Chapter 7  Vortex Dynamics

7.1  Vortex Lines and Vortex Tubes

Vortex Lines and Vortex Tubes
A vortex line is a line whose tangent is everywhere parallel to the vorticity vector.

Then, for any closed contour in a flow field, each point on the contour will have a vortex line passing through it, and the series of vortex lines defined by the closed contour from a vortex tube.

A vortex tube whose area is infinitesimally small is usually referred as vortex filament.

Vortex lines either must form a close loops or must terminate on the boundaries of the fluid. The boundary of the fluid may be either a solid surface or a free surface.

Similarity Between of the Stream Tube and Vortex Tube

Mass conservation in incompressible flow,
\[ \nabla \cdot \vec{V} = 0 \]
Consider the volume \( \mathcal{V} \) of a stream tube,
\[ \int_{\mathcal{V}} \nabla \cdot \vec{V} d\mathcal{V} = 0 \]
Using Gauss divergence theorem:
\[
\int_{\mathcal{L}} \int_{s_1} \int_{s_1} \nabla \cdot \vec{V} d\mathcal{V} = \int_{s_2} \vec{V} \cdot d\vec{A} + \int_{s_2} \vec{V} \cdot d\vec{A} + \int_{s_1} \vec{V} \cdot d\vec{A} + \int_{s_1} \vec{V} \cdot d\vec{A}
\]
\[
= \int_{s_1} \vec{V} \cdot d\vec{A} + \int_{s_2} \vec{V} \cdot d\vec{A}
\]
\[
= -Q_1 + Q_2
\]

For vorticity field

Since \( \nabla \cdot \vec{\Omega} = 0 \)

Consider the volume \( \mathcal{V} \) of a vortex tube,
\[ \iiint_{\mathcal{V}} \nabla \cdot \Omega \, dV = 0 \]

Using Gauss divergence theorem:

\[
\iiint_{\mathcal{V}} \nabla \cdot \Omega \, dV = \int_{s_1} \int_{s_2} \int_{s_3} \int_{s_4} \int_{s_5} \int_{s_6} \int_{s_7} \int_{s_8} \nabla \cdot \Omega \, dA = \int_{s_1} \int_{s_2} \int_{s_3} \int_{s_4} \int_{s_5} \int_{s_6} \int_{s_7} \int_{s_8} \nabla \cdot \Omega \, dA
\]

\[= \int_{s_1} \int_{s_2} \int_{s_3} \int_{s_4} \int_{s_5} \int_{s_6} \int_{s_7} \int_{s_8} \nabla \cdot \Omega \, dA = -\Gamma_1 + \Gamma_2 = 0 \]

i.e., \[ \Gamma_1 = \Gamma_2 \]

Circulation at each cross section of a vortex tube is the same.

If \( \Omega_1 \) is the averaged vorticity at \( A_1 \) and \( \Omega_2 \) is the averaged vorticity at \( A_2 \),

Then, \[ \iiint_{\mathcal{V}} \nabla \cdot \Omega \, dV = \int_{s_1} \int_{s_2} \int_{s_3} \int_{s_4} \int_{s_5} \int_{s_6} \int_{s_7} \int_{s_8} \nabla \cdot \Omega \, dA = -A_1 \Omega_1 + A_2 \Omega_2 = 0 \]

i.e., \[ A_1 \Omega_1 = A_2 \Omega_2 = \text{const} \]

Implication:

- Vortex tube, vortex filament or a vortex line cannot begin or end abruptly in a fluid. It should form a closed vortex ring, or end at infinity or at a solid or free surface.
- According to Kelvin’s theorem, \( \frac{D\Gamma}{Dt} = 0 \) for ideal flow and body force field is conservative, it can be deduced that a surface which is a vortex sheet at one instant time remains as a vortex sheet for all time.

Vortex are common in nature, the difference between a real vortex as opposed to a theoretic vortex is that the former has a core of fluid which is rotating as “solid”, although the associated “swirl” outside is the same as the flow outside the point vortex. This happens because the rate of change of velocity radially becomes so large (as \( r \) tends to go to 0) that the viscous effects become significant and the assumption of inviscid flow breaks down. Thus, the central core of the flow starts to rotate as a solid body and vortex type flow, in which the velocity varies inversely as radius, exists outside the core.

A tornado in the atmosphere is an example of a vortex in nature. Because of viscous effects, a funnel-shaped core is formed whose diameter increasing upward. The vortex originates near the ground so, as the air within it moves upward, the circulation around the core remains constant while the radius increases. Thus, velocities near the ground will be much higher than they are
aloft; the resulting high pressure variations associated with the flow around structure is one of the main causes of damage of structure.

Many other examples of vortices are commonly observed. Behind an airplane, there are two trailing vortices of opposite directions of circulations. The commonly observed smoke rings illustrate a vortex filament that closes itself.
7.2 Conservation of Circulation Theorem

Consider a closed curve $C$ attached to the fluid at time $t_0$ and let it move with the fluid.

Let $\Gamma$ be the circulation of the fluid particles of fixed identity in $C$.

\[ \Gamma = \oint_C \mathbf{V} \cdot d\mathbf{l} \] the line integral is over space at a given time.

\[ \frac{d\Gamma}{Dt} = \oint_C \frac{D\mathbf{V}}{Dt} \cdot d\mathbf{l} + \oint_C \mathbf{V} \cdot \frac{D(d\mathbf{l})}{Dt} \]

Let $\mathbf{f}$ be conservative (i.e. $\mathbf{f} = \nabla U$), and rewrite the Euler equation:

\[ \frac{D\mathbf{V}}{Dt} = -\frac{\nabla (P/\rho)}{\rho} + \nabla U \]

Since

\[ \begin{align*}
\mathbf{a}_1 + d\mathbf{l}_1 &= d\mathbf{l} + b\mathbf{b}_1 \\
\mathbf{a}_1 &= \mathbf{V}\Delta t \\
b\mathbf{b}_1 &= (\mathbf{V} + \Delta\mathbf{V})\Delta t
\end{align*} \]

\[ \frac{D(d\mathbf{l})}{Dt} = \lim_{\Delta t \to 0} \frac{d\mathbf{l}_1 - d\mathbf{l}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{V}\Delta t}{\Delta t} = \Delta\mathbf{V} \]

Therefore,

\[ \frac{d\Gamma}{Dt} = \oint_C \left[ -\frac{\nabla (P/\rho)}{\rho} + \nabla U \right] \cdot d\mathbf{l} + \oint_C \mathbf{V} \cdot \frac{D(d\mathbf{l})}{Dt} = \oint_C \nabla \left( \frac{P}{\rho} + U \right) \cdot d\mathbf{l} + \oint_C \mathbf{V} \cdot \mathbf{V} = \oint_C \left( d\left( \frac{P}{\rho} + U \right) + \frac{\mathbf{V}^2}{2} \right) \]

\[ = \oint_C d\left( \frac{P}{\rho} + U + \frac{\mathbf{V}^2}{2} \right) = 0 \]
Conservation of circulation theorem states that, the circulation remains a constant along any closed fluid curves in a conservative body force field under the condition of ideal flows (inviscid, non-heat conducting, incompressible, homogenous). The theorem is also called Kelvin’s Theorm.

Implication of Kelvin Theorem:

- If the fluid starts from rest, or if the fluid in some region is uniform and parallel, the rotation in this region is zero.

- Kelvin’s theorem leads to the important conclusion that the entire flow remains irrotational if the fluid is homogenous and body force is conservative in the absence of heat conduction, viscous forces.
7.3 Rate of Change of Vorticity in an Ideal Fluid

Consider the Euler equation

$$\frac{\partial \vec{V}}{\partial t} + \nabla\left(\frac{\vec{V} \cdot \vec{V}}{2}\right) - \vec{V} \times (\nabla \times \vec{V}) = -\nabla\left(\frac{P}{\rho}\right) + \vec{f}$$

Or

$$\frac{\partial (\vec{V})}{\partial t} + \nabla\left(\frac{\vec{V} \cdot \vec{V}}{2}\right) - \vec{V} \times \vec{\Omega} = -\nabla\left(\frac{P}{\rho}\right) + \vec{f}$$

Taking curl of the above equation, and notice that the order of time and space derivative does not matter;

$$\frac{\partial (\nabla \times \vec{V})}{\partial t} + \nabla \times \nabla\left(\frac{\vec{V} \cdot \vec{V}}{2}\right) - \nabla \times (\vec{V} \times \vec{\Omega}) = -\nabla \times \nabla\left(\frac{P}{\rho}\right) + \nabla \times \vec{f}$$

Since Curl of a gradient is zero, and if the body force is conservative, the above equation can be reduced to

$$\frac{\partial \vec{\Omega}}{\partial t} - \nabla \times (\vec{V} \times \vec{\Omega}) = 0$$

Use the vector identity

$$\nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) + (\vec{B} \cdot \nabla)\vec{A} - \vec{B}(\nabla \cdot \vec{A}) - (\vec{A} \cdot \nabla)\vec{B}$$

Then,

$$\nabla \times (\vec{V} \times \vec{\Omega}) = \vec{V}(\nabla \cdot \vec{\Omega}) + (\vec{\Omega} \cdot \nabla)\vec{V} - \vec{\Omega}(\nabla \cdot \vec{V}) - (\vec{V} \cdot \nabla)\vec{\Omega}$$

$$= (\vec{\Omega} \cdot \nabla)\vec{V} - (\vec{V} \cdot \nabla)\vec{\Omega}$$

Therefore,
\[ \frac{\partial \Omega}{\partial t} = (\Omega \cdot \nabla) \vec{V} + (\vec{V} \cdot \nabla) \Omega = 0 \]

\[ \Rightarrow \frac{D \Omega}{D t} = (\Omega \cdot \nabla) \vec{V} = 0 \]

\[ \Rightarrow \frac{D \Omega}{D t} = (\Omega \cdot \nabla) \vec{V} \]

Consider \( d\phi = \nabla \phi \cdot d\vec{S} = \nabla \phi \cdot \hat{e}_s \, dS \quad \Rightarrow \quad \frac{d\phi}{dS} = \hat{e}_s \cdot \nabla \phi \)

That is, the rate if change of \( \phi \) along \( S \) is given by operating \( (\hat{e}_s \cdot \nabla) \) on \( \phi \).

Similarly \( \frac{d\vec{A}}{dS} = (\hat{e}_s \cdot \nabla) \vec{A} \) yields the derivative of \( \vec{A} \) respect to distance in the direction of \( \hat{e}_s \).

Therefore, above equation becomes

\[ \frac{D \Omega}{D t} = (\hat{e}_\Omega \cdot \nabla) \vec{V} = 0 \]

Where \( (\hat{e}_\Omega \cdot \nabla) \vec{V} \) is rate of change of \( \vec{V} \) with respect to distance in the direction of \( \hat{\Omega} \).

Case 1.

If \( \| \Omega \| = 0 \) (i.e., \( \Omega = 0 \), then the material derivative of \( \Omega \) is also zero.

Case 2:

In two-dimensional flows:

\[ V_3 = 0 \text{ and } \vec{V} = V_1 \hat{e}_1 + V_2 \hat{e}_2 \]

\[ \Omega = \nabla \times \vec{V} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & 0 \\ h_1 V_1 & h_2 V_2 & 0 \end{vmatrix} = \frac{h_3 \hat{e}_3}{h_1 h_2 h_3} \left[ \frac{\partial (h_2 V_2)}{\partial q_1} - \frac{\partial (h_1 V_1)}{\partial q_2} \right] = \hat{e}_3 \| \Omega \| \]

\[ \Omega \cdot \nabla = (\hat{e}_3 \| \Omega \|) \cdot \left( \hat{e}_1 \frac{\partial}{\partial q_1} + \hat{e}_2 \frac{\partial}{\partial q_2} + 0 \right) = 0 \]

Then
\[
\frac{D\tilde{\Omega}}{Dt} - \|\tilde{\Omega}\| (\mathbf{\tilde{e}}_t \cdot \nabla) \mathbf{\tilde{v}} = 0 \quad \Rightarrow \quad \frac{D\tilde{\Omega}}{Dt} = 0
\]

Therefore, the material rate of change of the vorticity is zero whenever the vorticity is zero or if the flow is two-dimensional in an ideal fluid under the action of irrotational body forces.