Energy Optimal Speed Profile for Arrival of Tandem Tilt-Wing eVTOL Aircraft with RTA Constraint

Priyank Pradeep[1] and Peng Wei[2]

***Abstract*— The electric vertical takeoff and landing (eVTOL) aircraft can alleviate transportation congestion on the ground by utilizing three-dimensional airspace efficiently. However, the endurance of Lithium-ion Polymer (Li-Po) batteries imposes severe constraints on the operational time span of an eVTOL on urban air mobility (UAM) passenger transportation mission. This research focuses on the formulation of a fixed final time multiphase optimal control problem with energy consumption as the performance index for a tandem tilt-wing eVTOL aircraft. The proposed multiphase optimal control problem formulation and the numerical solution enable the eVTOL aircraft to meet the given required time of arrival (RTA) with the optimal speed profile for the most energy efficient arrival. The problem formulation is validated in a UAM passenger transport use case with Airbus Vahana eVTOL aircraft, and the Uber Elevate proposed vertiport concept in numerical simulations.**

***Index Terms*— Concept of Operation (CONOP), Electric Vertical Takeoff and Landing (eVTOL) aircraft, Required Time of Arrival (RTA), Urban Air Mobility (UAM).**

I. INTRODUCTION

In 2014, road-traffic congestion resulted in a wastage of 6.9 billion person-hours and 3.1 billion gallons of excess fuel consumed in the USA alone [1]. Transportation as a whole accounted for approximately 33 % of CO_2 emissions in the USA, of which 80 % are from cars and trucks traveling on the roadway system [2]. A study in the American Journal of Preventative Medicine, for example, found that those who commute more than 10 miles were at increased odds of elevated blood pressure [3]. The electric vertical takeoff and landing (eVTOL) aircraft can alleviate transportation congestion on the ground by utilizing three-dimensional airspace efficiently, just as skyscrapers allowed cities to use limited land more efficiently [4]. Recently, technological advances have made it possible to build eVTOL aircraft. Over a dozen companies, for example, Airbus, Aurora, Ehang, Jobby Aviation, Lilium, Volocopter, etc., with many different design approaches, are passionately working to make eVTOLs a reality [4], [5].

II. BACKGROUND AND MOTIVATION

A. Background

The possibility of urban air mobility (UAM) has been explored by NASA, Uber, Airbus and university researchers since 2016 [5]. Most of the UAM operations of the eVTOL aircraft are in a constrained environment (due to limitations in vertiport capacity and battery endurance). To the best of our knowledge, no significant research work has been carried out for the energy efficient arrival with the required time of arrival (RTA) for various types of eVTOL aircraft in the UAM context [5], [6]. This paper aims to work towards filling the research gap to enable safe and energy efficient arrival operations of a tandem tilt-wing eVTOL aircraft given the limited vertiport capacity and endurance of Lithium-ion polymer (Li-Po) batteries [5].

Considerable efficiency improvements are possible utilizing distributed electric propulsion (DEP) technology because it enables fixed-wing VTOL aircraft to provide lift with far higher efficiency than rotors especially in cruise phase [4]. Engine failure accounts for 18 % of general aviation accidents when combined with fuel management errors. The use of DEP, controllers, and a redundant battery bus architecture avoids the problems of catastrophic engine failure by having full propulsion system redundancy [4], [7]. In addition to the improved cruise efficiency, a tandem tilt-wing has an advantage of lower power consumption in hover than a tilt-rotor as the impact of the rotor downwash on the wing is substantially reduced [7], [8]. A tilt-wing has an additional benefit as the induced airflow behind the rotors reduces the angle-of-attack on the wing in hover and low-speed forward flight [7].

B. Motivation

Electric propulsion is the preferred propulsion choice for the urban VTOL aircraft because of the zero operational emissions [4]. However, the specific energy (the amount of energy per unit weight provided by the battery) of Lithium-ion polymer (Li-Po) batteries today is insufficient for long-range commutes [5], [12]. Also, from the certification point of view, eVTOL aircraft may require landing with reserve battery charge/usage time (analogous to reserve fuel for conventional aircraft). This research paper focuses on optimal (minimum battery usage) speed profile computation under arrival time-constraint for the operational success of tandem tilt-wing eVTOL aircraft [5].

III. MULTIPHASE OPTIMAL CONTROL PROBLEM FORMULATION

A. Aircraft Model

In this paper, the aircraft dynamics are modeled based on the tandem tilt-wing eVTOL (Airbus Vahana) from Airbus A^3 [7]. This eVTOL aircraft has two tandem tilt-wings with eight rotors.

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Fig. 1. Airbus Vahana: Tandem tilt-wing configuration during the cruise phase [7]

B. Trajectory Optimization

In this research, the longitudinal flight dynamics for the tandem tilt-wing aircraft are decoupled from the lateral flight dynamics as the eVTOL has a symmetrical: i) wing structure and ii) placements of rotors about the longitudinal axis; like fixed-wing aircraft and quadrotors respectively. Hence, the decoupling logic used by researchers for fixed-wing aircraft [13], [14], [15], [16], [17], conventional rotorcraft [8], [9], [10], [11] and quadrotors [5], [12] are assumed to be applicable for the tandem tilt-wing eVTOL as well.

In this paper, the following assumptions have been made:

- The lateral trajectory is a geodesic path.
- The vertical trajectory of arrival consists of a portion of the cruise, transition, and descent phases.
- Only, a part of the cruise phase is considered to study energy efficient delay absorption while airborne.
- The transition phase involves tandem tilt of rotors and wings for the transition from cruise speed to hover for the vertical descent per the concept of operation (CONOP) of Airbus Vahana [7].
- In the transition phase, the rotation of rotors and wings from cruise to vertical descent configuration occurs in negligible time with ignorable mechanical energy losses.

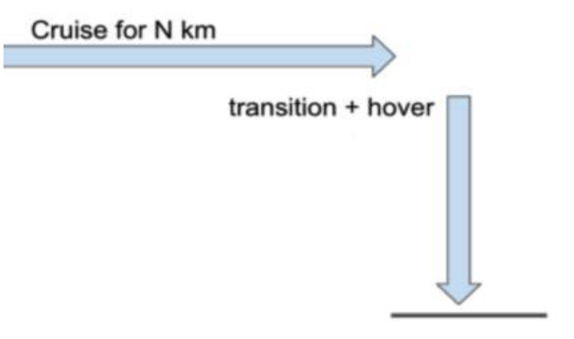

Fig. 2. Vertical trajectory of the eVTOL's arrival [7]

- The transition phase, consists of the following sub-phases: i) deceleration to hover speed and ii) hover above the vertiport at the cruise altitude.
- The descent phase includes vertical descent.

Therefore, only the speed profile of the eVTOL aircraft is free for the optimization. However, since the arrival time constraint has been imposed on the eVTOL aircraft, the trajectory optimization problem involves computation of an energy efficient speed profile for the eVTOL aircraft with fixed final time [5].

C. Cruise Flight Dynamics

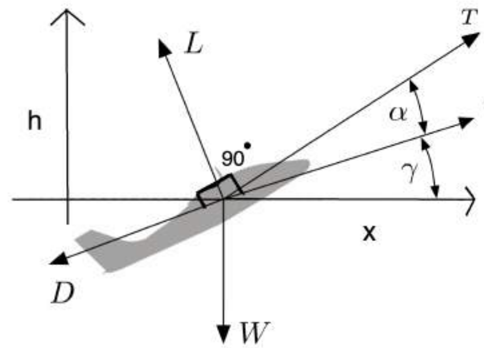

Fig. 3. Free body diagram of the eVTOL [17]

The decoupling of the longitudinal and lateral flight dynamics allows us to solve the vertical trajectory generation problem as a two-dimensional flight dynamics problem in the vertical plane. Therefore, the equations of motion for the tandem tilt-wing eVTOL aircraft (shown in Fig. 1) in aerodynamic frame of reference are as follows [13], [14], [15], [16], [17]:

$$\frac{dV}{dt} = \frac{T\cos\alpha - D - W\sin\gamma}{m} \tag{1}$$

$$\frac{dx}{dt} = V\cos\gamma \tag{2}$$

$$\frac{dh}{dt} = V\sin\gamma \tag{3}$$

$$T = \sum_{i=1}^{8} T_i \tag{4}$$

where $[x, h]$ is the position vector (along track distance, altitude) of the center of mass relative to the initial position, α is the angle between the thrust vector and the aerodynamic velocity, γ is the aerodynamic flight path angle, T is the net thrust produced by DEP of the eVTOL, D is the net drag, L is the net Lift, T_i is the thrust produced by the i^{th} rotor, m is the mass, V is the true airspeed, W is the weight of the eVTOL aircraft and g is the acceleration due to gravity.

D. Path Constraints in Cruise Phase

The path constraints for the tandem tilt-wing eVTOL aircraft in the cruise phase are as follows:

i) Assuming the cruise phase consists of constant level flight segments:

$$T\sin\alpha + L - W\cos\gamma = 0 \tag{5}$$

$$\frac{dh}{dt} = 0 \tag{6}$$

$$\gamma = 0 \tag{7}$$

ii) Lower and upper bounds on the cruise speed:

$$1.3V_{\text{stall}} \leq V \leq V_{\text{max cruise}} \tag{8}$$

E. Drag Model in Cruise

The cruise performance of the tandem tilt-wing eVTOL aircraft is based on a traditional quadratic drag polar with a span efficiency factor (e) of 1.3 [7]. The drag model used in this research is based on Airbus Vahana's drag model as follows [7]:

$$S_{\text{ref}} = \frac{2L}{\rho V_{\text{stall}}^2 C_{\text{Lmax}}} \tag{9}$$

where $C_{\text{Lmax}} = 1.1$, $V_{\text{stall}} = 35$ m/s, ρ is the density of air at the cruising altitude and S_{ref} is the reference area of the eVTOL (both wings combined).

$$AR = \frac{b_{\text{ref}}^2}{S_{\text{ref}}} \tag{10}$$

where AR is the equivalent aspect ratio of the eVTOL aircraft.

$$C_{\text{D0}} = (C_{\text{D0}})_{\text{wing}} + (C_{\text{D0}})_{\text{fuselage}} \tag{11}$$

where C_{D0} is the overall parasite drag coefficient based on the lumped drag model.

$$L = \frac{\rho V^2 S_{\text{ref}} C_L}{2} \tag{12}$$

where L is the lift, C_L is the coefficient of lift and V is the true airspeed.

$$D = \frac{\rho V^2 S_{\text{ref}}}{2}(C_{\text{D0}} + \frac{C_L^2}{\pi ARe}) \tag{13}$$

where D is the net drag, AR is the aspect ratio and e is the span efficiency.

F. Power Consumption Model in Cruise

The power consumption model by DEP of the eVTOL aircraft in cruise phase is derived from the standard lift, drag, and propulsion models assuming quasi-steady and constant altitude of the aircraft [18]. The equation governing the power consumption by DEP of the tandem tilt-wing eVTOL aircraft in cruise phase is as follows [18]:

$$P = \frac{TV\cos\alpha}{\eta_{\text{prop}}} \tag{14}$$

where T is the net thrust produced by DEP of the eVTOL, η_{prop} is the efficiency of the propeller, α is the angle between the thrust vector and the aerodynamic velocity and V is the true air speed of the eVTOL aircraft.

Therefore, the total energy consumed (E_{cruise}) by DEP in cruise phase is as follows:

$$E_{\text{cruise}} = \int_0^{t_f^{\text{cruise}}} P dt \tag{15}$$

G. Transition Flight Dynamics

In the transition phase, the eVTOL is assumed to decelerate from cruise speed to hover at a constant altitude [7]. Hence, transition phase consists of deceleration of the eVTOL followed by hover above the vertiport at the cruise altitude. In this research, we have assumed the following: i) rotation of wings and rotors from cruise configuration (Fig. 1) to vertical descent configuration (Fig. 4) occurs in negligible duration of time and ii) deceleration of the eVTOL aircraft occurs solely due to the aerodynamic drag. Hence, the equations of motion in transition phase is given by:

$$\frac{dV}{dt} = -\frac{D}{m} \tag{16}$$

$$\frac{dx}{dt} = V \tag{17}$$

$$T = \sum_{i=1}^{8} T_i \tag{18}$$

H. Path Constraints in Transition Phase

The path constraints for the tandem tilt-wing eVTOL aircraft in transition phase are as follows:

i) Assuming transition phase consisting of constant level flight segments:

$$T - W = 0 \tag{19}$$

$$\frac{dh}{dt} = 0 \tag{20}$$

$$\gamma = 0 \tag{21}$$

ii) Lower and upper bounds on the cruise speed:

$$0 \leq V \leq V_{\text{max cruise}} \tag{22}$$

I. Drag Model in Transition Phase

The net drag on the aircraft is assumed to be equivalent to the parasite drag on the fuselage and wings of the aircraft. Therefore, the net drag on the aircraft is calculated as follows [11], [21], [5], [12]:

$$D = \rho V^2 S_{\text{ref}} \frac{(C_{\text{D0}})_{\text{fuselage}} + C_{\text{D}}\sin\alpha}{2} \tag{23}$$

where $C_{\text{D}} = 1$ [11] and $S_{\text{ref}}\sin\alpha$ is the instantaneous equivalent flat plate area of the wings (combined) during the rotation of wings (transition phase).

J. Power Consumption in Transition Phase

The equation governing the power consumption by DEP of the tandem tilt-wing eVTOL aircraft in transition phase i.e. deceleration and hover (assuming DEP produce vertically upward thrust) is calculated as follows:

$$P = \frac{T v_{\text{h}}}{\eta_{\text{prop}}} \tag{24}$$

where T is the net thrust produced by DEP of the eVTOL, η_{prop} is the efficiency of the propeller and v_{h} is the induced velocity in hover.

Using momentum theory [19], [8], the induced velocity (v_{h}) in hover is given by:

$$v_{\text{h}} = \sqrt{\frac{T_i}{2\rho A}} \tag{25}$$

where T_i is the thrust produced by the i^{th} rotor, A is the rotor disk area (πR^2), R is the radius of the rotor and ρ is the density of the air.

In this research, as stated earlier we have ignored mechanical energy losses during the rotation of wings and rotors from cruise to descent configuration. Therefore, the total energy consumed ($E_{\text{transition}}$) by DEP in transition phase is as follows:

$$E_{\text{transition}} = \int_0^{t_f^{\text{transition}}} P dt \tag{26}$$

K. Vertical Descent Flight Dynamics

Fig. 4. Airbus Vahana: Tandem tilt-wing configuration during the descent phase [7]

The equations of motion for the tandem tilt-wing eVTOL aircraft (shown in Fig. 3) during the vertical descent are as follows [8], [9], [10], [11]:

$$\frac{dV}{dt} = \frac{T - D - W}{m} \tag{27}$$

$$\frac{dh}{dt} = V \tag{28}$$

$$T = \sum_{i=1}^{8} T_i \tag{29}$$

where h is the altitude of the center of mass, T is the net thrust, D is the parasite drag, T_{i} is the thrust produced by the i^{th} rotor, m is the mass, V is the true airspeed, W is the weight of the eVTOL aircraft and g is the acceleration due to gravity.

L. Path Constraint in Descent

In order to avoid Vortex Ring State (VRS) during the descent phase, the following additional path constraint is imposed on the descent phase of the problem [22], [5], [12]:

$$-0.28 \leq \frac{V}{v_{\text{h}}} \leq 0 \tag{30}$$

where v_{h} is the induced velocity in hover.

M. Drag Model in Descent

The net drag on the aircraft is assumed to be equivalent to the parasite drag on the fuselage of the aircraft. Therefore, the net drag on the aircraft is calculated as follows [11], [21]:

$$D = \frac{\rho V^2 C_{\text{D}} F_{\text{top}}}{2} \tag{31}$$

where F_{top} is the equivalent top flat plate area of the fuselage and $C_{\text{D}} = 1$ [11].

N. Power Consumption in Descent

Consider a rotor in vertical descent at true airspeed V, the solution for induced velocity (v_i) is [19], [8]:

$$v_i = \frac{v_{\text{h}}^2}{(-V + v_i)} \tag{32}$$

The power consumed by DEP in descent phase is as follows [19], [8]:

$$P = \frac{T(-V + v_i)}{\eta_{\text{prop}}} \tag{33}$$

where T is the net thrust produced by the eVTOL. Hence, the total energy consumed by DEP during the vertical descent is given by:

$$E_{\text{descent}} = \int_0^{t_f^{\text{descent}}} P dt \tag{34}$$

O. Phase link

Each of the three phases in the trajectory is linked to the adjoining phases by a set of linkage conditions [23]. These constraints force the position (along-track and altitude) and velocity (horizontal component and vertical component) to be continuous [12], [5]. Hence, the problem is subject to phase link constraints on state variables (X) as follows [23]:

$$X^{\text{phase-1}}(t_f^{\text{phase-1}}) = X^{\text{phase}}(t_0^{\text{phase}}) \tag{35}$$

P. Performance Index

The performance index (Lagrange type) of the multiphase optimal control problem for the vertical trajectory optimization of the eVTOL aircraft is as follows [5], [12]:

$$J = \sum_{N=1}^{3} \int_{t_0^N}^{t_f^N} (\sum_{i=1}^{8} P_i) dt \tag{36}$$

where i is the i^{th} rotor, P_i is the power consumption by the i^{th} rotor, and N is the vertical flight phase (N = 1 for cruise, N = 2 for transition and N = 3 for descent (arrival)).

IV. NUMERICAL STUDY

The equations of motion of the tandem tilt-wing eVTOL are continuous-time nonlinear differential equations [5], [12]. Therefore, a numerical-optimization technique (direct method) has been used in this research to solve the trajectory optimization problem.

A. GPOPS-II

GPOPS-II is a commercially available general-purpose MATLAB software for solving multiphase optimal control problems using variable-order Gaussian quadrature collocation methods. The software employs a Legendre-Gauss-Radau quadrature orthogonal collocation (Pseudospectral) method where the continuous-time optimal control problem is transcribed to a large sparse nonlinear programming problem (NLP) [23]. GPOPS-II has been used for solving the multiphase optimal control problems (maximum area coverage path and minimum energy path). The IPOPT has been used as the solver to solve the problems transcribed to NLP by GPOPS-II [24]. The numerical simulation was performed on a MacBook Pro with 2.8 GHz Intel Core i7 Processor.

B. Initial and Final Conditions

The initial condition (IC) and final condition (FC) for the multi-phase optimal control problems are as shown in Table I, II and III.

TABLE I
Initial and Final Conditions in Cruise

State Variable	IC	FC
Altitude	500 m	500 m
Along-track distance	0 m	Free

TABLE II
Initial and Final Conditions in Transition

State Variable	IC	FC
Altitude	500 m	500 m
Along-track distance	Free	50000 m

TABLE III
Initial and Final Conditions in Descent

State Variable	IC	FC
Altitude	500 m	5 m
Along-track distance	50000 m	50000 m

C. Performance Data

The performance data of the eVTOL aircraft are as shown in Table IV based on design range of 60 km [7].

TABLE IV
Performance Data of Airbus Vahana [7]

Performance Parameter	Value
m	752.2 kg
b_{ref}	6.87 m
S_{ref}	8.93 m^2
AR	5.29
e	1.3
c_{ref}	0.65 m
$(C_{D0})_{\text{wing}}$	0.012
$(C_{D0})_{\text{fuselage}}$	0.039
$(\frac{L}{D})_{\text{cruise}}$	5.92
η_{prop}	0.8
F_{top}	5.8 m^2
R	0.95 m
V_{stall}	35 m/s
$V_{\text{max cruise}}$	80 m/s

D. Results of Minimum Energy Trajectory for Various Constant Cruise Speed Missions with RTA Constraint (1500 Sec)

The energy consumptions (Megajoule (MJ)), ground speed profiles (m/s) and flight time distributions (sec) for Airbus Vahana on various constant cruise speed missions (45.5, 50, 60, 70 and 80 m/s) with RTA constraint (1500 sec) are shown in Fig. 5, Fig. 6 and Fig. 7. From the figures, it can be seen that while the tandem-tilt eVTOL aircraft is airborne; the delay absorption is most energy efficiently managed between cruise and hover by distributing the maximum possible delay to cruise phase followed by absorption of remaining delay in hover at the cruise altitude directly above the vertiport per the CONOP.

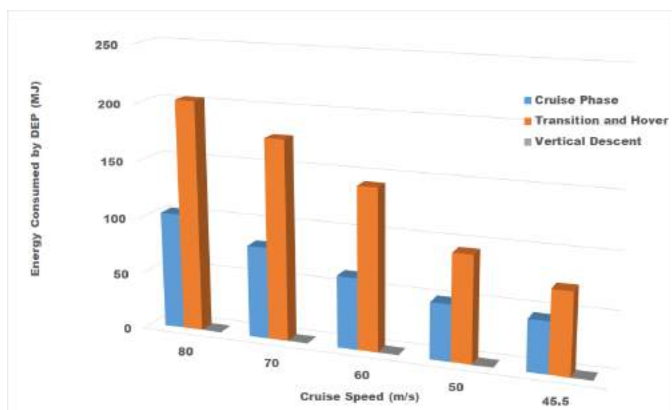

Fig. 5. **Energy consumed by DEP for various cruise speeds with fixed RTA (1500 sec)**

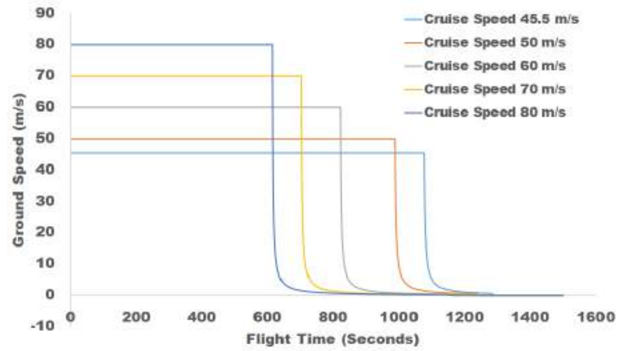

Fig. 6. **Ground speed profiles for various cruise speeds with fixed RTA (1500 sec)**

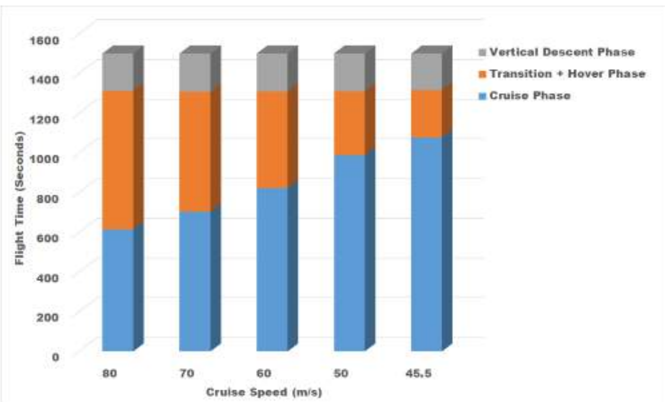

Fig. 7. **Flight time distribution for various cruise speeds with fixed RTA (1500 sec)**

V. CONCLUSION

In this research, a multiphase optimal control problem with energy consumption as the performance index is formulated for a tandem-tilt eVTOL aircraft on urban air mobility (UAM) passenger transportation mission. Further, we present an optimal control framework to perform energy efficient arrival for a tandem-tilt urban eVTOL aircraft given the required time of arrival (RTA) constraint. The formulated vertical trajectory optimization problem is numerically solved using pseudospectral method for a specific eVTOL aircraft, i.e., Airbus Vahana. For the given CONOP of Airbus Vahana, the computational results show that while the tandem-tilt eVTOL aircraft is airborne; the delay absorption is most energy efficiently managed between cruise and hover by distributing the maximum possible delay to cruise phase followed by absorption of the remaining delay in hover at the cruise altitude directly above the vertiport.

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