2 \psi$$
1.14 Gauss Divergence Theorem

Recall that: \( \nabla \cdot \vec{B} = \text{Div} \vec{B} = \lim_{\Delta V \to 0} \left[ \frac{\int \int \int \vec{B} \cdot d\vec{A}}{\Delta V} \right] \)

can be approximated as: \( \nabla \cdot \vec{B} = \frac{1}{\Delta V} \int \int \int \vec{B} \cdot d\vec{A} \) or \( (\nabla \cdot \vec{B}) \Delta V = \int \int \int \vec{B} \cdot d\vec{A} \) for an element control volume.

Now consider a finite control volume \( V \) in space subdivided into many smaller elemental sub-volumes.

Suppose \( \nabla \cdot \vec{B} \) for all the sub volume are evaluated and summed:

\[
\sum_{i=1}^{N} (\nabla \cdot \vec{B})_i \Delta V_i \approx \sum_{i=1}^{N} \int \int \int \vec{B} \cdot d\vec{A}
\]

\[
\lim_{\Delta V \to 0} \sum_{i=1}^{N} (\nabla \cdot \vec{B})_i \Delta V_i \approx \lim_{\Delta V \to 0} \sum_{i=1}^{N} \int \int \int \vec{B} \cdot d\vec{A}
\]

\[
\int \int \int (\nabla \cdot \vec{B}) \, dV = \lim_{\Delta V \to 0} \sum_{i=1}^{N} \int \int \int \vec{B} \cdot d\vec{A}
\]

The flow of \( \vec{B} \) through the common faces of adjacent volumes canceled because the inflow through one face equals the outflow through the other.

Thus, if we now sum the net outflow of \( \vec{B} \) of all the sub-volumes, only faces on the surface enclosing the region will contribute to the summation.

State in integral form the above statement becomes:

\[
\lim_{\Delta V \to 0} \sum_{i=1}^{N} \int \int \int \vec{B} \cdot d\vec{A} = \int \int \int \vec{B} \cdot d\vec{A}
\]

Thus, Gauss divergence theorem states:

\[
\int \int \int (\nabla \cdot \vec{B}) \, dV = \int \int \int \vec{B} \cdot d\vec{A}
\]
1.15 Stokes Theorem

For a curve $C$ in a three-dimensional space, let us assume there is a function $f(\vec{R}) = f(x, y, z)$ defined everywhere on $C$.

Let us make $N$ sub-divisions between the two points $P_1$ and $P_2$.

Then: \[ \int_C f(x, y, z) \, dl = \lim_{\Delta l_i \to 0} \sum_{i=1}^{N} f(x_i, y_i, z_i) \Delta l_i \]

If we specify $l$ as the arc length parameter $S$, then $f$ can be parametrically represented in forms of the arc length $S$.

i. e., \[ \int_C f(x, y, z) \, dl = \int_{s_1}^{s_2} f(x(s), y(s), z(s)) \, ds \]

Line integral for a vector valued function $\vec{B}$

\[ \int_C \vec{B} \cdot d\vec{l} = \int_C \vec{B} \cdot \hat{e}_T \, dl \]

As before, we let $l$ as the arc length parameter $S$, then

\[ \int_C \vec{B} \, dl = \int_C [\vec{B}(x(s), y(s), z(s)) \frac{d\vec{R}}{dS}] dS \]

Since

$\vec{B} = B_1 \hat{e}_1 + B_2 \hat{e}_2 + B_3 \hat{e}_3$

$\frac{d\vec{R}}{dS} = dS_1 \hat{e}_1 + dS_2 \hat{e}_2 + dS_3 \hat{e}_3$

$\frac{d\vec{R}}{dS} \, dS = dS_1 \hat{e}_1 + dS_2 \hat{e}_2 + dS_3 \hat{e}_3$
Component of Curl in a direction $\hat{e}$

i.e. $\hat{e} \cdot (\nabla \times \vec{B}) = ?$

From the integral definition:

$$\nabla \times \vec{B} = \lim_{\Delta V \to 0} \left[ \frac{\iint d\hat{A} \times \vec{B}}{\Delta V} \right]$$

Thus:

$$\hat{e} \cdot (\nabla \times \vec{B}) = \lim_{\Delta V \to 0} \left[ \frac{\iint \hat{e} \cdot (d\hat{A} \times \vec{B})}{\Delta V} \right] = \lim_{\Delta V \to 0} \left[ \frac{\iint \hat{e} \cdot (\vec{e}_n \times \vec{B}) \, dA}{\Delta V} \right]$$

By using the identity

$$\vec{C} \cdot (\vec{A} \times \vec{B}) = (\vec{C} \times \vec{A}) \cdot \vec{B}$$

for the right hand, we can write:

$$\hat{e} \cdot (\nabla \times \vec{B}) = \lim_{\Delta V \to 0} \left[ \frac{\iint \hat{e} \cdot (\vec{e}_n \times \vec{B}) \, dA}{\Delta V} \right] = \lim_{\Delta V \to 0} \left[ \frac{\iint \vec{B} \cdot (\hat{e} \times \vec{e}_n) \, dA}{\Delta V} \right]$$

To evaluate this integral, we propose a volume elements as a cylinder (not necessarily of circular cross section) with its axis parallel to $\hat{e}$.

Thus,

$$\hat{e} \cdot (\nabla \times \vec{B}) = \lim_{\Delta V \to 0} \left[ \frac{\iint \vec{B} \cdot (\hat{e} \times \vec{e}_n) \, dA}{\Delta V} \right]$$

$$= \lim_{\Delta V \to 0} \left[ \frac{\iint \vec{B} \cdot (\hat{e} \times \vec{e}_n) \, dA}{\Delta V} \right] + \lim_{\Delta V \to 0} \left[ \frac{\iint \vec{B} \cdot (\hat{e} \times \vec{e}_n) \, dA}{\Delta V} \right] + \lim_{\Delta V \to 0} \left[ \frac{\iint \vec{B} \cdot (\hat{e} \times \vec{e}_n) \, dA}{\Delta V} \right]$$
CHAPTER 1 REVIEW OF VECTOR ALGEBRA

At the top and bottom, \( \hat{e} \) and \( \hat{e}_n \) are parallel, therefore, \( \hat{e} \times \hat{e}_n = 0 \).

\[
\hat{e} \cdot (\nabla \times \vec{B}) = \lim_{\Delta \mathcal{A} \to 0} \left[ \frac{\oint_{\Delta \mathcal{A}} (\hat{e} \times \hat{e}_n) \, d\mathcal{A}}{\Delta V} \right] = \lim_{\Delta \mathcal{A} \to 0} \left[ \frac{\oint_{\Delta \mathcal{A} = \text{side}} (\hat{e} \times \hat{e}_n) \, d\mathcal{A}}{\Delta V} \right]
\]

\( \hat{e} \times \hat{e}_n = \hat{e}_t \) at the side of the volume, therefore,

\[
\hat{e} \cdot (\nabla \times \vec{B}) = \lim_{\Delta \mathcal{A} \to 0} \left[ \frac{\oint_{\Delta \mathcal{A} = \text{side}} (\vec{B} \cdot \hat{e}_t) \, d\mathcal{A}}{\Delta V} \right]
\]

By assuming \( \hat{e}_t \cdot \vec{B} \) is constant along the axis on the side surface

Then,

\[
\hat{e} \cdot (\nabla \times \vec{B}) = \lim_{\Delta \mathcal{A} \to 0} \left[ \frac{1}{\Delta A_{\text{bottom} C}} \oint_{C} (\vec{B} \cdot \hat{e}_t) \, ds \right] = \lim_{\Delta \mathcal{A} \to 0} \left[ \frac{1}{\Delta A_{\text{bottom} C}} \oint_{C} (\vec{B} \cdot \hat{e}_t) \, ds \right]
\]

We divide \( \Delta \mathcal{A} \) into a large number of tiny surface regions, say, \( N \) of them,

Given the \( i \)-th region

\[
\hat{e}_{n_i} \cdot (\nabla \times \vec{B}) \big|_{C_i} = \frac{1}{\Delta A_{C_i}} \oint_{C_i} (\vec{B} \cdot \hat{e}_t) \, ds_i
\]

\[
[\hat{e}_{n_i} \cdot (\nabla \times \vec{B})] \Delta A_i \approx \oint_{C_i} (\vec{B} \cdot \hat{e}_t) \, ds_i = \oint_{C_i} \vec{B} \cdot d\vec{S}_i
\]

Adding the \( N \)-equations for \( i = 1 \) to \( N \), the left hand side becomes the surface integral as the portion become infinity tiny as show below:

\[
\sum_{i=1}^{N} [\hat{e}_{n_i} \cdot (\nabla \times \vec{B})] \Delta A_i = \lim_{\Delta \mathcal{A} \to 0} \sum_{i=1}^{N} \oint_{C_i} \vec{B} \cdot d\vec{S}_i
\]

\[
\implies \iint_{\Delta \mathcal{A}} (\nabla \times \vec{B}) \cdot d\vec{A} = \lim_{\Delta \mathcal{A} \to 0} \sum_{i=1}^{N} \oint_{C_i} \vec{B} \cdot d\vec{S}_i
\]
Consider surface region 1 and 2 which have the portion DC of their boundaries in common. At any point on the segments DC, note that the $d\vec{S}$ for $C_1$ and $C_2$ are oppositely orientated while the $\vec{B}$ is uniquely defined. So, the contribution from the DC portion of $C_2$ be exactly cancels the contribution of $C_1$. Similarly, we have cancellation from all the $C_i$’s except for segments along the boundary curve $C$, such as AB, which are not shared.

State in integral form:

$$\lim_{\Delta A \to 0} \sum_{i=1}^{N} \oint_{C_i} \vec{B} \cdot d\vec{S}_i \approx \oint_{C} \vec{B} \cdot d\vec{S}$$

Therefore,

$$\iint_{\Delta S} (\nabla \times \vec{B}) \cdot d\vec{A} = \oint_{C} \vec{B} \cdot d\vec{S}$$