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A NEW LOOK AT THE PROBLEM OF PLASTIC SPIN BASED ON STABILITY ANALYSIS

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ABSTRACT

A fundamental contradiction in the theory of constitutive equations is revealed for polycrystalline solids: in the general theory it is impossible to exclude rotation relative to some fixed privileged configuration when the principle of material frame-indifference is applied. Two equivalent methods of overcoming the above contradiction are developed. In the first one it is assumed that after excluding the rigid body rotation relative to the fixed privileged configuration the constitutive equations depend additionally on some rotational internal variable(s), for which some equation(s) will be derived. In the second one we formulate and solve the problem of finding some variable privileged configuration, similar to Mandel's isoclinic configuration. Consequently, we arrive at the problem of determination of an equation for plastic spin. It is shown using a simple example that the problem of slip-system indeterminacy for a single crystal includes a plastic spin problem as well. A new approach for determination of single or multiple plastic spins is suggested based on stability analysis. Using the postulate of realizability (Levitas, 1995a) a principle of minimum of dissipation rate at the time $t+\Delta t$ is derived. The unique equations for plastic spin(s) for polycrystals and for one simple model of a single crystal are obtained as a stable homogeneous solution of the boundary-value problem, based on the above extremum principle. For polycrystals the necessity of one additional scalar constraint equation is substantiated. It is shown that the extremum principle and equation for plastic spin is derived using the same assumption as for the derivation of the associated flow rule, namely the postulate of realizability. A number of concrete expressions for plastic spin are derived for polycrystals with initial and strain induced anisotropy, represented by internal variables and material tensors of arbitrary order, with multiple spins, as well as for a simple model of a single crystal. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords: A. plastic deformation, B. constitutive behaviour, B. finite strain, B. crystal plasticity, B. polycrystalline material, C. stability and bifurcation.

1. INTRODUCTION

One of the methods to satisfy the principle of material frame-indifference (PMFI) for anisotropic polycrystalline plastic materials consists in the consideration of constitutive equations in some rotating frame of reference which is equivalent to the exclusion of rotation with respect to some configuration. There are two main non-equivalent ways to exclude the rotations. In the first one, rotations are excluded relative to some fixed privileged configuration V_0 (Green and Naghdi, 1965; Casey and Naghdi 1980; Fressengeas and Molinari, 1983; Levitas, 1987, 1992, 1996). In configuration V_0 the particles are in an equilibrium state under zero stresses and deformations and such a state is usually reached after annealing and recrystallization

at high temperature. Only under such an assumption can be obtain the classical result (Truesdell and Noll, 1965)

$$\mathbf{T} = \mathbf{R} \cdot \hat{\mathbf{\Phi}}(\mathbf{U}(t')) \cdot \mathbf{R}^t \tag{1}$$

when we apply the PMFI. Here $\mathbf{T}$ is the Cauchy stress, $\hat{\mathbf{\Phi}}$ is the functional, $\mathbf{R}$ is the orthogonal rotation tensor and $\mathbf{U}$ the positive definite symmetric right stretch tensor in the polar decomposition of the deformation gradient $\mathbf{F} = \mathbf{R} \cdot \mathbf{U}$ with respect to configuration V_0. An objective corotational derivative associated with a skew-symmetric spin tensor $\mathbf{M} = \dot{\mathbf{R}} \cdot \mathbf{R}^t$ relative to the reference configuration V_0 appears in all evolution equations.

Another approach is related to the introduction of the concept of plastic spin (Mandel, 1973; Kratochvil, 1973; Loret, 1983; Dafalias, 1984, 1985; Van der Giessen, 1991; Besseling and Van der Giessen, 1993; Rubin, 1994). A triad of directors (the analogue of a crystal lattice for a single crystal) which characterizes the orientation is ascribed to each point of the medium, and it is postulated that all time derivatives must be taken with respect to a variable privileged configuration described by these directors, i.e. rotations are excluded with respect to directors. Rotation of the directors must be described by an additional constitutive equation. Configurations obtained when we fix directors (i.e. which rotate together with directors) are called isoclinic configurations (Mandel, 1973), see also Besseling and Van der Giessen (1993). A non-symmetric tensor of the plastic deformation gradient appears in the isoclinic configuration and in addition to the flow rule for the plastic strain rate a constitutive equation for the plastic spin has to be given (alternatively to the constitutive equation for spin of directors).

Mandel (1973) insists that plasticity theories which do not contain the triad of directors primarily describe only isotropic materials. Casey and Naghdi (1980) do not agree with this suggestion; they show that by using the invariance requirements it is possible to exclude a plastic rotation with respect to configuration V_0 from constitutive relations and to obtain a symmetric plastic strain gradient rather than a non-symmetric one. At the same time Mandel asserts that as the rotation of a triad of directors is determined by the prehistory of the deformation gradient and temperature, then with a functional representation of the constitutive eqn (1) the role of the directors disappears. The aims of the present paper are:

- To show that in the general theory we *cannot* exclude rotations with respect to *fixed* privileged configuration. Even with a functional representation we should introduce some variable privileged configuration (rotational variable) and exclude rotation with respect to it.
- To find an extremum principle and equation describing the rotation of such a variable privileged configuration for rigid-plastic materials.

As shown in Section 2, the first method based on the exclusion of rotations relative to fixed privileged configuration V_0 has one important drawback. It is always possible to prepare a *number of equivalent* privileged configurations by some thermomechanical deformation process, including annealing. Considering the same deformation process relative to two equivalent privileged configurations and assuming the same consti-

tutive equations when we exclude rotations with respect to each privileged configuration, we cannot obtain the same constitutive equations for the case with rotations. This contradiction shows that in the general theory we *cannot* exclude rotations with respect to some *fixed* privileged configuration.

We will study two methods of overcoming the contradiction revealed:

- Assume that after excluding the rigid body rotation (RBR) relative to fixed configuration V_0 material behaviour depends additionally on some rotational internal variable, for which some equation will be derived.
- Formulate and solve the problem of finding some variable privileged configuration, similar to Mandel's (1973) isoclinic configuration.

As will be shown both these approaches are equivalent, so we come to the problem of the determination of a "constitutive equation" for plastic spin or rotation of a triad of directors. The privileged isoclinic configuration for polycrystals is introduced by Mandel (1973) as the counterpart of a crystal lattice for single crystals. For a single crystal, plastic spin is in principle determined by kinematics, i.e. for plastic spin it is not the constitutive equation (as for a polycrystal) which is prescribed, but the kinematic constraint (the rate of plastic deformation gradient describes the combination of simple shears). In Section 3 it is demonstrated using the simple Asaro (1983) model that in the case of multiplicity of slip systems an additional equation for plastic spin is required, as for polycrystals. The situation that in some cases we need (e.g. at angle between slip direction $\beta = \pi/2$) but in other cases ($\beta \neq \pi/2$) do not need an equation for plastic spin creates the impression that this is not a *constitutive equation*. The same can be assumed for polycrystals as well.

In the paper we will suggest a new view of the problem of plastic spin for both single and polycrystals. It is assumed that a dissipation function or yield surface depends on one or several rotational internal variables (orthogonal tensors). We will not postulate constitutive equations for them. The unique equations for the corresponding spin tensors are obtained as *a stable homogeneous solution of the boundary-value problem* (*BVP*). To formulate the stability criterion and corresponding extremum principle, the previously formulated *postulate of realizability* (Levitas, 1995a) is applied. In Section 4 the postulate of realizability is used to derive the associated flow rule and related extremum principles for rigid-plastic materials. In Section 5 this postulate is applied to the choice of a unique stable homogeneous solution of the BVP. A principle of *minimum of dissipation rate at time* $t+\Delta t$ *is* derived which is independent of the type of boundary conditions (displacement controlled, stress controlled or mixed). The unique equations for one or multiple plastic spins for single and polycrystals are obtained, based on the above extremum principle. For polycrystals one additional scalar constraint equation is necessary which guarantees that the plastic spin is zero when the plastic deformation rate is zero. Thus both equations for the deformation rate and plastic spin are derived using the same assumption—the postulate of realizability. In Section 6 equations for plastic and lattice spins are derived for the tension and combined plane loading of a single crystal. In Section 7 various types of polycrystalline materials are considered: initially anisotropic, with anisotropic hardening, depending on internal variables and deformation gradient. In Section 8 the situation with the multiple spins is analyzed. Simple formulas are derived

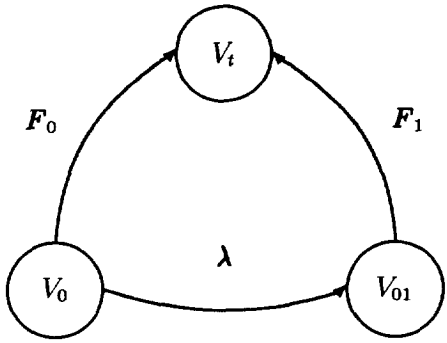

Fig. 1. Consideration of deformation relative to two equivalent privileged configurations V_0 and V_{01}. Configuration V_{01} is obtained by some thermomechanical (with annealing) or mechanical deformation process.

for the two-spin description, one of which is related to the back stress tensor and the second one to initial anisotropy. Some preliminary results are reported in Levitas (1995b, 1997b).

Direct tensor notations are used throughout this paper. Vectors and tensors are denoted in boldface type; **mn** is the dyadic product of vectors **m** and **n**; $(\mathbf{A}\cdot\mathbf{B})_{ik} = A_{ij}B^{jk}$ and $\mathbf{A}:\mathbf{B} = A_{ij}B^{ji}$ are the contraction of tensors over one and two indices; in the formulas $\cdot$ is performed first and then : (e.g. $\mathbf{A}:\mathbf{B}\cdot\mathbf{K} = \mathbf{A}:(\mathbf{B}\cdot\mathbf{K})$). Let superscript t and -1 denote transposition and inverse operation, **I** the unit tensor second order, $:=$ means equals per definition, $(\mathbf{A})_s := 1/2\,(\mathbf{A}+\mathbf{A}^t)$, $(\mathbf{A})_a := 1/2\,(\mathbf{A}-\mathbf{A}^t)$. The modulus (amplitude) of tensor **A** is defined as $|\mathbf{A}| = (\mathbf{A}:\mathbf{A}^t)^{1/2}$; in particular, for a skew-symmetric tensor **W** we have $|\mathbf{W}| = (W_{ij}W^{ij})^{1/2}$. Superscript Δ denotes a tensor after superposition of RBR, $\mathbf{d}^*$ means arbitrary (virtual) deformation rate tensor, superscript 0 is used for all possible solutions of the BVP at time t and subscript Δ denotes parameters at time $t+\Delta t$.

2. CONTRADICTIONS ARISING UNDER EXCLUDING ROTATION WITH RESPECT TO FIXED PRIVILEGED CONFIGURATION

We will show that the method based on the exclusion of rotations relative to a fixed privileged configuration V_0 when we apply the PMFI is contradictory if we do not introduce an additional rotational variable. Let us consider the following thermomechanical process as both a thought experiment and simultaneous calculation of the stress–strain state. Let us plastically deform some specimen made from an *initially isotropic* material with anisotropic hardening starting from configuration V_0. Then after removing stresses, annealing and the occurrence of complete recrystallization the material has the *same properties* as in configuration V_0, but occupies another configuration V_{01} (Fig. 1). But we do not know about the existence of a new privileged configuration. When we continue a thermomechanical process and deform a material plastically, we should continue to calculate the objective corotational derivative or

memory functional eqn (1) with respect to configuration V_0. Another investigator obtaining completely annealed isotropic material in a configuration V_{01} knows nothing about configuration V_0, because it should not influence material behaviour (we knew nothing about the existence of completely annealed states before configuration V_0 either). Producing the same deformation process as we do with respect to V_{01}, he will, in contrast to us, calculate the corotational derivative or memory functional in eqn (1) with respect to configuration V_{01}. Our and his results of the Cauchy stress measurements will be the same (because we both produce the same deformation process starting with the same privileged configuration V_{01}). Our and his results of stress calculation will differ, because the rotation tensor and corotational derivative are not invariant under a change of reference configuration (see below).

For a model with kinematic hardening, a purely mechanical counterpart of the same contradiction can be shown. Let the back stress tensor $\mathbf{L}$ during cyclic loading become equal to zero several times. All the states with zero back stress are undistinguishable and equivalent; with respect to which one should the rotation be excluded? With respect to the first one? But we cannot define experimentally and conceptually which state with $\mathbf{L} = \mathbf{0}$ was the first. The last one? Then the spin tensor $\mathbf{M}$ will have jumps by passing through the state with $\mathbf{L} = \mathbf{0}$ and the results of stress calculation for $\mathbf{L}$ history with states with $\mathbf{L} = \mathbf{0}$ and with infinitesimal $\mathbf{L}$ will have a finite difference.

Let us illustrate the above reasoning with formulae. In some frame of reference κ, let us consider the motion of a small uniformly deformable volume, described by the function $\mathbf{r} = \mathbf{r}(\mathbf{r}_0, t)$, where $\mathbf{r}$ and $\mathbf{r}_0$ are the position vectors of the points of the volume at time t in the actual configuration V_t and in the reference configuration V_0. A superposition of RBR is described by the equation

$$\mathbf{r}^{\Delta} = \mathbf{Q} \cdot \mathbf{r}, \tag{2}$$

where $\mathbf{Q}$ is the proper orthogonal tensor.

To exclude rotation with respect to configuration V_0 we consider the frame of reference δ rotating together with the small volume in which $\mathbf{r}_\delta = \mathbf{R}^t \cdot \mathbf{r}$, i.e. these relations are obtained from eqn (2) at $\mathbf{Q} = \mathbf{R}^t$. Indeed, the motion considered with respect to this frame of reference is pure deformation without rotation relative to configuration V_0, because for $\mathbf{Q} = \mathbf{R}^t$ we obtain

$$\mathbf{F}_\delta = \frac{\partial \mathbf{r}_\delta}{\partial \mathbf{r}_0} = \mathbf{R}^t \cdot \mathbf{F} = \mathbf{U} = \mathbf{U}_\delta, \quad \mathbf{R}_\delta = \mathbf{R}^t \cdot \mathbf{R} = \mathbf{I}.$$

As under superposed RBR described by eqn (2)

$$\mathbf{T}^{\Delta} = \mathbf{Q} \cdot \mathbf{T} \cdot \mathbf{Q}^t; \quad \mathbf{d}^{\Delta} = \mathbf{Q} \cdot \mathbf{d} \cdot \mathbf{Q}^t; \quad \mathbf{L}^{\Delta} = \mathbf{Q} \cdot \mathbf{L} \cdot \mathbf{Q}^t,$$

then in the frame δ

$$\mathbf{T}_\delta = \mathbf{R}_\tau^t \cdot \mathbf{T} \cdot \mathbf{R}_\tau; \quad \mathbf{d}_\delta = \mathbf{R}_\tau^t \cdot \mathbf{d} \cdot \mathbf{R}_\tau; \quad \mathbf{L}_\delta = \mathbf{R}_\tau^t \cdot \mathbf{L} \cdot \mathbf{R}_\tau,$$

where $\mathbf{d}$ is the deformation rate. Evidently, tensors $\mathbf{T}_\delta$, $\mathbf{L}_\delta$ and $\mathbf{d}_\delta$ are invariant under superposed RBR described by eqn (2), because the frame δ is rotating together with the particle.

Let there be two equivalent preferred reference configurations V_0 and V_{01}, in which the material is isotropic and has the same properties, and configuration V_{01} can be obtained after some thermomechanical or mechanical deformation process with the deformation gradient $\tilde{\boldsymbol{\lambda}}(t')$ and the temperature variation $\theta(t')$ with respect to configuration V_0, ending with $\tilde{\boldsymbol{\lambda}} = \boldsymbol{\lambda}$ and $\theta = \theta_0$. As the material is isotropic in V_{01}, the tensor $\boldsymbol{\lambda}$ is determined to within the RBR. Then we continue the deformation process, which will be considered simultaneously with respect to configurations V_0 and V_{01}. The following kinematic relations are valid:

$$\mathbf{F}_0 = \mathbf{F}_1 \cdot \boldsymbol{\lambda}; \quad \mathbf{U}_0^2 = \mathbf{F}_0^t \cdot \mathbf{F}_0 = \boldsymbol{\lambda} \cdot \mathbf{F}_1^t \cdot \mathbf{F}_1 \cdot \boldsymbol{\lambda} = \boldsymbol{\lambda} \cdot \mathbf{U}_1^2 \cdot \boldsymbol{\lambda}; \tag{3}$$

$$\mathbf{R}_0 \cdot \mathbf{U}_0 = \mathbf{R}_1 \cdot \mathbf{U}_1 \cdot \boldsymbol{\lambda} \Rightarrow \mathbf{R}_1 = \mathbf{R}_0 \cdot \tilde{\mathbf{R}}; \quad \tilde{\mathbf{R}} := \mathbf{U}_0 \cdot \boldsymbol{\lambda}^{-1} \cdot \mathbf{U}_1^{-1}; \tag{4}$$

$$\mathbf{M}_1 := \dot{\mathbf{R}}_1 \cdot \mathbf{R}_1^t = \mathbf{M}_0 + \mathbf{R}_0 \cdot \dot{\tilde{\mathbf{R}}} \cdot \tilde{\mathbf{R}}^t \cdot \mathbf{R}_0^t; \quad \mathbf{M}_0 := \dot{\mathbf{R}}_0 \cdot \mathbf{R}_0^t, \tag{5}$$

where subscripts 0 and 1 denote the tensors defined with respect to configuration V_0 and V_{01} correspondingly and $\mathbf{M}$ is the skew-symmetric spin tensor. Let us consider the model with kinematic hardening $\dot{\mathbf{L}} = A\mathbf{d}$. Under superposed RBR this equation appears as

$$\overline{\mathbf{Q} \cdot \mathbf{L} \cdot \mathbf{Q}^t}^{\cdot} = A\mathbf{Q} \cdot \mathbf{d} \cdot \mathbf{Q}^t. \tag{6}$$

If configurations V_0 and V_{01} are equivalent, then with exclusion of rotations relative to each of them (i.e. with substitutions $\mathbf{Q} = \mathbf{R}_1^t$ and $\mathbf{Q} = \mathbf{R}_0^t$) the same equations $\dot{\mathbf{L}}_{\delta 1} = A\mathbf{d}_{\delta 1}$ and $\dot{\mathbf{L}}_{\delta 0} = A\mathbf{d}_{\delta 0}$ should be valid, i.e.

$$\overline{\mathbf{R}_1^t \cdot \mathbf{L} \cdot \mathbf{R}_1}^{\cdot} = A\mathbf{R}_1^t \cdot \mathbf{d} \cdot \mathbf{R}_1; \quad \overline{\mathbf{R}_0^t \cdot \mathbf{L} \cdot \mathbf{R}_0}^{\cdot} = A\mathbf{R}_0^t \cdot \mathbf{d} \cdot \mathbf{R}_0 \tag{7}$$

or

$$\dot{\mathbf{L}} + \mathbf{L} \cdot \mathbf{M}_1 + \mathbf{M}_1^t \cdot \mathbf{L} = A\mathbf{d}; \quad \dot{\mathbf{L}} + \mathbf{L} \cdot \mathbf{M}_0 + \mathbf{M}_0^t \cdot \mathbf{L} = A\mathbf{d}. \tag{8}$$

As according to eqn (5) $\mathbf{M}_1 \neq \mathbf{M}_0$, then eqns $(8)_1$ and $(8)_2$ cannot be equivalent, which is a contradiction.

Let us assume that the same equation for the simple solid is valid with exclusion of rotation with respect to configurations V_0 and V_{01}:

$$\mathbf{T}_{\delta 1} := \mathbf{R}_1^t \cdot \mathbf{T} \cdot \mathbf{R}_1 = \hat{\boldsymbol{\Phi}}(\mathbf{U}_1(t')); \quad \mathbf{T}_{\delta 0} := \mathbf{R}_0^t \cdot \mathbf{T} \cdot \mathbf{R}_0 = \hat{\boldsymbol{\Phi}}(\mathbf{U}_0(t')), \tag{9}$$

where we omit the temperature. We intend to prove that in the general case

$$\mathbf{T} = \mathbf{R}_1 \cdot \hat{\boldsymbol{\Phi}}(\mathbf{U}_1(t')) \cdot \mathbf{R}_1^t \neq \mathbf{R}_0 \cdot \hat{\boldsymbol{\Phi}}(\mathbf{U}_0(t')) \cdot \mathbf{R}_0^t, \tag{10}$$

despite the fact that the Cauchy stress $\mathbf{T}$ must be independent of the choice of reference configuration, and this is a conceptual contradiction. To prove this, it is sufficient to find one corroborating example and we can again use a simple model with kinematic hardening. The integral (functional) representation of such a model in the frames of reference δ_1 and δ_0, making allowance for eqn (7), is

$$\mathbf{T}_{\delta 1} = \int_0^t \left(\sqrt{\frac{2}{3}} \sigma_Y \frac{\mathbf{d}_{\delta 1}(\tau)}{|\mathbf{d}_{\delta 1}(\tau)|} \delta(t-\tau) + A\mathbf{d}_{\delta 1}(\tau) \right) \mathrm{d}\tau; \quad \mathbf{d}_{\delta 1} = (\dot{\mathbf{U}}_1 \cdot \mathbf{U}_1^{-1})_s, \tag{11}$$

$$\mathbf{T}_{\delta 0} = \int_0^t \left(\sqrt{\frac{2}{3}} \sigma_Y \frac{\mathbf{d}_{\delta 0}(\tau)}{|\mathbf{d}_{\delta 0}(\tau)|} \delta(t-\tau) + A\mathbf{d}_{\delta 0}(\tau) \right) \mathrm{d}\tau; \quad \mathbf{d}_{\delta 0} = (\dot{\mathbf{U}}_0 \cdot \mathbf{U}_0^{-1})_s, \tag{12}$$

where σ_Y is the yield stress and $\delta(t-\tau)$ the Dirac delta function. The first term in eqns (11) and (12) characterizes isotropic plastic resistance and the second term is the back stress tensor. For plastic incompressible materials the deformation rate and, according to eqns (11) and (12), tensor $\mathbf{T}$ are deviators. It is evident that eqns (11) and (12) in the frame κ

$$\mathbf{T} = \sqrt{\frac{2}{3}} \sigma_Y \frac{\mathbf{d}}{|\mathbf{d}|} + \mathbf{R}_1 \cdot \left(\int_0^t A(\mathbf{R}_1'(\tau) \cdot \mathbf{d}(\tau) \cdot \mathbf{R}_1(\tau)\, \mathrm{d}\tau) \right) \cdot \mathbf{R}_1' \tag{13}$$

and

$$\mathbf{T} = \sqrt{\frac{2}{3}} \sigma_Y \frac{\mathbf{d}}{|\mathbf{d}|} + \mathbf{R}_0 \cdot \left(\int_0^t A(\mathbf{R}_0'(\tau) \cdot \mathbf{d}(\tau) \cdot \mathbf{R}_0(\tau)\, \mathrm{d}\tau) \right) \cdot \mathbf{R}_0' \tag{14}$$

are not equivalent since $\mathbf{R}_1 \neq \mathbf{R}_0$.

The contradiction found is not related to a specific model, but has fundamental reasons. The first one is related to the way a privileged configuration is created. Let such a configuration be prepared using some mechanical or thermomechanical deformation process $\mathscr{A}$, and we would like to develop theory which is valid for an arbitrary thermomechanical process. Then for thermomechanical processes which include process $\mathscr{A}$ we can obtain multiple undistinguished privileged configurations, and the above examples demonstrate the fact that in any case we do not see any general way to overcome the contradiction.

Of course, if the privileged state can be created by annealing only and we develop the theory for a much lower temperature, then the contradiction does not arise, but this is a limited theory.

The second reason is connected with the impossibility of introducing a fading memory of the "old" privileged configuration V_0 in terms of rotations.

If there are several equivalent preferred configurations, then for stress–strain relation eqn (9) or in the evolution equation for an internal variable (7) for excluded rotations we can take this fact into account by choosing a proper constitutive functional or evolution equation, i.e. we can introduce a fading memory of an old preferred configuration in terms of strain. But for the case with rotation [eqns (8) and (10)], as there are no constitutive equations for rotation or spin tensors, we have no degrees of freedom to introduce such a fading memory in terms of rotation.

Let us draw a conclusion. If we exclude rotation relative to some fixed privileged configuration V_0, then it is always possible to create a number of completely equivalent privileged configurations V_i by some thermomechanical deformation process.

Assuming the same constitutive equations for rotations excluded relative to each

of the privileged configurations, we cannot obtain the same equations for the case with rotations. This contradiction shows that in the general theory we *cannot exclude rotations with respect to some fixed privileged configuration.*

We will study the two ways of overcoming the contradiction found above, as mentioned in the Introduction.

Remark. If we assume that the variable privileged configuration coincides with the actual configuration, then there is no necessity for an additional rotational variable, $\mathbf{M}_0 = \mathbf{W}$ is the spin tensor in the actual configuration and an objective derivative in eqn (8) is the Jaumann rate. But in this case the well-known shear stress oscillations emerge in simple shearing even though their introduction into the model was not desired. Consequently, this way is contradictory. It will follow from the equation for rotational variable obtained that the actual configuration is the privileged one for isotropic materials only [this also coincides with Mandel's (1973) results].

Note that if we use an objective derivative related to the velocity gradient $\mathbf{l}$ rather than a corotational one, e.g. $\dot{\mathbf{L}}+\mathbf{L}\cdot\mathbf{l}+\mathbf{l}^t\cdot\mathbf{L} = A\mathbf{d}$, then the result is independent of the choice of reference configuration. But this equation cannot be obtained from the initial equation $\dot{\mathbf{L}} = A\mathbf{d}$ by the application of PMFI, i.e. this way is not related to the exclusion of the RBR and will not be considered in the paper.

3. TENSION OF SINGLE CRYSTAL—THE AMBIGUITY OF SPIN TENSORS

Let us consider plane tension of an infinite rigid-plastic monocrystal with two mutually orthogonal sliding directions $\mathbf{m}$ and $\mathbf{n}$ (Fig. 2). The kinematics can be described using the following formulas (Asaro, 1983):

$$\mathbf{F} = \mathbf{R}_l\cdot\mathbf{F}_p\,; \quad \mathbf{l} = \dot{\mathbf{F}}\cdot\mathbf{F}^{-1} = \dot{\mathbf{R}}_l\cdot\mathbf{R}_l^t+\mathbf{R}_l\cdot\dot{\mathbf{F}}_p\cdot\mathbf{F}_p^{-1}\cdot\mathbf{R}_l^t = \mathbf{W}_l+\mathbf{l}_p\,; \tag{15}$$

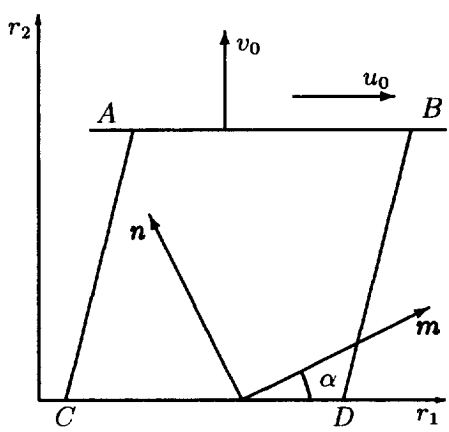

Fig. 2. Scheme of straining.

$$\mathbf{l}_p = \dot{\gamma}_1 \mathbf{mn} + \dot{\gamma}_2 \mathbf{nm}; \quad \mathbf{d}_p = \dot{\gamma}(\mathbf{mn} + \mathbf{nm}); \tag{16}$$

$$\mathbf{W}_p = \tfrac{1}{2}(\dot{\gamma}_1 - \dot{\gamma}_2)(\mathbf{mn} - \mathbf{nm}); \quad \mathbf{d} = \mathbf{d}_p; \quad \mathbf{W} = \mathbf{W}_l + \mathbf{W}_p; \tag{17}$$

$$W := W_{21} = W_l + \tfrac{1}{2}(\dot{\gamma}_2 - \dot{\gamma}_1); \quad \dot{\gamma} := \tfrac{1}{2}(\dot{\gamma}_1 + \dot{\gamma}_2), \tag{18}$$

where $\mathbf{F}$ and $\mathbf{F}_p$ are the total and plastic deformation gradients, $\mathbf{R}_l$ the lattice rotation, $\mathbf{l}$ and $\mathbf{l}_p$ the total and plastic velocity gradients, $\mathbf{W}$, $\mathbf{W}_l$ and $\mathbf{W}_p$ the total, lattice and plastic spins, $\mathbf{d}$ and $\mathbf{d}_p$ the total and plastic deformation rates, γ_i the shear strain along the i-th slip direction. The following boundary conditions are prescribed.

On the line CD ($r_2 = 0$): the vertical velocity $v = 0$; the shear stress $\tau = 0$.

On the line AB ($r_2 = h$): the vertical velocity $v = v_0$; the shear stress $\tau = 0$.

Here h is the height of the crystal. Schmid's law

$$|\tau_i| = k(\gamma) \tag{19}$$

applies, where τ_i and k are the resolved and the critical resolved shear stresses along each slip direction. It is very easy to find a trivial homogeneous solution of the BVP. Using Mohr circles and the incompressibility condition [$\mathbf{I} : \mathbf{d}_p = 0$, see eqn $(16)_2$] we obtain

$$d_{22} = -d_{11} = \dot{\gamma} \sin 2\alpha; \quad d_{12} = \dot{\gamma} \cos 2\alpha; \quad \tau_\alpha := \tau(\alpha) = 0.5\sigma \sin 2\alpha, \tag{20}$$

where α is the angle between axis r_1 and vector $\mathbf{m}$ and σ is the tensile stress. The velocity field compatible with the boundary conditions and plastic incompressibility is as follows:

$$v = \frac{v_0}{h} r_2; \quad u = \frac{u_0}{h} r_2 - \frac{v_0}{h} r_1 + \text{const}, \tag{21}$$

where u_0 is some constant. The velocity field eqn (21) determines all the components of the deformation rate $\mathbf{d}$

$$d_{22} = -d_{11} = \frac{v_0}{h} = \dot{\gamma} \sin 2\alpha; \quad d_{12} = \frac{u_0}{2h} = \dot{\gamma} \cos 2\alpha = d_{22} \cot \alpha = \frac{v_0}{h} \cot 2\alpha;$$

$$u_0 = 2v_0 \cot 2a, \tag{22}$$

as well as the velocity gradient l_{ij} and spin W

$$l_{21} = 0; \quad W = -d_{21}; \quad l_{12} = 2d_{12}. \tag{23}$$

The lattice spin can be calculated using eqns $(17)_3$, (22) and (23):

$$W_l := \dot{\alpha} = W - 0.5(\dot{\gamma}_2 - \dot{\gamma}_1) = 0.5\dot{\gamma}_1(1 - \cos 2\alpha) - 0.5\dot{\gamma}_2(1 + \cos 2\alpha). \tag{24}$$

Conditions (19) and $(20)_3$ determine the tensile stress $\sigma = \pm 2k(\gamma)/\sin 2\alpha$; the other components of the Cauchy stress $\mathbf{T}$ are zero.

Thus we know tensors $\mathbf{T}$, $\mathbf{d}$, $\mathbf{d}_p$, $\mathbf{W}$, $\mathbf{l}$ and consequently have found the homogeneous solution of the BVP, which fulfills all the equations of continuum mechanics. But the parameters $\dot{\gamma}_1$ and $\dot{\gamma}_2$, $\mathbf{W}_p$ and $\dot{\alpha}$ are undetermined separately. Because none of the above parameters contribute to the rate of dissipation $\mathscr{D} := \mathbf{T} : \mathbf{d}$, the principle of minimum of $\mathscr{D}$ (Taylor, 1938; Chin and Mammel, 1969) yields nothing new. For

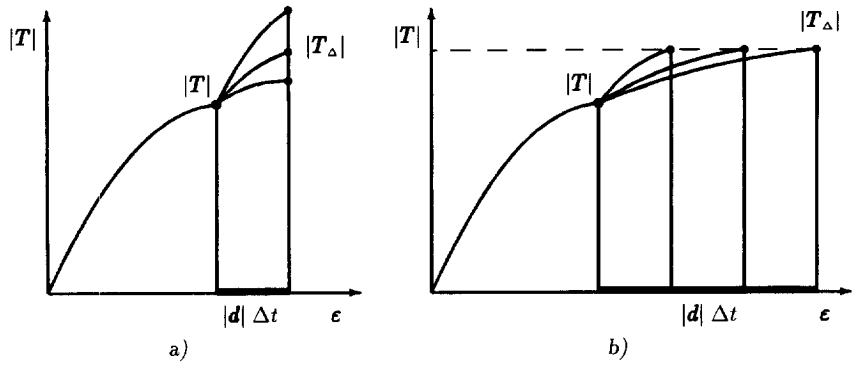

Fig. 3. Non-unique stress–strain diagrams for different spins: (a) under prescribed displacements; (b) under prescribed stresses.

different solutions at time t we obtain different stress σ–strain $\ln h/h_0$ diagrams during the time $(t, t+\Delta t)$ and different stresses σ_Δ at time $t+\Delta t$ [Fig. 3(a)], where h_0 is the initial height of the crystal and subscript Δ means that the parameter is determined at time $t+\Delta t$. A similar ambiguity occurs in the given problem with a stress controlled boundary condition [Fig. 3(b)].

The non-uniqueness in the given model results from the mutual orthogonality of the slip directions and the dependence of k on γ, but not on γ_1 and γ_2 separately. The above model is a particularly degenerate case of the Asaro (1983) model, which was not studied due to its ambiguity. In reality, slip-system indeterminacy is a well-known feature of single crystals (see, e.g. Taylor, 1938; Asaro, 1983; Anand and Kothari, 1996 and references) and consequently, plastic and lattice spins are undetermined not only for polycrystals, but for single crystals as well. The usual way to avoid this non-uniqueness is to replace the rate-independent plastic model by a rate-dependent viscoplastic one (Asaro, 1983; Asaro and Needleman, 1985; Kalidindi *et al.*, 1992; Boukadia *et al.*, 1993). But if we assume $k = k(\gamma, \dot{\gamma})$ (of course, this assumption does not correspond to the method used by the aforementioned authors), the viscoplastic solution will also be ambiguous. It will not help either to find the equation for spin for polycrystals. The algebraic method to remove the multiplicity of slip systems developed by Anand and Kothari (1996) cannot be used for the solution of our problem either.

Fuh and Havner (1989) developed a technique to remove the ambiguity based on the postulate of minimum plastic spin. Their predictions are in good correlation with many experimental results and coincide in most cases (but not always) with rate sensitive solutions (Tóth *et al.*, 1990). The minimum plastic spin approach can be applied to our problem, but we do not see a way to generalize it consistently for polycrystals.

Let us summarize the results. For a single crystal the triad of directors describes an orientation of the crystal lattice and consequently it has a clear physical interpretation. In principle, the equation for plastic spin is determined unambiguously by kinematics [see e.g. eqn $(17)_1$], i.e. for plastic spin it is not the constitutive equation (as for a

polycrystal) that is specified, but the kinematic constraint (rate of plastic deformation gradient $\dot{\mathbf{F}}_p$ describes the combination of simple shears). The above example demonstrates that in the case of multiplicity of slip systems an additional equations for plastic spin is required, as for polycrystals. The situation that in some cases we need an equation for plastic spin but in other cases do not create the impression that this is *not a constitutive equation*. The same can be assumed for polycrystals as well.

In the paper we will suggest a new view of the problem of plastic spin for both single and polycrystals. We assume that the dissipation function or yield surface depends on one or several orthogonal rotation tensors. We will not prescribe constitutive equations for these tensors. To avoid the ambiguity a new approach is suggested based on *stability analysis*. The unique equations for corresponding spin tensors for polycrystals and the above model of monocrystal are obtained as a stable homogeneous solution of the BVP. The stability analysis is based on a new extremum principle, derived with the help of the previously formulated postulate of realizability (Levitas, 1995a).

4. FORMULATION OF THE PROBLEM

Let us apply the postulate of realizability (Levitas, 1995a) to the derivation of an equation for the plastic spin for the general case of a uniformly deformed representative volume of the rigid-plastic material. Assume that in a fixed frame of reference the constitutive equation is as follows:

$$\mathbf{T} = \mathbf{T}(\mathbf{d}, \mathbf{F}, \bar{\mathbf{R}}), \tag{25}$$

where $\bar{\mathbf{R}}$ is some orthogonal rotation tensor. Suppose that, under a superposed RBR described by eqn (2), eqn (25) transforms as

$$\mathbf{Q}\cdot\mathbf{T}\cdot\mathbf{Q}' = \mathbf{T}(\mathbf{Q}\cdot\mathbf{d}\cdot\mathbf{Q}', \mathbf{Q}\cdot\mathbf{F}, \mathbf{Q}\cdot\bar{\mathbf{R}}) = \mathbf{Q}\cdot\mathbf{T}(\mathbf{d}, \mathbf{F}, \bar{\mathbf{R}})\cdot\mathbf{Q}', \tag{26}$$

i.e. dependence in eqn (25) satisfies the PMFI. To rewrite eqn (25) in the rotating frame of reference δ we put $\mathbf{Q} = \mathbf{R}'$, i.e.

$$\mathbf{T}_\delta = \mathbf{T}(\mathbf{d}_\delta, \mathbf{F}_\delta, \boldsymbol{\varphi}); \tag{27}$$

$$\mathbf{T}_\delta := \mathbf{R}'\cdot\mathbf{T}\cdot\mathbf{R}; \quad \mathbf{d}_\delta := \mathbf{R}'\cdot\mathbf{d}\cdot\mathbf{R}; \quad \mathbf{F}_\delta := \mathbf{R}'\cdot\mathbf{F} = \mathbf{U}; \quad \boldsymbol{\varphi} := \mathbf{R}'\cdot\bar{\mathbf{R}}. \tag{28}$$

For the rate of dissipation we admit $\mathscr{D}(\mathbf{d}, \mathbf{F}, \bar{\mathbf{R}}) := \mathbf{T}(\mathbf{d}, \mathbf{F}, \bar{\mathbf{R}}) : \mathbf{d} \geqslant 0$. By definition, for rate-independent plastic materials $\mathbf{T}(\mathbf{d}, \mathbf{F}, \bar{\mathbf{R}})$ is a homogeneous function of degree zero in $\mathbf{d}$, so $\mathbf{T}(\mathbf{d}, \mathbf{F}, \bar{\mathbf{R}}) = \mathbf{T}(\boldsymbol{\kappa}, \mathbf{F}, \bar{\mathbf{R}})$, where $\boldsymbol{\kappa} = \mathbf{d}/|\mathbf{d}|$ is the directing tensor. Consequently,

$$\mathscr{D}(\mathbf{d}, \mathbf{F}, \bar{\mathbf{R}}) := \mathbf{T}(\mathbf{d}, \mathbf{F}, \bar{\mathbf{R}}) : \mathbf{d} = |\mathbf{d}|\mathbf{T}(\boldsymbol{\kappa}, \mathbf{F}, \bar{\mathbf{R}}) : \boldsymbol{\kappa} = |\mathbf{d}|\mathscr{D}(\boldsymbol{\kappa}, \mathbf{F}, \bar{\mathbf{R}})$$

is a homogeneous function of degree one in $\mathbf{d}$. We do not exclude the possibility that the deformation rate $\mathbf{d}$ must satisfy some kinematic constraint equations, e.g. the incompressibility condition or that the deformation rate describes a combination of simple shears [as in eqn $(16)_2$]. We will not indicate these constraints explicitly, but will call the deformation rate $\mathbf{d}$ which meets the kinematic constraints the admissible one.

When varying all admissible directing tensors $\boldsymbol{\kappa} \in \mathscr{S}^6$ at fixed $\mathbf{F}$ and $\bar{\mathbf{R}}$, where $\mathscr{S}^6 = \{\boldsymbol{\kappa} \in \mathscr{R}^6 : |\boldsymbol{\kappa}| = 1\}$, the ends of vectors $\mathbf{T}(\boldsymbol{\kappa}, \mathbf{F}, \bar{\mathbf{R}})$ corresponding to them describe the yield surface $\varphi(\mathbf{T}, \mathbf{F}, \bar{\mathbf{R}}) = 0$ in the $\mathbf{T}$-space. For all $\mathbf{T}$ for which $\varphi\ (\mathbf{T}, \mathbf{F}, \bar{\mathbf{R}}) < 0$, assume that $\mathbf{d} = \mathbf{0}$ (at $\mathbf{d} = \mathbf{0}$ vector $\boldsymbol{\kappa}$ and function $\mathbf{T}(\boldsymbol{\kappa}, \mathbf{F}, \bar{\mathbf{R}})$ are undetermined). So we have $\mathbf{T} = \mathbf{T}(\mathbf{d}, \mathbf{F}, \bar{\mathbf{R}})$ and the set of vectors $\mathbf{T}(\mathbf{0}, \mathbf{F}, \bar{\mathbf{R}})$, which are not related to any $\mathbf{d}$.

In the frame of reference δ the yield surface has the form $\varphi(\mathbf{T}_\delta, \mathbf{F}_\delta, \boldsymbol{\varphi}) = 0$. Consequently as for a single crystal, the orthogonal tensor $\boldsymbol{\varphi}$ characterizes the rotation of a yield surface $\varphi(\mathbf{T}_\delta, \mathbf{F}_\delta, \boldsymbol{\varphi}) = 0$ in the space $\mathbf{T}_\delta$. This means that the tensor $\boldsymbol{\varphi}$ can in principle be measured. Some experiments demonstrating rotation of the yield surface and its description can be found in the papers by Rees (1986) and Cho and Dafalias (1996). It is natural to assume that

$$\mathbf{d}_\delta = \mathbf{0} \quad (\text{i.e. } \dot{\mathbf{F}}_\delta = \mathbf{0}) \Rightarrow \dot{\boldsymbol{\varphi}} = \mathbf{0}, \tag{29}$$

i.e. the yield surface does not rotate in the frame δ without plastic flow.

Let us introduce after Mandel (1973) an isoclinic frame of reference χ in which $\bar{\mathbf{R}} = \mathbf{I}$. To do this we put $\mathbf{Q} = \bar{\mathbf{R}}^t$ in eqns (2) and (26):

$$\mathbf{T}_\chi = \mathbf{T}(\mathbf{d}_\chi, \mathbf{F}_\chi); \quad \mathbf{T}_\chi := \bar{\mathbf{R}}^t \cdot \mathbf{T} \cdot \bar{\mathbf{R}} = \boldsymbol{\varphi}^t \cdot \mathbf{T}_\delta \cdot \boldsymbol{\varphi}; \tag{30}$$

$$\mathbf{d}_\chi = \bar{\mathbf{R}}^t \cdot \mathbf{d} \cdot \bar{\mathbf{R}} = \boldsymbol{\varphi}^t \cdot \mathbf{d}_\delta \cdot \boldsymbol{\varphi}; \quad \mathbf{F}_\chi = \bar{\mathbf{R}}^t \cdot \mathbf{F} = \boldsymbol{\varphi}^t \cdot \mathbf{F}_\delta. \tag{31}$$

In the frame of reference χ, $\varphi = \varphi(\mathbf{T}_\chi, \mathbf{F}_\chi) = 0$, i.e. by definition of the frame of reference χ the yield surface does not rotate in it. Kinematic decompositions in the fixed frame of reference, frames δ and χ have the following form:

$$\mathbf{F} = \mathbf{R} \cdot \mathbf{F}_\delta = \bar{\mathbf{R}} \cdot \mathbf{F}_\chi = \mathbf{R} \cdot \boldsymbol{\varphi} \cdot \mathbf{F}_\chi; \tag{32}$$

$$\mathbf{l} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} = \dot{\mathbf{R}} \cdot \mathbf{R}^t + \mathbf{R} \cdot \dot{\mathbf{F}}_\delta \cdot \mathbf{F}_\delta^{-1} \cdot \mathbf{R}^t = \dot{\bar{\mathbf{R}}} \cdot \bar{\mathbf{R}}^t + \bar{\mathbf{R}} \cdot \dot{\mathbf{F}}_\chi \cdot \mathbf{F}_\chi^{-1} \cdot \bar{\mathbf{R}}^t; \tag{33}$$

$$\mathbf{d} = (\dot{\mathbf{F}} \cdot \mathbf{F}^{-1})_s = \mathbf{R} \cdot \mathbf{d}_\delta \cdot \mathbf{R}^t = \bar{\mathbf{R}} \cdot \mathbf{d}_\chi \cdot \bar{\mathbf{R}}^t; \tag{34}$$

$$\mathbf{W} = (\dot{\mathbf{F}} \cdot \mathbf{F}^{-1})_a = \dot{\mathbf{R}} \cdot \mathbf{R}^t + \mathbf{R} \cdot \mathbf{W}_{p\delta} \cdot \mathbf{R}^t = \dot{\bar{\mathbf{R}}} \cdot \bar{\mathbf{R}}^t + \bar{\mathbf{R}} \cdot \mathbf{W}_{p\chi} \cdot \bar{\mathbf{R}}^t; \tag{35}$$

$$\mathbf{W}_{p\delta} := (\dot{\mathbf{F}}_\delta \cdot \mathbf{F}_\delta^{-1})_a; \quad \mathbf{W}_{p\chi} := (\dot{\mathbf{F}}_\chi \cdot \mathbf{F}_\chi^{-1})_a; \tag{36}$$

$$\mathbf{W} = \boldsymbol{\Omega} + \mathbf{W}_p; \quad \boldsymbol{\Omega} := \dot{\bar{\mathbf{R}}} \cdot \bar{\mathbf{R}}^t; \quad \mathbf{W}_p := \bar{\mathbf{R}} \cdot \mathbf{W}_{p\chi} \cdot \bar{\mathbf{R}}^t. \tag{37}$$

Tensors $\mathbf{W}_p$ and $\boldsymbol{\Omega}$ are the plastic spin and spin of some privileged orientation in the fixed frame of reference. It follows from the chain

$$\begin{aligned} &\mathbf{d} = \mathbf{0} \Rightarrow \mathbf{d}_\delta = \mathbf{0}\,[\text{eqn}(34)] \Rightarrow \dot{\mathbf{F}}_\delta = \mathbf{0}, \\ &\dot{\boldsymbol{\varphi}} = \mathbf{0}\,[\text{eqn}(29)] \Rightarrow \dot{\mathbf{F}}_\chi = \mathbf{0}\,[\text{eqn}(31)] \Rightarrow \mathbf{W}_p = \mathbf{0}\,[\text{eqn}(37)], \end{aligned} \tag{38}$$

that the condition $\mathbf{W}_p = \mathbf{0}$ at $\mathbf{d} = \mathbf{0}$ should be met. Under superposed RBR eqn (2)

$$\bar{\mathbf{R}}^\Delta = \mathbf{Q} \cdot \bar{\mathbf{R}}; \quad \boldsymbol{\Omega}^\Delta = \dot{\mathbf{Q}} \cdot \mathbf{Q}^t + \mathbf{Q} \cdot \boldsymbol{\Omega} \cdot \mathbf{Q}^t;$$

$$\mathbf{W}_p^{\Delta} = \mathbf{Q}\cdot\mathbf{W}_p\cdot\mathbf{Q}^t; \quad \mathbf{W}^{\Delta} = \dot{\mathbf{Q}}\cdot\mathbf{Q}^t + \mathbf{Q}\cdot\mathbf{W}\cdot\mathbf{Q}^t, \tag{39}$$

i.e. eqn $(37)_1$ is objective. Let us consider two formulations of the problem.

1. In the case of rotations excluded with respect to fixed privileged configuration V_0 (in the rotating frame of reference δ) let us introduce in relation $\mathbf{T}_\delta(\mathbf{d}_\delta, \mathbf{F}_\delta)$ an additional rotational internal variable $\boldsymbol{\varphi}$. It is necessary to find the equation for the orthogonal tensor $\boldsymbol{\varphi}$ based on the stability analysis.
2. Assume that there exists some privileged variable frame of reference χ defined by $\mathbf{Q} = \bar{\mathbf{R}}^t$ in eqn (2), in which the constitutive equation has the form of eqn (30). It is necessary to find such a privileged frame of reference based on the stability analysis.

In the second formulation the whole initial information is included in the function $\mathbf{T}(\mathbf{d}_\chi, \mathbf{F}_\chi)$, i.e. it is assumed that the dependence of $\mathbf{T}$ on $\bar{\mathbf{R}}$ in eqn (25) is that which follows from eqns (30) and (31), i.e. $\bar{\mathbf{R}}^t\cdot\mathbf{T}\cdot\bar{\mathbf{R}} = \mathbf{T}(\bar{\mathbf{R}}^t\cdot\mathbf{d}\cdot\bar{\mathbf{R}}, \bar{\mathbf{R}}^t\cdot\mathbf{F})$. But we obtained the same result starting with eqn (25) at $\bar{\mathbf{R}}^{\Delta} = \mathbf{Q}\cdot\bar{\mathbf{R}}$, which satisfies PMFI eqn (26), i.e. formulation 2 is equivalent to eqns (25)–(26).

The first formulation is more general, because we can introduce an arbitrary dependence $\mathbf{T}(\mathbf{d}_\delta, \mathbf{F}_\delta, \boldsymbol{\varphi})$ on $\boldsymbol{\varphi}$ which cannot be obtained from eqns (25)–(26), e.g.

$$\mathbf{T}_\delta = k\frac{\mathbf{d}_\delta}{|\mathbf{d}_\delta|}(1 + a|\boldsymbol{\varphi}|). \tag{40}$$

Equation (40) is objective. But in the fixed frame of reference it has the form of

$$\mathbf{T} = k\frac{\mathbf{d}}{|\mathbf{d}|}(1 + a|\boldsymbol{\varphi}|), \tag{41}$$

i.e. it is not equivalent to eqns (25)–(26). In this paper we will consider such a dependence $\mathbf{T}$ of $\boldsymbol{\varphi}$ which can be obtained from eqns (25)–(26). As both problem formulations follow from eqns (25)–(26), they are equivalent in the present study. The reason for the consideration of two formulations is that in the first case we follow the methods developed in various papers and books (Green and Naghdi, 1965; Casey and Naghdi (1980); Fressengeas and Molinari, 1983; Levitas, 1987, 1992a, 1996), i.e. exclude rotation with respect to V_0, and then introduce an additional rotational internal variable. In the second formulation we seek the new variable privileged configuration.

5. APPLICATION OF THE POSTULATE OF REALIZABILITY

5.1. *Derivation of the flow rule*

First we will formulate and apply the postulate of realizability to derive the flow rule (Levitas, 1995a). This will be helpful for our understanding of the general idea as well as to see that both the associated flow rule and the equation for plastic spin can be derived using the same assumption. It is evident that if for a given $\mathbf{T}$ at arbitrary fixed $\mathbf{F}$, $\bar{\mathbf{R}}$ an inequality

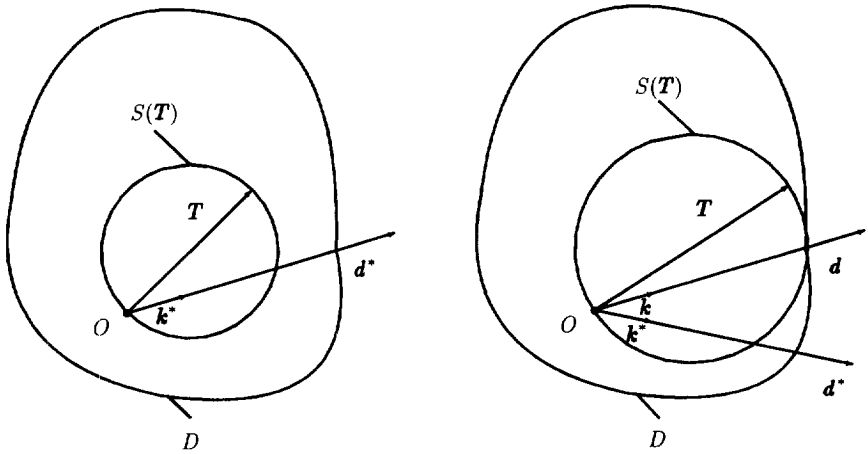

Fig. 4. On the application of the postulate of realizability for the derivation of flow rule: (a) sphere $S(\mathbf{T})$ is inside the surface D; (b) sphere $S(\mathbf{T})$ touches the surface D.

$$\mathbf{T}:\mathbf{d}^* - \mathscr{D}(\mathbf{d}^*, \mathbf{F}, \bar{\mathbf{R}}) < 0, \quad \forall \mathbf{d}^* \neq \mathbf{0} \tag{42}$$

is valid, then $\mathbf{d} = \mathbf{0}$. Indeed, if $\mathbf{d} \neq \mathbf{0}$ then $\mathbf{T} = \mathbf{T}(\mathbf{d}, \mathbf{F}, \bar{\mathbf{R}})$ for this $\mathbf{d}$, and $\mathbf{T}:\mathbf{d} = \mathbf{T}(\mathbf{d}, \mathbf{F}, \bar{\mathbf{R}}):\mathbf{d} = \mathscr{D}(\mathbf{d}, \mathbf{F}, \bar{\mathbf{R}})$, which contradicts inequality (42). Since $\mathbf{T}:\mathbf{d}^* - \mathscr{D}(\mathbf{d}^*, \mathbf{F}, \bar{\mathbf{R}}) = |\mathbf{d}^*|\ (\mathbf{T}:\boldsymbol{\kappa}^* - \mathscr{D}(\boldsymbol{\kappa}^*, \mathbf{F}, \bar{\mathbf{R}}))$, where $\boldsymbol{\kappa}^* = (\mathbf{d}^*/|\mathbf{d}^*|) \in \mathscr{S}^6$, eqn (42) admits an equivalent presentation $\mathbf{T}:\boldsymbol{\kappa}^* - \mathscr{D}(\boldsymbol{\kappa}^*, \mathbf{F}, \bar{\mathbf{R}}) < 0$, $\forall \boldsymbol{\kappa}^* \in \mathscr{S}^6$. In a geometrical interpretation (Fig. 4) vectors $S(\mathbf{T}) := \{(\mathbf{T}:\boldsymbol{\kappa}^*)\boldsymbol{\kappa}^* : \boldsymbol{\kappa}^* \in \mathscr{S}^6 : \mathbf{T}:\boldsymbol{\kappa}^* \geqslant 0\}$ describe in space $\mathscr{R}^6$ the sphere, plotted on vector $\mathbf{T}$ as the diameter. Vectors $D := \{\mathscr{D}(\boldsymbol{\kappa}^*, \mathbf{F}, \bar{\mathbf{R}})\boldsymbol{\kappa}^* : \boldsymbol{\kappa}^* \in \mathscr{S}^6\}$ at fixed $\mathbf{F}$ and $\bar{\mathbf{R}}$ describe some surface. Inequality (42) means that the sphere $S(\mathbf{T})$ is inside the surface D. Condition $\mathbf{T}:\mathbf{d} = \mathscr{D}(\mathbf{d}, \mathbf{F}, \bar{\mathbf{R}})$ can be met when sphere $S(\mathbf{T})$ and surface D have common points, i.e. at their intersection or touching. Touching is the first opportunity for plastic flow to begin and we assume that this opportunity is realized. As our main postulate we will use the **postulate of realizability**:

Let us start from the plastic equilibrium state $\mathbf{d} = \mathbf{0}$ *and vary the* $\mathbf{T}$*-vector continuously; for each* $\mathbf{T}$ *vary all admissible* $\mathbf{d}^*$. *If in the course of this variation of* $\mathbf{T}$ *the condition*

$$\mathbf{T}:\mathbf{d} - \mathscr{D}(\mathbf{d}, \mathbf{F}, \bar{\mathbf{R}}) = 0 \tag{43}$$

is fulfilled for the first time for some $\mathbf{d} \neq \mathbf{0}$, *then plastic flow will occur with this* $\mathbf{d}$ (*if condition* (43) *is not violated in the course of the plastic flow*).

If, in the course of $\mathbf{T}$-variation, condition (43) is satisfied for one or simultaneously for several tensors $\mathbf{d}$, then for arbitrary other $\mathbf{d}^*$ the inequality (42) should hold, as in the opposite case for this $\mathbf{d}^*$ condition (43) had to be met before it was satisfied for $\mathbf{d}$. Taking into account that for $\mathbf{d} \neq \mathbf{0}$, $\mathbf{T} = \mathbf{T}(\mathbf{d}, \mathbf{F}, \bar{\mathbf{R}})$, we obtain the extremum principle

$$\mathbf{T}(\mathbf{d}, \mathbf{F}, \bar{\mathbf{R}}) : \mathbf{d} - \mathscr{D}(\mathbf{d}, \mathbf{F}, \bar{\mathbf{R}}) = 0 > \mathbf{T}(\mathbf{d}, \mathbf{F}, \bar{\mathbf{R}}) : \mathbf{d}^* - \mathscr{D}(\mathbf{d}^*, \mathbf{F}, \bar{\mathbf{R}}). \tag{44}$$

From principle (44) for smooth $\mathscr{D}$ we derive the normality rule

$$\mathbf{T}(\mathbf{d}, \mathbf{F}, \bar{\mathbf{R}}) = \frac{\partial \mathscr{D}}{\partial \mathbf{d}}, \tag{45}$$

which is equivalent to the associated flow rule $\mathbf{d}(\mathbf{T}, \mathbf{F}, \bar{\mathbf{R}}) = h(\partial\varphi/\partial\mathbf{T})$, where h is a scalar.

Remark. For more complex models (materials with structural changes and concave yield surfaces) the postulate of realizability results in nonassociated flow rule (Levitas, 1995a).

To summarize, we conclude that the method proposed consists of two points:

- First, we prove that when some energetic inequality is valid for all admissible parameters, then a dissipative process (plastic flow) cannot occur, i.e. this inequality is sufficient for plastic equilibrium ($\mathbf{d} = \mathbf{0}$).
- Second, we assume that when some energetic equality is satisfied *the first time* and plastic flow can occur, it will occur.

The postulate of realizability represents a very simple and natural assumption which expresses explicitly our understanding of the concept of stability. In fact, if some dissipative process (plastic flow) can occur from an energetic point of view, but does not occur, such a situation is not stable. Various fluctuations provoke the beginning of a dissipative process and when all the equations are satisfied, the dissipative process will occur.

Various applications of the postulate of realizability to plasticity, irreversible thermodynamics, phase-transition theory and stability analysis (Levitas, 1995a, 1997a) give the impression that it expresses a general essential property of dissipative systems.

5.2. *Equation and extremum principle for plastic spin*

We assume that there is no constitutive equation for spin $\mathbf{\Omega}$. Then each instant of a deformation process is a *bifurcation* point, because a number of solutions with various spins $\mathbf{\Omega}$ are possible. It is natural to determine the unique $\mathbf{\Omega}$ as the stable homogeneous solution of the BVP. We will use the postulate of realizability to formulate the corresponding concept of stability.

5.2.1. *Displacement controlled boundary conditions.* Let at time t tensors $\mathbf{F}$, $\bar{\mathbf{R}}$, $\mathbf{l}$ and $\mathbf{T}$ be known, but tensor $\mathbf{\Omega}$ and consequently the plastic spin $\mathbf{W}_p$ not. Let us determine $\mathbf{\Omega}$ at prescribed $\mathbf{l}$, i.e. at displacement controlled boundary conditions. As tensor $\mathbf{\Omega}$ is not an indifferent one [see eqn (39)], it is more convenient to formulate the problem with respect to indifferent plastic spin $\mathbf{W}_p = \mathbf{W} - \mathbf{\Omega}$ (tensor $\mathbf{W}$ is prescribed). As will be shown, we should assume the existence of one scalar constraint equation $q(\mathbf{d}, \mathbf{W}_p) = 0$, which for example limits $|\mathbf{W}_p|$ [the counterpart of eqn (24)]. Such a constraint is also necessary to satisfy the condition $\mathbf{W}_p = \mathbf{0}$ at $\mathbf{d} = \mathbf{0}$.

Let us designate all the possible homogeneous solutions of the BVP at time t by

$\mathbf{W}_p^0 \in q(\mathbf{d}, \mathbf{W}_p^0) = 0$. To find the stable solution at time t we need to study the deformation process during the time $[t, t+\Delta t]$. For each solution $\mathbf{W}_p^0$ the following equation

$$\int_t^{t+\Delta t} \mathbf{T} : \mathbf{d}\, d\tau - \int_t^{t+\Delta t} \mathscr{D}(\mathbf{d}, \mathbf{F}, \bar{\mathbf{R}})\, d\tau = 0 \tag{46}$$

is valid, which is the consequence of the definition of the dissipation function. Here Δt is the arbitrary small time increment. For each set of functions $\mathbf{d}(\tau)$, $\mathbf{W}(\tau)$, $\mathbf{F}(\tau)$, $\bar{\mathbf{R}}^0(\tau) = \bar{\mathbf{R}}(t) + \mathbf{\Omega}^0(t) \cdot \bar{\mathbf{R}}(t)(\tau - t)$, $\tau \in [t, t+\Delta t]$ we obtain a corresponding set of stress tensors $\mathbf{T}(\tau) = \mathbf{T}(\mathbf{d}(\tau), \mathbf{F}(\tau), \bar{\mathbf{R}}(\tau)) \in \mathscr{B}$. We assume that all parameters $(\mathbf{d}, \mathbf{W}, \mathbf{T}, \ldots)$ are continuous functions of time τ. As tensor $\mathbf{\Omega}^0(t) = \mathbf{W}(t) - \mathbf{W}_p^0(t)$ is not uniquely defined, the stress tensor $\mathbf{T}(\tau)$ is undefined as well and we will consider it as an independent parameter. It is easy to prove the following statement:

If for given $\mathbf{T}_\diamond(\tau)$ *at arbitrary fixed* $\mathbf{d}(\tau) \neq \mathbf{0}$, $\mathbf{W}(\tau)$ *and* $\mathbf{F}(\tau)$ *an inequality*

$$\int_t^{t+\Delta t} \mathbf{T}_\diamond : \mathbf{d}\, d\tau - \int_t^{t+\Delta t} \mathscr{D}(\mathbf{d}, \mathbf{F}, \bar{\mathbf{R}} + (\mathbf{W}(t) - \mathbf{W}_p^0(t)) \cdot \bar{\mathbf{R}}(t)(\tau - t))\, d\tau < 0$$

$$\forall \mathbf{W}_p^0 \in q(\mathbf{d}, \mathbf{W}_p^0) = 0 \tag{47}$$

is valid, then $\mathbf{d}(\tau) = \mathbf{0}$, *i.e. the deformation process is impossible.*

The proof is very simple: for an actual process with $\mathbf{d}(\tau) \neq \mathbf{0}$ eqn (46) should be met, which contradicts condition eqn (47). It is clear that such a tensor $\mathbf{T}_\diamond(\tau)$ does not belong to the set $\mathscr{B}$, otherwise it is always possible to find $\mathbf{\Omega}^0(t)$ which corresponds to given $\mathbf{T}_\diamond(\tau)$, i.e. $\mathbf{T}_\diamond(\tau) = \mathbf{T}(\mathbf{d}(\tau), \mathbf{F}(\tau), \bar{\mathbf{R}}(\tau))$ for this $\mathbf{\Omega}^0(t)$, and eqn (46) will be valid according to the definition of function $\mathscr{D}$. Thus, tensor $\mathbf{T}_\diamond(\tau)$ produces less work than any $\mathbf{T}(\tau) \in \mathscr{B}$.

For small Δt it is easy to calculate the integrals in eqns (46) and (47). Thus, eqn (46) results in

$$0.5(\mathbf{T} : \mathbf{d} + \mathbf{T}_\Delta : \mathbf{d}_\Delta)\, \Delta t - 0.5(\mathscr{D}(\mathbf{d}, \mathbf{F}, \bar{\mathbf{R}}) + \mathscr{D}(\mathbf{d}_\Delta, \mathbf{F}_\Delta, \bar{\mathbf{R}}_\Delta))\, \Delta t = 0 \tag{48}$$

and

$$\mathbf{T}_\Delta : \mathbf{d}_\Delta - \mathscr{D}(\mathbf{d}_\Delta, \mathbf{F}_\Delta, \bar{\mathbf{R}} + (\mathbf{W} - \mathbf{W}_p) \cdot \bar{\mathbf{R}}\, \Delta t) = 0. \tag{49}$$

As at time t tensors $\mathbf{F}$, $\bar{\mathbf{R}}$, $\mathbf{l}$ and $\mathbf{T}$ are known, the deformation process during the time $(t, t+\Delta t)$ will be completely defined by tensors $\mathbf{l}_\Delta$, $\mathbf{T}_\Delta$ and $\bar{\mathbf{R}}_\Delta$. The above statement can be reformulated in the following form:

If for given $\mathbf{T}_\Delta$ *at arbitrary prescribed* $\mathbf{d}_\Delta \neq \mathbf{0}$ *and* $\mathbf{l}$ *(and consequently* $\mathbf{F}_\Delta$*) an inequality*

$$\mathbf{T}_\Delta : \mathbf{d}_\Delta - \mathscr{D}(\mathbf{d}_\Delta, \mathbf{F}_\Delta, \bar{\mathbf{R}} + (\mathbf{W} - \mathbf{W}_p^0) \cdot \bar{\mathbf{R}}\, \Delta t) < 0 \quad \forall \mathbf{W}_p^0 \in q(\mathbf{d}, \mathbf{W}_p^0) = 0 \tag{50}$$

is valid, then $\mathbf{d}_\Delta = \mathbf{0}$, *i.e. the deformation process is impossible.*

The proof is the same as above. To choose unique $\mathbf{W}_p$ let us apply the **postulate of realizability**:

Let us for arbitrary prescribed tensors $\mathbf{l}$, $\mathbf{d}_\Delta \neq \mathbf{0}$ *and* Δt *start with stress tensor* $\mathbf{T}_\Delta$ *(or process* $\mathbf{T}(\tau)$*), for which a deformation process is impossible. Let us vary the vector* $\mathbf{T}_\Delta$ *continuously, and for each* $\mathbf{T}_\Delta$ *vary all admissible* $\mathbf{W}_p^0$. *As only the condition*

$$\mathbf{T}_\Delta : \mathbf{d}_\Delta - \mathscr{D}(\mathbf{d}_\Delta, \mathbf{F}_\Delta, \bar{\mathbf{R}} + (\mathbf{W} - \mathbf{W}_p) \cdot \bar{\mathbf{R}}\, \Delta t) = 0 \tag{51}$$

is fulfilled for the first time for some $\mathbf{W}_p$, *then the deformation process will occur with this* $\mathbf{W}_p$.

If in the course of continuous variation of $\mathbf{T}_\Delta$ and checking of all admissible $\mathbf{W}_p^0$ the condition (51) is fulfilled for the first time for some $\mathbf{W}_p$, then for all other $\mathbf{W}_p^0 \neq \mathbf{W}_p$ inequality (50) is valid, because eqn (51) is met for $\mathbf{W}_p$ the first time. Consequently the following extremum principle is derived

$$\mathbf{T}_\Delta : \mathbf{d}_\Delta - \mathscr{D}(\mathbf{d}_\Delta, \mathbf{F}_\Delta, \bar{\mathbf{R}} + (\mathbf{W} - \mathbf{W}_p) \cdot \bar{\mathbf{R}}\, \Delta t) = 0 > \mathbf{T}_\Delta : \mathbf{d}_\Delta - \mathscr{D}(\mathbf{d}_\Delta, \mathbf{F}_\Delta, \bar{\mathbf{R}} + (\mathbf{W} - \mathbf{W}_p^0) \cdot \bar{\mathbf{R}}\, \Delta t)$$
$$\forall \mathbf{W}_p^0 \neq \mathbf{W}_p, \quad q(\mathbf{d}, \mathbf{W}_p^0) = 0; \quad q(\mathbf{d}, \mathbf{W}_p) = 0. \tag{52}$$

In the opposite case when at time t one of the solutions $\mathbf{\Omega}^0 \neq \mathbf{\Omega}$ is realized and the postulate of realizability is violated, then for $\mathbf{T}_\Delta^0 = \mathbf{T}(\mathbf{d}_\Delta, \mathbf{F}_\Delta, \bar{\mathbf{R}} + \mathbf{\Omega}^0 \cdot \bar{\mathbf{R}}\, \Delta t)$ we can find some admissible $\mathbf{\Omega}$ for which

$$\mathbf{T}_\Delta^0 : \mathbf{d}_\Delta - \mathscr{D}(\mathbf{d}_\Delta, \mathbf{F}_\Delta, \bar{\mathbf{R}} + \mathbf{\Omega}^0 \cdot \bar{\mathbf{R}}\, \Delta t) = 0 < \mathbf{T}_\Delta^0 : \mathbf{d}_\Delta - \mathscr{D}(\mathbf{d}_\Delta, \mathbf{F}_\Delta, \bar{\mathbf{R}} + \mathbf{\Omega} \cdot \bar{\mathbf{R}}\, \Delta t). \tag{53}$$

In this situation eqn (49) will be satisfied for the first time for this $\mathbf{\Omega}$ and according to the postulate of realizability solution $\mathbf{\Omega}$ should be realized. Thus, at $\Delta t \neq 0$ the jump from $\mathbf{\Omega}^0$ to $\mathbf{\Omega}$ will occur. Consequently, the solution $\mathbf{\Omega}^0$ is *unstable* and using the postulate of realizability we can choose a *unique stable solution* among all the possible ones. From principles (52) or (53) the principle of minimum of dissipation rate at time $t + \Delta t$ follows:

$$\mathscr{D}(\mathbf{d}_\Delta, \mathbf{F}_\Delta, \bar{\mathbf{R}} + (\mathbf{W} - \mathbf{W}_p) \cdot \bar{\mathbf{R}}\, \Delta t) < \mathscr{D}(\mathbf{d}_\Delta, \mathbf{F}_\Delta, \bar{\mathbf{R}} + (\mathbf{W} - \mathbf{W}_p^0) \cdot \bar{\mathbf{R}}\, \Delta t)$$
$$\text{at} \quad q(\mathbf{d}, \mathbf{W}_p^0) = 0. \tag{54}$$

In the one-dimensional case principle (54) means that we choose the lowest curve in the stress–strain diagram (Fig. 3(a)). At small Δt the extremum principle (54) can be transformed into

$$\frac{\partial \mathscr{D}}{\partial \bar{\mathbf{R}}^t} : \dot{\bar{\mathbf{R}}} = \frac{\partial \mathscr{D}}{\partial \bar{\mathbf{R}}^t} : (\mathbf{W} - \mathbf{W}_p) \cdot \bar{\mathbf{R}} < \frac{\partial \mathscr{D}}{\partial \bar{\mathbf{R}}^t} : (\mathbf{W} - \mathbf{W}_p^0) \cdot \bar{\mathbf{R}} \quad \text{at} \quad q(\mathbf{d}, \mathbf{W}_p^0) = 0. \tag{55}$$

Additional transformations

$$\frac{\partial \mathscr{D}}{\partial \bar{\mathbf{R}}^t} : (\mathbf{W} - \mathbf{W}_p) \cdot \bar{\mathbf{R}} = \bar{\mathbf{R}} \cdot \frac{\partial \mathscr{D}}{\partial \bar{\mathbf{R}}^t} : (\mathbf{W} - \mathbf{W}_p)$$
$$= \left(\bar{\mathbf{R}} \cdot \frac{\partial \mathscr{D}}{\partial \bar{\mathbf{R}}^t} \right)_a : (\mathbf{W} - \mathbf{W}_p) = \mathbf{a} : (\mathbf{W} - \mathbf{W}_p); \quad \mathbf{a} := \left(\bar{\mathbf{R}} \cdot \frac{\partial \mathscr{D}}{\partial \bar{\mathbf{R}}^t} \right)_a, \tag{56}$$

lead to

$$\mathbf{a} : (\mathbf{W} - \mathbf{W}_p) < \mathbf{a} : (\mathbf{W} - \mathbf{W}_p^0) \quad \text{at} \quad q(\mathbf{d}, \mathbf{W}_p^0) = 0, \tag{57}$$

and

$$\mathbf{a} : \mathbf{W}_p > \mathbf{a} : \mathbf{W}_p^0 \quad \text{or} \quad \mathbf{a} : \mathbf{W}_p^0 \to \max \quad \text{at} \quad q(\mathbf{d}, \mathbf{W}_p^0) = 0. \tag{58}$$

As the extremum principle (58) is linear in $\mathbf{W}_p^0$, it is clear that without an additional constraint the solution cannot be found. That is why we assumed the existence of the constraint equation $q(\mathbf{d}, \mathbf{W}_p^0) = 0$ from the very beginning. For the single crystal, due to the known structure of the expression for $\mathbf{W}_p$ [see eqn $(17)_1$], an additional equation is not necessary.

Let q be a nonlinear function of $\mathbf{W}_p$. In this case using principle (58) the following equation for $\mathbf{W}_p$ can be derived:

$$\mathbf{a} = \eta \frac{\partial q}{\partial \mathbf{W}_p^t}, \tag{59}$$

where η is a scalar determined from condition $q = 0$; the sign of η is determined from eqn $(58)_2$. If we assume as the simplest case that q depends on $\mathbf{d}$ and $\mathbf{W}_p$ separately and is an isotropic function of $\mathbf{W}_p$, i.e. $q = f(\mathbf{d}) - |\mathbf{W}_p| = 0$, then

$$\mathbf{W}_p = \eta \mathbf{a} \quad \text{or} \quad \mathbf{W}_p = \frac{\mathbf{a}}{|\mathbf{a}|} f(\mathbf{d}) \quad \text{at} \quad \mathbf{a} \neq \mathbf{0}. \tag{60}$$

Equation $(60)_1$ is valid at arbitrary $\mathbf{a}$. Equation $(60)_2$ can be used at $\mathbf{a} \neq \mathbf{0}$ which we will assume in the following similar equations as well. We will use this formula for plastic spin, because it is consistent with some known results (see below). In the given case scalar function η completely determines q [at the given $\mathscr{D}(\mathbf{d}, \mathbf{F}, \bar{\mathbf{R}})$]; examples of function η for the model with kinematic hardening are given for instance in papers by Dafalias (1985), Paulun and Pechersky (1987), Van der Giessen (1991). Note that the function q (and consequently η) can depend on $\bar{\mathbf{R}}, \mathbf{F}, \mathbf{T}, \ldots$, as well. But the more general eqn (59) can be applied as well. If q is a linear function of $\mathbf{W}_p$, then the minimum in the principle (55) is reached at the boundary of the set $\mathscr{C}$ of all admissible $\mathbf{W}_p$.

If for some $\mathbf{d}$, $\mathbf{F}$ and $\mathbf{W}_p(\partial\mathscr{D}/\partial\bar{\mathbf{R}}') = \mathbf{a} = \mathbf{0}$ and this point corresponds to the minimum of $\mathscr{D}$, then $\mathbf{W}_p = \mathbf{0}$. In the case of maximum $\mathscr{D}$ again $\mathbf{W}_p = \mathbf{0}$, but this state is unstable. Arbitrary small or finite deviations from the point with $\mathbf{a} = \mathbf{0}$ will increase.

Remark. Let us explain why it is impossible to derive equation $\mathbf{a} = \eta(\partial q/\partial \mathbf{\Omega}')$ instead of eqn (59) from principle (57). The expressions in the extremum principle (54) are independent of superposed RBR, but in extremum principle (55) [or (57)] they are not, because $(\partial\mathscr{D}/\partial\bar{\mathbf{R}}') \cdot \dot{\bar{\mathbf{R}}}$ and $\mathbf{a} : \mathbf{\Omega}$ are transformed into

$$\left(\frac{\partial \mathscr{D}}{\partial \bar{\mathbf{R}}'} \cdot \mathbf{Q}'\right) \cdot (\dot{\mathbf{Q}} \cdot \bar{\mathbf{R}} + \mathbf{Q} \cdot \dot{\bar{\mathbf{R}}}) \quad \text{and} \quad (\mathbf{Q} \cdot \mathbf{a} \cdot \mathbf{Q}') : (\dot{\mathbf{Q}} \cdot \mathbf{Q}' + \mathbf{Q} \cdot \mathbf{\Omega} \cdot \mathbf{Q}'). \tag{61}$$

This does not mean that principle eqn (57) is not correct; it is valid for arbitrary $\mathbf{Q}$ (because it follows from the frame indifferent principle eqns (54) by removing some equal frame dependent terms). But principle (57) cannot be used to derive equations between $\mathbf{\Omega}$ and $\mathbf{d}$, because such a relationship will be frame dependent (even the constraint equation

$$q(\mathbf{d}, \mathbf{\Omega}) = 0 = q(\mathbf{Q} \cdot \mathbf{d} \cdot \mathbf{Q}', \dot{\mathbf{Q}} \cdot \mathbf{Q}' + \mathbf{Q} \cdot \mathbf{\Omega} \cdot \mathbf{Q}') \tag{62}$$

is not objective). The expressions in extremum principle eqn (57) are frame indifferent, so that is why it is used to derive the required equation.

5.2.2. *Stress controlled and mixed boundary conditions.* Assume that we known tensors $\mathbf{F}$, $\bar{\mathbf{R}}$, $\mathbf{W}$ and $\mathbf{T}$ at time t and tensors $\mathbf{T}_\Delta$ and $\mathbf{W}_\Delta$ at time $t+\Delta t$. In this case we need to vary $\mathbf{d}$ and $\mathbf{\Omega}$, as well as $\mathbf{d}_\Delta$. For actual parameters

$$\mathbf{T}_\Delta : \mathbf{d}_\Delta - \mathscr{D}(\mathbf{d}_\Delta, \mathbf{F}+\mathbf{l}\cdot\mathbf{F}\,\Delta t, \bar{\mathbf{R}}+(\mathbf{W}-\mathbf{W}_p)\cdot\bar{\mathbf{R}}\,\Delta t) = 0. \tag{63}$$

Consider $\mathbf{d} = |\mathbf{d}|\mathbf{k}$ and $\mathbf{d}_\Delta = |\mathbf{d}_\Delta|\mathbf{k}_\Delta$. According to the associated flow rule for smooth yield surface tensor $\mathbf{T}$ completely determines a directing tensor $\mathbf{k}$, and the tensor $\mathbf{T}_\Delta$ defines the unique $\mathbf{k}_\Delta$. As $\mathbf{T}$ and $\mathscr{D}$ are homogeneous functions of degree zero and one in $\mathbf{d}$ correspondingly, then $|\mathbf{d}_\Delta|$ can be omitted in eqn (63):

$$\mathbf{T}_\Delta : \mathbf{k}_\Delta - \mathscr{D}(\mathbf{k}_\Delta, \mathbf{F}+(\mathbf{k}|\mathbf{d}|+\mathbf{W})\cdot\mathbf{F}\,\Delta t, \bar{\mathbf{R}}+(\mathbf{W}-\mathbf{W}_p)\cdot\bar{\mathbf{R}}\,\Delta t) = 0. \tag{64}$$

If for given $\mathbf{T}_\Delta$ and $\mathbf{W}$ and at some fixed $|\mathbf{d}|$ and Δt an inequality

$$\mathbf{T}_\Delta : \mathbf{k}_\Delta - \mathscr{D}(\mathbf{k}_\Delta, \mathbf{F}+(\mathbf{k}|\mathbf{d}|+\mathbf{W})\cdot\mathbf{F}\,\Delta t, \bar{\mathbf{R}}+(\mathbf{W}-\mathbf{W}_p^0)\cdot\bar{\mathbf{R}}\,\Delta t) < 0$$
$$\forall \mathbf{W}_p^0 \in q(|\mathbf{d}|\mathbf{k}, \mathbf{W}_p^0) = 0 \tag{65}$$

is valid, then $\mathbf{d} = \mathbf{0}$ and a deformation process is impossible. The proof is the same as for the case with prescribed $\mathbf{d}$. Let us apply the **postulate of realizability**:

Let us at arbitrary prescribed $\mathbf{T}_\Delta$, $\mathbf{W}$ *and* Δt *start with* $|\mathbf{d}|$, *for which inequality* (65) *is valid and a deformation process is impossible. Let us vary* $|\mathbf{d}|$ *continuously and for each* $|\mathbf{d}|$ *vary all admissible* $\mathbf{W}_p^0$. *If in the course of such a variation condition* (64) *is fulfilled for the first time for some* $\mathbf{W}_p$, *then a deformation process will occur with this* $\mathbf{W}_p$.

As for the displacement controlled case, for the actual tensor $\mathbf{W}_p$ eqn (63) is valid, for all other admissible $\mathbf{W}_p^0$ the inequality (65) is valid. Consequently we get the extremum principle (52) and following from it eqns (54)–(60). In the one-dimensional case [Fig. 3(b)] we again choose the lowest curve (with maximal $\mathbf{d}\,\Delta t$).

The non-essential difference consists in the fact that tensor $\mathbf{T}_\Delta$, after finding $\mathbf{W}_p$ for prescribed $\mathbf{d}\,\Delta t$ and $\mathbf{W}\,\Delta t$, should be found from equation $\mathbf{T}_\Delta = \mathbf{T}(\mathbf{k}_\Delta, \mathbf{F}_\Delta, \bar{\mathbf{R}}_\Delta)$, and for prescribed $\mathbf{T}_\Delta$, $\mathbf{W}$ and Δt scalar $|\mathbf{d}|$ has to be defined from eqn (64).

For mixed boundary conditions (some components of $\mathbf{T}_\Delta$ and $\mathbf{d}$ tensors are prescribed) the results will be the same, because in fact we vary $\mathbf{W}_p$ at fixed $\mathbf{T}_\Delta$ and $\mathbf{d}$, some of the components of $\mathbf{T}_\Delta$ and $\mathbf{d}$ are prescribed and the remaining ones are fixed arbitrarily.

Remark. It is possible to repeat the same consideration for a finite volume of rigid-plastic material (as in the paper by Levitas (1995a) for the case without plastic spin) under arbitrary nonuniform boundary conditions and to derive the same principle of minimum of dissipation rate at time $t+\Delta t$ for each material particle. The important point of such a derivation is that extremum principle (54) is independent of the type of boundary condition, because for a nonuniform state we cannot prescribe boundary conditions for internal particles. Consequently the extremum principle (54) and eqn (60) are valid for an arbitrary material point and deformation process, i.e. as a constitutive equation.

Note that alternative extremum principles for the description of the stable post-bifurcation deformation process by Petryk (1991) and Bažant (1989) are analysed in a paper by Levitas (1995a).

6. PLASTIC AND LATTICE SPINS FOR A SINGLE CRYSTAL

6.1. *Tension of a single crystal*

Principle (54) results in

$$\mathscr{D}_\Delta = \mathbf{T}_\Delta : \mathbf{d}_\Delta = \sigma_\Delta d_{22\Delta} = \pm \frac{2k(\gamma_\Delta)}{\sin 2\alpha_\Delta} \frac{v_{0\Delta}}{h_\Delta} \to \min, \tag{66}$$

under constraint eqn (24), where only $\dot{\alpha}$ is variable. If in this concrete problem we do not consider superposed RBR, we can determine $\dot{\alpha}$ from the principle (66).

For tensile stress $\sigma > 0$, velocity $v_0 > 0$ as well (since $\mathscr{D} > 0$) and condition (66) results in

$$\frac{1}{\sin 2\alpha_\Delta} \to \min \quad \text{at} \quad \sin 2\alpha_\Delta > 0; \tag{67}$$

$$-\frac{1}{\sin 2\alpha_\Delta} \to \min \quad \text{at} \quad \sin 2\alpha_\Delta < 0; \tag{68}$$

or at infinitesimal Δt [see principle (55)]

$$-\cos 2\alpha\dot{\alpha} \to \min \quad \text{at} \quad \sin 2\alpha > 0; \tag{69}$$

$$\cos 2\alpha\dot{\alpha} \to \min \quad \text{at} \quad \sin 2\alpha < 0. \tag{70}$$

Principles (69) and (70) are equivalent to

$$\begin{aligned} \dot{\alpha} \to \min \quad \text{at} \quad & \cos 2\alpha < 0 \quad \text{and} \quad \sin 2\alpha > 0; \\ & \cos 2\alpha > 0 \quad \text{and} \quad \sin 2\alpha < 0; \end{aligned} \tag{71}$$

$$\begin{aligned} \dot{\alpha} \to \max \quad \text{at} \quad & \cos 2\alpha > 0 \quad \text{and} \quad \sin 2\alpha > 0; \\ & \cos 2\alpha < 0 \quad \text{and} \quad \sin 2\alpha < 0; \end{aligned} \tag{72}$$

or

$$\begin{aligned} \dot{\alpha} \to \min \quad \text{at} \quad & \cos 2\alpha \sin 2\alpha < 0; \\ \dot{\alpha} \to \max \quad \text{at} \quad & \cos 2\alpha \sin 2\alpha > 0. \end{aligned} \tag{73}$$

Constraint eqn (24) is linear in $\dot{\alpha}$, so the extremum in principle (73) is reached at the boundary of set $\mathscr{C}$ of admissible $\dot{\alpha}$. Let us define the boundary. As the shear stresses τ_1 and τ_2 in both slip systems are the same (due to the symmetry of the stress tensor and orthogonality of slip systems), then from conditions $\tau_1\dot{\gamma}_1 \geqslant 0$ and $\tau_2\dot{\gamma}_2 \geqslant 0$ it follows that $\dot{\gamma}_1$ and $\dot{\gamma}_2$ (and consequently $\dot{\gamma}$) should have the same sign or one of them

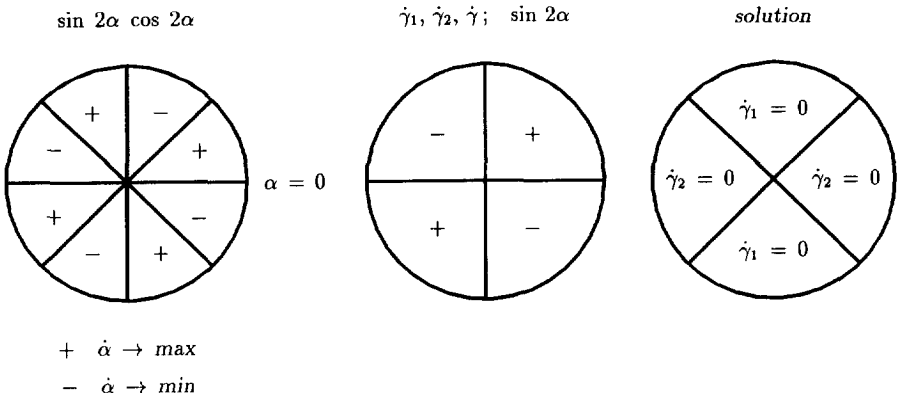

Fig. 5. Signs of some parameters and the solution of the extremum problem (73) for the tension of a single crystal for different intervals of α.

is zero (Fig. 5). Consequently the boundaries of the set $\mathscr{C}$ are determined by conditions $\dot{\gamma}_1 = 0$ and $\dot{\gamma}_2 = 0$ in eqn (24). From eqn $(20)_1$ it is clear that the sign of $\dot{\gamma}$ (consequently $\dot{\gamma}_1$ and $\dot{\gamma}_2$) is the same as the sign of $\sin 2\alpha$. Considering all the combinations of extrema in principles (73) and signs of $\dot{\gamma}_1$ and $\dot{\gamma}_2$ in eqn (24) we obtain the solution (Fig. 5):

$$\text{at} \quad -\frac{\pi}{4} < \alpha < \frac{\pi}{4} \quad \text{and} \quad \frac{3\pi}{4} < \alpha < \frac{5\pi}{4} \tag{74}$$

$$\dot{\gamma}_2 = 0; \quad \dot{\gamma}_1 = 2\dot{\gamma} = \frac{2v_0}{h \sin 2\alpha} = -2W_p; \quad \dot{\alpha} = \frac{v_0}{h}\frac{(1-\cos 2\alpha)}{\sin 2\alpha}. \tag{75}$$

$$\text{At} \quad \frac{\pi}{4} < \alpha < \frac{3\pi}{4} \quad \text{and} \quad \frac{5\pi}{4} < \alpha < \frac{7\pi}{4} \tag{76}$$

$$\dot{\gamma}_1 = 0; \quad \dot{\gamma}_2 = 2\dot{\gamma} = \frac{2v_0}{h \sin 2\alpha} = 2W_p; \quad \dot{\alpha} = -\frac{v_0}{h}\frac{(1+\cos 2\alpha)}{\sin 2\alpha}. \tag{77}$$

Equation $(75)_3$ can be easily integrated: $\sin\alpha = \sin\alpha_0(h/h_0)$. Similarly for eqn $(77)_3$ we obtain $\cos\alpha = \cos\alpha_0(h/h_0)$. At $\alpha = \pm(\pi/4)$ and $\alpha = \pm(3\pi/4)$ $\dot{\alpha} = 0$ and $\dot{\gamma}_1 = \dot{\gamma}_2$.

Let us summarize the scenario of crystal lattice rotation. Four orientations, namely $\alpha = \pm(\pi/4)$ and $\alpha = \pm(3\pi/4)$, correspond to stable equilibrium, the lattice orientation is fixed in them, both slip systems are symmetric and work equivalently, and any small or finite deviation from this orientation decreases. At $\alpha = (\pi/2)k$, $k = 0, 1, 2, 3$, shear stress in both slip systems is zero, so $\dot{\gamma}_1 = \dot{\gamma}_2 = \dot{h} = \dot{\alpha} = 0$. At small or finite deviation from these orientations deformation can proceed and the lattice rotates to the nearest minimum of $\mathscr{D}$, i.e. to $\alpha = \pm(\pi/4)$, $\alpha = \pm(3\pi/4)$. Thus, we have obtained a unique solution, in particular for lattice and plastic spins, using the extremum principle to choose the stable solution. The compression of a crystal can be considered in a similar way.

It is difficult to make an exact comparison of our results with the results of the application of the minimum spin principle by Fuh and Havner (1989). These authors

presented a number of applications of their principle without an explicit general mathematical formulation of the principle. But it seems to us that for the given simplest problem the results coincide. In fact, according to the extremum principle (73), a crystal lattice rotates with a maximum possible spin to the nearest orientation among $\alpha = \pm(\pi/4)$, $\alpha = \pm(3\pi/4)$. As the plastic spin is the difference between prescribed total spin and lattice spin, it should tend to a minimum. But again, we do not know a reasonable formulation of the minimum spin principle for polycrystals.

6.2. *Combined loading of a single crystal*

If on the lines AB and CD (Fig. 2) $\tau(h) \neq 0$ then Schmid's law results in

$$\tau_\alpha = 0.5\sigma \sin 2\alpha + \tau \cos 2\alpha = \pm k(\gamma). \tag{78}$$

In this case eqns (15)–(24) and (66)–(77) are also valid and $\tau \neq 0$ only reduces the value of σ [see eqn (78)]. This result follows from the fact that an expression

$$\mathscr{D} = \pm 2k\dot{\gamma} = \pm \frac{2k}{\sin 2\alpha} \frac{v_0}{h} \tag{79}$$

at prescribed v_0 is independent of value $\tau(h)$.

Let us consider the other type of boundary condition.

On the line AB $(r_2 = h)$: the horizontal velocity $u = u_0 + Ar_1$; the tensile stress $\sigma = \sigma(h)$.

On the line CD $(r_2 = 0)$: the horizontal velocity $u = Ar_1$; the vertical velocity $v_0 = 0$.

The constant A could not be independent of u_0, because the field $u = (2\cos 2\alpha r_2 - \sin 2\alpha r_1)\dot{\gamma}$, compatible with kinematical eqn (20) satisfies the boundary conditions at

$$\dot{\gamma} = \frac{u_0}{2h\cos 2\alpha}, \quad A = -0.5\frac{u_0}{h}\tan 2\alpha. \tag{80}$$

Direct calculations lead to

$$d_{12} = \dot{\gamma}\cos 2\alpha = \frac{u_0}{2h}, \quad d_{22} = -d_{11} = \dot{\gamma}\sin 2\alpha = \frac{u_0}{2h}\tan 2\alpha,$$

$$\mathscr{D} = \mathbf{T}:\mathbf{d} = \pm \frac{k}{\cos 2\alpha}\frac{u_0}{h}. \tag{81}$$

The shear stress is determined from eqn (78). As above, we have found the homogeneous solution of the BVP, but parameters $\dot{\gamma}_i$, W_p and $\dot{\alpha}$ are undetermined. Let us find the stable solution, using extremum principle (54), i.e.

$$\pm \frac{2k(\gamma_\Delta)}{\cos 2\alpha_\Delta}\frac{u_{0\Delta}}{h_\Delta} \to \min, \tag{82}$$

under the constraint eqn (24), where only $\dot{\alpha}$ is variable.

Assuming $u_0 > 0$ (the case $u_0 < 0$ can be analyzed similarly) we get

$$\frac{1}{\cos 2\alpha_\Delta} \to \min \quad \text{at} \quad \cos 2\alpha_\Delta > 0\,; \tag{83}$$

$$-\frac{1}{\cos 2\alpha_\Delta} \to \min \quad \text{at} \quad \cos 2\alpha_\Delta < 0\,; \tag{84}$$

or at infinitesimal Δt

$$\sin 2\alpha\dot{\alpha} \to \min \quad \text{at} \quad \cos 2\alpha > 0\,; \tag{85}$$

$$-\sin 2\alpha\dot{\alpha} \to \min \quad \text{at} \quad \cos 2\alpha < 0. \tag{86}$$

Principles (85) and (86) are equivalent to

$$\begin{aligned} \dot{\alpha} \to \min \quad \text{at} \quad & \sin 2\alpha > 0 \quad \text{and} \quad \cos 2\alpha > 0\,; \\ & \sin 2\alpha < 0 \quad \text{and} \quad \cos 2\alpha < 0\,; \end{aligned} \tag{87}$$

$$\begin{aligned} \dot{\alpha} \to \max \quad \text{at} \quad & \sin 2\alpha < 0 \quad \text{and} \quad \cos 2\alpha > 0\,; \\ & \sin 2\alpha > 0 \quad \text{and} \quad \cos 2\alpha < 0\,; \end{aligned} \tag{88}$$

or

$$\begin{aligned} \dot{\alpha} \to \min \quad \text{at} \quad & \cos 2\alpha \sin 2\alpha > 0\,; \\ \dot{\alpha} \to \max \quad \text{at} \quad & \cos 2\alpha \sin 2\alpha < 0. \end{aligned} \tag{89}$$

As in the previous problem the constraint eqn (24) is linear in $\dot{\alpha}$, and consequently the extremum in principle (89) is reached at the boundary of the set $\mathscr{C}$ of admissible $\dot{\alpha}$. Again the rates $\dot{\gamma}_1$ and $\dot{\gamma}_2$ (and consequently $\dot{\gamma}$) should have the same sign or one of them is zero and the boundaries of the set $\mathscr{C}$ are determined by conditions $\dot{\gamma}_1 = 0$ and $\dot{\gamma}_2 = 0$ in eqn (24).

From eqn $(81)_1$ it follows that the sign of $\dot{\gamma}$ (consequently $\dot{\gamma}_1$ and $\dot{\gamma}_2$) is the same as the sign of $\cos 2\alpha$. Considering all the combinations of extrema in principles (89) and signs of $\dot{\gamma}_1$ and $\dot{\gamma}_2$ in eqn (24) we obtain the solution (Fig. 6):

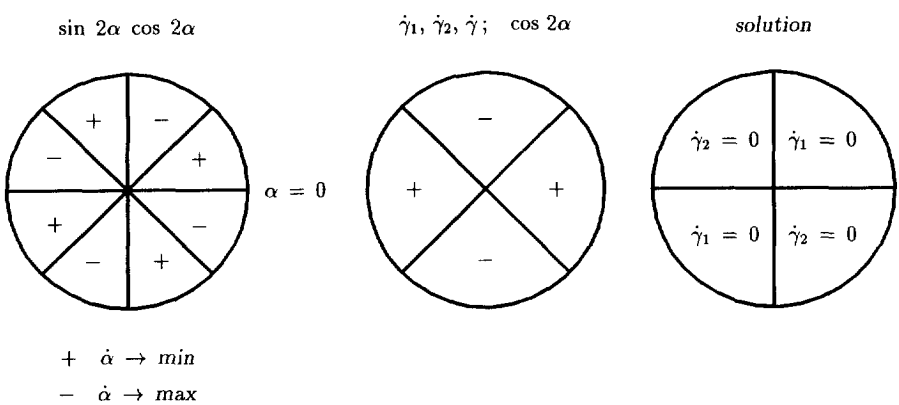

Fig. 6. Signs of some parameters and the solution of the extremum problem (89) for combined loading of a single crystal for different intervals of α.

at $\frac{\pi}{2} < \alpha < \pi$ and $\frac{3\pi}{2} < \alpha < 2\pi$

$$\dot{\gamma}_2 = 0; \quad \dot{\gamma}_1 = 2\dot{\gamma} = \frac{u_0}{h\cos 2\alpha} = -2W_p; \quad \dot{\alpha} = \frac{u_0}{2h}\frac{(1-\cos 2\alpha)}{\cos 2\alpha}. \tag{90}$$

At $0 < \alpha < \frac{\pi}{2}$ and $\pi < \alpha < \frac{3\pi}{2}$

$$\dot{\gamma}_1 = 0; \quad \dot{\gamma}_2 = 2\dot{\gamma} = \frac{u_0}{h\cos 2\alpha} = 2W_p; \quad \dot{\alpha} = -\frac{u_0}{2h}\frac{(1+\cos 2\alpha)}{\cos 2\alpha}. \tag{91}$$

At $\alpha = (\pi/2)k$, $k = 0, 1, 2, 3$, $\dot{\alpha} = 0$ and $\dot{\gamma}_1 = \dot{\gamma}_2$.

7. INITIALLY ANISOTROPIC POLYCRYSTAL

In the frame of reference δ in which $\mathbf{F}_\delta = \mathbf{U} = \mathbf{R}^t \cdot \mathbf{F}$ and $\mathbf{d}_\delta = (\dot{\mathbf{U}} \cdot \mathbf{U}^{-1})_s$ let us assume (see Appendix for designations)

$$\mathscr{D} := (\mathbf{d}_\delta : \mathbf{E}_\delta : \mathbf{d}_\delta)^{1/2}; \quad \mathbf{E}_\delta := \boldsymbol{\varphi} \overset{2}{\circ} (\boldsymbol{\varphi} \cdot \mathbf{E}_0 \cdot \boldsymbol{\varphi}^t) \overset{2}{*} \boldsymbol{\varphi}^t, \tag{92}$$

where $\mathbf{E}_0$ is the constant fourth order tensor. Then

$$\mathbf{T}_\delta = \frac{\partial \mathscr{D}}{\partial \mathbf{d}_\delta} = \mathscr{D}^{-1}\mathbf{E}_\delta : \mathbf{d}_\delta, \tag{93}$$

and for the yield conditions it follows that

$$\varphi = \mathbf{T}_\delta : \mathbf{E}_\delta^{-1} : \mathbf{T}_\delta - 1 = 0, \quad \mathbf{E}_\delta^{-1} := \boldsymbol{\varphi} \overset{2}{\circ} (\boldsymbol{\varphi} \cdot \mathbf{E}_0^{-1} \cdot \boldsymbol{\varphi}^t) \overset{2}{*} \boldsymbol{\varphi}^t. \tag{94}$$

Consequently tensor $\boldsymbol{\varphi}$ characterizes a rotation of the ellipsoidal yield surface in a stress space $\mathbf{T}_\delta$ and is in principle a measurable parameter. Alternative expressions of eqns (92)–(94) in the isoclinic frame of reference χ are

$$\mathscr{D} = (\mathbf{d}_\chi : \mathbf{E}_0 : \mathbf{d}_\chi)^{1/2}; \quad \mathbf{T}_\chi := \mathscr{D}^{-1}\mathbf{E}_0 : \mathbf{d}_\chi; \quad \varphi = \mathbf{T}_\chi : \mathbf{E}_0 : \mathbf{T}_\chi - 1 = 0, \tag{95}$$

or in the fixed frame of reference

$$\mathscr{D} = (\mathbf{d} : \mathbf{E} : \mathbf{d})^{1/2}; \quad \mathbf{E} := \bar{\mathbf{R}} \overset{2}{\circ} (\bar{\mathbf{R}} \cdot \mathbf{E}_0 \cdot \bar{\mathbf{R}}^t) \overset{2}{*} \bar{\mathbf{R}}^t; \tag{96}$$

$$\mathbf{T} = \mathscr{D}^{-1}\mathbf{E} : \mathbf{d}; \quad \varphi = \mathbf{T} : \mathbf{E} : \mathbf{T} - 1 = 0. \tag{97}$$

To calculate the tensor $\mathbf{a}$ let us determine

$$\frac{\partial \mathscr{D}}{\partial \bar{\mathbf{R}}^t} : \dot{\bar{\mathbf{R}}} = \frac{\partial \mathscr{D}}{\partial \mathbf{d}_\chi} : \frac{\mathrm{d}(\bar{\mathbf{R}}^t \cdot \mathbf{d} \cdot \bar{\mathbf{R}})}{\mathrm{d}t}\bigg|_{\mathbf{d}} = \mathbf{T}_\chi : (\bar{\mathbf{R}}^t \cdot \mathbf{d} \cdot \dot{\bar{\mathbf{R}}} + \dot{\bar{\mathbf{R}}}^t \cdot \mathbf{d} \cdot \bar{\mathbf{R}}) \tag{98}$$

$$= 2\mathbf{T}_\chi : \bar{\mathbf{R}}^t \cdot \mathbf{d} \cdot \dot{\bar{\mathbf{R}}} = 2\mathbf{T}_\chi \cdot \bar{\mathbf{R}}^t \cdot \mathbf{d} : \dot{\bar{\mathbf{R}}},$$

i.e.

$$\frac{\partial \mathscr{D}}{\partial \bar{\mathbf{R}}^t} = 2\mathbf{T}_\chi \cdot \bar{\mathbf{R}}^t \cdot \mathbf{d}, \quad \bar{\mathbf{R}} \cdot \frac{\partial \mathscr{D}}{\partial \bar{\mathbf{R}}^t} = 2\bar{\mathbf{R}} \cdot \mathbf{T}_\chi \cdot \bar{\mathbf{R}}^t \cdot \mathbf{d} = 2\mathbf{T} \cdot \mathbf{d} \tag{99}$$

and according to eqns (60) and $(37)_1$

$$\mathbf{W}_p = f(\mathbf{d}) \frac{(\mathbf{T} \cdot \mathbf{d} - \mathbf{d} \cdot \mathbf{T})}{|\mathbf{T} \cdot \mathbf{d} - \mathbf{d} \cdot \mathbf{T}|} = \eta(\mathbf{T} \cdot \mathbf{d} - \mathbf{d} \cdot \mathbf{T}), \quad \eta \geqslant 0, \tag{100}$$

$$\mathbf{\Omega} := \dot{\bar{\mathbf{R}}} \cdot \bar{\mathbf{R}}^t = \mathbf{W} - \eta(\mathbf{T} \cdot \mathbf{d} - \mathbf{d} \cdot \mathbf{T}). \tag{101}$$

For an isotropic material, tensors $\mathbf{T}$ and $\mathbf{d}$ are coaxial and according to eqns (100) and (101) $\mathbf{W}_p \equiv \mathbf{0}$, $\mathbf{\Omega} = \mathbf{W}$ and the privileged isoclinic configuration coincides with the actual configuration. When the vector $\mathbf{T}$ is directed along the symmetry aces of the ellipsoid, then vectors $\mathbf{T}$ and $\mathbf{d}$ are collinear and again $\mathbf{W}_p \equiv \mathbf{0}$. If the vector $\mathbf{T}$ is directed along the shortest axis of the yield surface, then the state with $\mathbf{W}_p = \mathbf{0}$ is stable; in other cases small or finite perturbations will lead to a deviation from state $\mathbf{W}_p = \mathbf{0}$.

It is easy to prove the following statement: *when the dissipation function can be represented as*

$$\mathscr{D} = \mathscr{D}(\mathbf{d}_\chi), \tag{102}$$

then eqn (100) *is valid.* Function $\mathscr{D}$ can depend on the number of fixed tensors of arbitrary rank, which characterize initial anisotropy, as well as on scalar hardening parameters.

The proof repeats all the transformations in eqns (98), (99), because we did not use any other properties of the dissipation function than assumed in eqn (102).

If in the isoclinic configuration $\chi\mathscr{D} = \mathscr{D}(\mathbf{d}_\chi, \mathbf{F}_\chi)$, then the calculation of tensor $\mathbf{a}$ results in

$$\mathbf{W}_p = f(\mathbf{d}) \frac{(\mathbf{T} \cdot \mathbf{d} - \mathbf{T} \cdot \mathbf{d}) + 0.5\left(\dfrac{\partial \mathscr{D}}{\partial \mathbf{F}} \cdot \mathbf{F}^t - \mathbf{F} \cdot \dfrac{\partial \mathscr{D}}{\partial \mathbf{F}^t}\right)}{\left|(\mathbf{T} \cdot \mathbf{d} - \mathbf{T} \cdot \mathbf{d}) + 0.5\left(\dfrac{\partial \mathscr{D}}{\partial \mathbf{F}} \cdot \mathbf{F}^t - \mathbf{F} \cdot \dfrac{\partial \mathscr{D}}{\partial \mathbf{F}^t}\right)\right|}. \tag{103}$$

When $\mathscr{D}$ depends on $\mathbf{F}$ in terms $\mathbf{U}$, then $(\partial\mathscr{D}/\partial\mathbf{F}) \cdot \mathbf{F}^t = (\partial\mathscr{D}/\partial\mathbf{U}) \cdot \mathbf{U}$. If $\mathscr{D}$ is an isotropic function of $\mathbf{U}$, tensors $\mathbf{U}$ and $\partial\mathscr{D}/\partial\mathbf{U}$ are coaxial and eqn (103) reduces to eqn (100).

8. THEORY WITH INTERNAL VARIABLES

Assume that history dependence of the $\mathscr{D}$ is taken into account with the help of an internal variable $\mathbf{L}$, which is for example the back stress tensor ($\mathbf{L} = \mathbf{L}^t$):

$$\mathscr{D} = \mathscr{D}(\mathbf{d}, \mathbf{L}, \bar{\mathbf{R}}) = \mathscr{D}(\mathbf{d}_\chi, \mathbf{L}_\chi). \tag{104}$$

In the frame of reference χ we assume the same equation as at small strains $\dot{\mathbf{L}}_\chi = A\mathbf{d}_\chi$. Then in the fixed frame of reference

$$\dot{\mathbf{L}} + \mathbf{L}\cdot\mathbf{\Omega} + \mathbf{\Omega}'\cdot\mathbf{L} = A\mathbf{d}, \quad \mathbf{\Omega} = \dot{\bar{\mathbf{R}}}\cdot\bar{\mathbf{R}}'. \tag{105}$$

To determine $\partial\mathscr{D}/\partial\bar{\mathbf{R}}'$ let us find the terms proportional to $\dot{\bar{\mathbf{R}}}$ in the expression for $\dot{\mathscr{D}}$:

$$\dot{\mathscr{D}} = \frac{\partial\mathscr{D}}{\partial\mathbf{d}_\chi}:\dot{\mathbf{d}}_\chi + \frac{\partial\mathscr{D}}{\partial\mathbf{L}_\chi}:\dot{\mathbf{L}}_\chi = \mathbf{T}_\chi:\left(\bar{\mathbf{R}}'\cdot\dot{\mathbf{d}}\cdot\bar{\mathbf{R}} + \left.\frac{\mathrm{d}(\bar{\mathbf{R}}'\cdot\mathbf{d}\cdot\bar{\mathbf{R}})}{\mathrm{d}t}\right|_{\mathbf{d}}\right) + \frac{\partial\mathscr{D}}{\partial\mathbf{L}_\chi}:A\,\mathbf{d}_\chi$$

$$= \mathbf{T}:\dot{\mathbf{d}} + 2\mathbf{T}_\chi\cdot\bar{\mathbf{R}}'\cdot\mathbf{d}:\dot{\bar{\mathbf{R}}} + \frac{\partial\mathscr{D}}{\partial\mathbf{L}}:A\mathbf{d}. \tag{106}$$

Equation (98) was used. As the term related to $\mathbf{L}$ does not produce the terms proportional to $\dot{\bar{\mathbf{R}}}$ in eqn (106), equations for determination $\mathbf{W}_p$ will be the same as for the case with $\mathscr{D} = \mathscr{D}(\mathbf{d}_\chi)$, i.e. eqns (99)–(101). The same equations are valid in the case of several internal variables with the evolution equations of a similar type to eqn (105).

If a material is initially isotropic and

$$\mathscr{D} = F(\mathbf{d}) + \mathbf{L}:\mathbf{d} = F(\mathbf{d}_\chi) + \mathbf{L}_\chi:\mathbf{d}_\chi, \tag{107}$$

where F is the isotropic function of $\mathbf{d}$, then

$$\mathbf{T} = \frac{\partial F}{\partial\mathbf{d}} + \mathbf{L}, \quad \mathbf{T}\cdot\mathbf{d} - \mathbf{d}\cdot\mathbf{T} = \left(\frac{\partial F}{\partial\mathbf{d}}\cdot\mathbf{d} - \mathbf{d}\cdot\frac{\partial F}{\partial\mathbf{d}}\right) + (\mathbf{L}\cdot\mathbf{d} - \mathbf{d}\cdot\mathbf{L}). \tag{108}$$

For isotropic F, tensors $\partial F/\partial\mathbf{d}$ and $\mathbf{d}$ are coaxial and can be permuted, so consequently the first bracket disappears. Thus for materials of type eqn (107) we arrive at the well-known and well-investigated (Dafalias, 1985; Loret, 1983; Paulun and Pechersky, 1987; Van der Giessen, 1991) equation:

$$\mathbf{W}_p = \eta(\mathbf{L}\cdot\mathbf{d} - \mathbf{d}\cdot\mathbf{L}), \quad \mathbf{\Omega} = \mathbf{W} - \eta(\mathbf{L}\cdot\mathbf{d} - \mathbf{d}\cdot\mathbf{L}). \tag{109}$$

The results of work by Dafalias (1985) and Paulun and Pecherski (1987) show that with proper choice of a scalar-valued function η eqns (105), (108) and (109) at $\partial F/\partial\mathbf{d} = k(\mathbf{d}/|\mathbf{d}|)$ allow us to describe some model situations and experimental effects.

9. THEORY WITH MULTIPLE SPINS

Let us assume that the dissipation function depends on several orthogonal tensors $\boldsymbol{\varphi}_i$ and $\boldsymbol{\beta}_j$ in the following form

$$\mathscr{D}(\mathbf{d}_\delta, \mathbf{E}_{0i}, \mathbf{L}_{\chi j}, \boldsymbol{\varphi}_i, \boldsymbol{\beta}_j) = \mathscr{D}(\mathbf{d}_\delta, \boldsymbol{\varphi}_i \times \mathbf{E}_{0i}, \boldsymbol{\beta}_i \cdot \mathbf{L}_{\chi j} \cdot \boldsymbol{\beta}_j^t)$$
$$= \mathscr{D}(\mathbf{d}, \tilde{\mathbf{R}}_i \times \mathbf{E}_{0i}, \tilde{\mathbf{R}}_j \cdot \mathbf{L}_{\chi j} \cdot \tilde{\mathbf{R}}_j^t) = \mathscr{D}(\mathbf{d}, \mathbf{E}_i, \mathbf{L}_j), \quad (110)$$

where

$$\bar{\mathbf{R}}_i = \mathbf{R} \cdot \boldsymbol{\varphi}_i, \quad \tilde{\mathbf{R}}_j = \mathbf{R} \cdot \boldsymbol{\beta}_j, \quad \mathbf{L}_j = \tilde{\mathbf{R}}_j \cdot \mathbf{L}_{\chi j} \cdot \tilde{\mathbf{R}}_j^t,$$
$$\mathbf{E}_i := \bar{\mathbf{R}}_i \times \mathbf{E}_{0i} := \mathbf{E}_{0i} \cdot \bar{\mathbf{R}}_i^t \overset{2}{*} \bar{\mathbf{R}}_i^t \overset{3}{*} \bar{R}_i^t \overset{4}{*} \bar{\mathbf{R}}_i^t, \quad \mathbf{E}_{0i} = \mathbf{const}_i \quad (111)$$

and operator $\mathbf{R} \times \mathbf{E}$ means that all the basis vectors of tensors $\mathbf{E}$ are multiplied by $\mathbf{R}$ from the left (see Appendix). The fixed tensors of arbitrary order $\mathbf{E}_{0i}$ characterize initial anisotropy, the internal variables $\mathbf{L}_j$ describe the strain induced anisotropy. Consequently each tensor $\mathbf{E}_{0i}$ or $\mathbf{L}_{\chi j}$ rotates with its own rotation tensor $\boldsymbol{\varphi}_i$ or $\boldsymbol{\beta}_j$. For tensors $\mathbf{L}_j$ the following evolution equations are given

$$\dot{\mathbf{L}}_{\chi j} = A_j \mathbf{d}_{\chi j}; \quad \mathbf{d}_{\chi j} := \tilde{\mathbf{R}}_j^t \cdot \mathbf{d} \cdot \tilde{\mathbf{R}}_j \quad (112)$$

or

$$\dot{\mathbf{L}}_j + \mathbf{L}_j \cdot \boldsymbol{\omega}_j + \boldsymbol{\omega}_j^t \cdot \mathbf{L}_j = A_j \mathbf{d}, \quad \boldsymbol{\omega}_j := \dot{\tilde{\mathbf{R}}}_j \cdot \tilde{\mathbf{R}}_j^t.$$

For each $\boldsymbol{\Omega}_i = \dot{\bar{\mathbf{R}}}_i \cdot \bar{\mathbf{R}}_i^t$ and $\boldsymbol{\omega}_j$ we can define a corresponding plastic spin by formulas

$$\mathbf{W} = \boldsymbol{\Omega}_i + \mathbf{W}_{pi}; \quad \mathbf{W} = \boldsymbol{\omega}_j + \tilde{\mathbf{W}}_{pj}. \quad (113)$$

To determine spins $\mathbf{W}_{pi}$ and $\tilde{\mathbf{W}}_{pj}$ we need the existence of scalar constraint equations $q_i(\mathbf{d}, \mathbf{W}_{pi}) = 0$ and $\tilde{q}_j(\mathbf{d}, \tilde{\mathbf{W}}_{pj}) = 0$ or their particular form

$$q_i = f_i(\mathbf{d}) - |\mathbf{W}_{pi}| = 0; \quad \tilde{q}_j = \tilde{f}_j(\mathbf{d}) - |\tilde{\mathbf{W}}_{pj}| = 0. \quad (114)$$

Using the postulate of realizability we can obtain, as in the case of one spin, the principle of minimum of dissipation rate at time $t + \Delta t$:

$$\mathscr{D}(\mathbf{d}_\Delta, \bar{\mathbf{R}}_{i\Delta} \times \mathbf{E}_{0i}, \tilde{\mathbf{R}}_{j\Delta} \cdot \mathbf{L}_{\chi\Delta} \cdot \tilde{\mathbf{R}}_{j\Delta}^t) < \mathscr{D}(\mathbf{d}_\Delta, \bar{\mathbf{R}}_{i\Delta}^0 \times \mathbf{E}_{0i}, \tilde{\mathbf{R}}_{j\Delta}^0 \cdot \mathbf{L}_{\chi\Delta} \cdot \tilde{\mathbf{R}}_{j\Delta}^{t0}); \quad (115)$$

$$\bar{\mathbf{R}}_{i\Delta}^0 := \bar{\mathbf{R}}_i + (\mathbf{W} - \mathbf{W}_{pi}^0) \cdot \bar{\mathbf{R}}_i \Delta t; \quad \tilde{\mathbf{R}}_{j\Delta}^0 := \tilde{\mathbf{R}}_j + (\mathbf{W} - \tilde{\mathbf{W}}_{pj}^0) \cdot \tilde{\mathbf{R}}_j \Delta t \quad (116)$$

under constraints eqn (114). For infinitesimal Δt for each i and j we obtain separate extremum principles:

$$-\frac{\partial \mathscr{D}}{\partial \mathbf{L}_j} : (\mathbf{L}_j \cdot (\mathbf{W} - \tilde{\mathbf{W}}_{pj}^0) + (\mathbf{W}^t - \tilde{\mathbf{W}}_{pj}^{t0}) \cdot \mathbf{L}_j) \to \min; \quad (117)$$

$$\frac{\partial \mathscr{D}}{\partial \mathbf{E}_i^t} \ldots \dot{\mathbf{E}}_i^0 \to \min, \quad (118)$$

where $\mathbf{E}_i^t = E_i^{\ldots nmlk} \mathbf{e}_k \mathbf{e}_l \mathbf{e}_m \mathbf{e}_n \ldots$ is the transposed tensor $\mathbf{E}_i = E_i^{klmn\ldots} \mathbf{e}_k \mathbf{e}_l \mathbf{e}_m \mathbf{e}_n \ldots$, $E_i^{klmn\ldots}$ are the components of the tensor $\mathbf{E}_i$ in the basis $\mathbf{e}_k \mathbf{e}_l \mathbf{e}_m \mathbf{e}_n \ldots$. From principle (117) in the same way as in eqns (98) and (99) we obtain

$$\left(\frac{\partial \mathscr{D}}{\partial \mathbf{L}_j}\cdot \mathbf{L}_j\right)_a : \tilde{\mathbf{W}}^0_{pj} \to \min; \quad \tilde{\mathbf{W}}_{pj} = -\tilde{f}_j(\mathbf{d})\frac{\left(\frac{\partial \mathscr{D}}{\partial \mathbf{L}_j}\cdot \mathbf{L}_j\right)_a}{\left|\left(\frac{\partial \mathscr{D}}{\partial \mathbf{L}_j}\cdot \mathbf{L}_j\right)_a\right|}. \tag{119}$$

Let us calculate $\dot{\mathbf{E}}_i$, e.g. for a fourth-order tensor, omitting for simplicity subscript i:

$$\begin{aligned}
\dot{\mathbf{E}} &= \mathbf{E}_0\cdot\dot{\bar{\mathbf{R}}}^t \overset{2}{*} \bar{\mathbf{R}}^t \overset{3}{*} \bar{\mathbf{R}}^t \overset{4}{*} \bar{\mathbf{R}}^t + \mathbf{E}_0\cdot\bar{\mathbf{R}}^t \overset{2}{*} \dot{\bar{\mathbf{R}}}^t \overset{3}{*} \bar{\mathbf{R}}^t \overset{4}{*} \bar{\mathbf{R}}^t \\
&\quad + \mathbf{E}_0\cdot\bar{\mathbf{R}}^t \overset{2}{*} \bar{\mathbf{R}}^t \overset{3}{*} \dot{\bar{\mathbf{R}}}^t \overset{4}{*} \bar{\mathbf{R}}^t + \mathbf{E}_0\cdot\bar{\mathbf{R}}^t \overset{2}{*} \bar{\mathbf{R}}^t \overset{3}{*} \bar{\mathbf{R}}^t \overset{4}{*} \dot{\bar{\mathbf{R}}}^t \\
&= (\mathbf{E}_0\cdot\bar{\mathbf{R}}^t \overset{2}{*} \bar{\mathbf{R}}^t \overset{3}{*} \bar{\mathbf{R}}^t \overset{4}{*} \bar{\mathbf{R}}^t)\cdot\boldsymbol{\Omega}^t + (\mathbf{E}_0\cdot\bar{\mathbf{R}}^t \overset{2}{*} \bar{\mathbf{R}}^t \overset{3}{*} \bar{\mathbf{R}}^t \overset{4}{*} \bar{\mathbf{R}}^t) \overset{2}{*} \boldsymbol{\Omega}^t \\
&\quad + (\mathbf{E}_0\cdot\bar{\mathbf{R}}^t \overset{2}{*} \bar{\mathbf{R}}^t \overset{3}{*} \bar{\mathbf{R}}^t \overset{4}{*} \bar{\mathbf{R}}^t) \overset{3}{*} \boldsymbol{\Omega}^t + (\mathbf{E}_0\cdot\bar{\mathbf{R}}^t \overset{2}{*} \bar{\mathbf{R}}^t \overset{3}{*} \bar{\mathbf{R}}^t \overset{4}{*} \bar{\mathbf{R}}^t) \overset{4}{*} \boldsymbol{\Omega}^t \\
&= (\mathbf{E}\cdot\mathbf{C}_a + \mathbf{E} \overset{2}{*} \mathbf{C}_a + \mathbf{E} \overset{3}{*} \mathbf{C}_a + \mathbf{E} \overset{4}{*} \mathbf{C}_a) : \boldsymbol{\Omega}^t = \mathbf{A} : \boldsymbol{\Omega}^t;
\end{aligned} \tag{120}$$

$$\mathbf{A} := \mathbf{E}\cdot\mathbf{C}_a + \mathbf{E} \overset{2}{*} \mathbf{C}_a + \mathbf{E} \overset{3}{*} \mathbf{C}_a + \mathbf{E} \overset{4}{*} \mathbf{C}_a = \sum_{k=1}^{4} \mathbf{E} \overset{k}{*} \mathbf{C}_a, \tag{121}$$

where $\mathbf{C}_a$ is the antisymmetrizing tensor of fourth order ($\mathbf{C}_a : \mathbf{B} := 0.5\,(\mathbf{B} - \mathbf{B}^t)$ for an arbitrary second rank tensor $\mathbf{B}$). Substituting eqn (121) in the principle (118) we get for each i:

$$\frac{\partial \mathscr{D}}{\partial \mathbf{E}_i^t}\cdot\cdot\cdot\cdot \mathbf{A}_i : \mathbf{W}^0_{pi} \to \max; \quad \mathbf{W}_{pi} = f_i(\mathbf{d})\frac{\frac{\partial \mathscr{D}}{\partial \mathbf{E}_i^t}\cdot\cdot\cdot\cdot \mathbf{A}_i}{\left|\frac{\partial \mathscr{D}}{\partial \mathbf{E}_i^t}\cdot\cdot\cdot\cdot \mathbf{A}_i\right|}. \tag{122}$$

As an example we consider an initially anisotropic polycrystal with kinetmatic hardening, i.e.

$$\mathscr{D} = (\mathbf{d} : \mathbf{E} : \mathbf{d})^{1/2} + \mathbf{L} : \mathbf{d} = (\mathbf{d}_\chi : \mathbf{E}_0 : \mathbf{d}_\chi)^{1/2} + \mathbf{L} : \mathbf{d}; \tag{123}$$

$$\dot{\mathbf{L}} + \mathbf{L}\cdot\boldsymbol{\omega} + \boldsymbol{\omega}^t\cdot\mathbf{L} = A\mathbf{d}; \quad \mathbf{d}_\chi = \bar{\mathbf{R}}^t\cdot\mathbf{d}\cdot\bar{\mathbf{R}}, \tag{124}$$

i.e. tensor $\mathbf{E}_0$ characterizing initial anisotropy and the back stress tensor $\mathbf{L}$ rotate with the spins $\dot{\bar{\mathbf{R}}}\cdot\bar{\mathbf{R}}^t$ and $\boldsymbol{\omega}$, respectively. For plastic spin $\tilde{\mathbf{W}}_p$ related to $\mathbf{L}$ we can use eqn $(119)_2$ directly

$$\tilde{\mathbf{W}}_p = \tilde{f}(\mathbf{d})\frac{\mathbf{L}\cdot\mathbf{d} - \mathbf{d}\cdot\mathbf{L}}{|\mathbf{L}\cdot\mathbf{d} - \mathbf{d}\cdot\mathbf{L}|}. \tag{125}$$

For the spin $\mathbf{W}_p$ connected to $\mathbf{E}_0$ we can also directly apply eqn (122) at $\partial\mathscr{D}/\partial\mathbf{E}^t = \mathbf{dd}$, but a more vivid relation can be obtained. According to eqn (45)

$$\mathbf{T} = \frac{\mathbf{E} : \mathbf{d}}{(\mathbf{d} : \mathbf{E} : \mathbf{d})^{1/2}} + \mathbf{L}; \quad \mathbf{T}_\chi = \frac{\mathbf{E}_0 : \mathbf{d}_\chi}{(\mathbf{d}_\chi : \mathbf{E}_0 : \mathbf{d}_\chi)^{1/2}} + \mathbf{L}_\chi; \tag{126}$$

$$\mathbf{T}_\chi - \mathbf{L}_\chi = \frac{\partial \mathscr{D}_1}{\partial \mathbf{d}_\chi}; \quad \mathscr{D}_1 := (\mathbf{d}_\chi : \mathbf{E}_0 : \mathbf{d}_\chi)^{1/2}. \tag{127}$$

Then

$$\frac{\partial \mathscr{D}}{\partial \bar{\mathbf{R}}^t} : \dot{\bar{\mathbf{R}}} = \frac{\partial \mathscr{D}_1}{\partial \bar{\mathbf{R}}^t} : \dot{\bar{\mathbf{R}}} = \frac{\partial \mathscr{D}_1}{\partial \mathbf{d}_\chi} : \left. \frac{\mathrm{d}(\bar{\mathbf{R}}^t \cdot \mathbf{d} \cdot \bar{\mathbf{R}})}{\mathrm{d}t} \right|_{\mathbf{d}}$$

$$= (\mathbf{T}_\chi - \mathbf{L}_\chi) : (\bar{\mathbf{R}}^t \cdot \mathbf{d} \cdot \dot{\bar{\mathbf{R}}} + \dot{\bar{\mathbf{R}}}^t \cdot \mathbf{d} \cdot \bar{\mathbf{R}}) = 2(\mathbf{T}_\chi - \mathbf{L}_\chi) \cdot \bar{\mathbf{R}}^t \cdot \mathbf{d} : \dot{\bar{\mathbf{R}}}; \tag{128}$$

$$\bar{\mathbf{R}} \cdot \frac{\partial \mathscr{D}}{\partial \bar{\mathbf{R}}^t} = 2\bar{\mathbf{R}} \cdot (\mathbf{T}_\chi - \mathbf{L}_\chi) \cdot \bar{\mathbf{R}}^t \cdot \mathbf{d} = 2(\mathbf{T} - \mathbf{L}) \cdot \mathbf{d}; \tag{129}$$

$$\mathbf{W}_p = f(\mathbf{d}) \frac{(\mathbf{T} - \mathbf{L}) \cdot \mathbf{d} - \mathbf{d} \cdot (\mathbf{T} - \mathbf{L})}{|(\mathbf{T} - \mathbf{L}) \cdot \mathbf{d} - \mathbf{d} \cdot (\mathbf{T} - \mathbf{L})|}. \tag{130}$$

We have taken into account the fact that the term $\mathbf{L} : \mathbf{d}$ is independent of $\bar{\mathbf{R}}$ and used a transformation similar to that given in eqns (98) and (99).

Remark. As all the tensors in eqns (101), (105), (109), (124), (125) and (130) are independent of the choice of reference configuration, the contradiction mentioned in Section 2 and related to the appearance of several equivalent fixed preferred configurations (e.g. with $\mathbf{L} = \mathbf{0}$) does not arise. The deformation gradient in eqn (103) depends on the choice of reference configuration. The question of whether it is possible in this case to avoid the contradiction revealed in Section 2 will be treated elsewhere.

10. COMPARISON WITH THE KNOWN APPROACHES TO THE PLASTIC SPIN PROBLEM

A paper by Mandel (1973) contains two important points:

- The triad of directors is considered as some averaged orientation of a single crystal lattice, i.e. the preferred configuration is related to some microstructure.
- As the rotation of a triad of directors is determined by the prehistory of the deformation gradient and temperature, with a functional representation of the constitutive equations the role of the directors disappears.

In our approach we in fact use Mandel's isoclinic configuration, but there are some fundamental differences. Our argument for introducing some variable privileged configuration *is not related to a single crystal and microstructure*. We found the purely formal continuum mechanical contradiction that in the general theory with application of PMFI we *cannot exclude rotation with respect to some fixed privileged configuration*. This contradiction and consequently the necessity for the introduction of some variable privileged configuration is independent of the way we describe the material: with the help of an internal variable or constitutive functional.

The next question is the derivation of the constitutive equation for plastic spin. As plastic spin does not contribute to the rate of dissipation, it is impossible to derive for

it some extremum principle or constitutive equation using thermodynamics (as for the plastic deformation rate). The only known macroscopic way is to use the representation theorem (Dafalias, 1984; Loret, 1983). For example, if $\mathbf{W}_p = \mathbf{W}_p(\mathbf{T}, \mathbf{L})$, where $\mathbf{L}$ is the symmetrical second order tensor (an internal variable), then

$$\mathbf{W}_p = \eta_1(\mathbf{L}\cdot\mathbf{T})_a + \eta_2(\mathbf{L}^2\cdot\mathbf{T}^2)_a + \eta_3(\mathbf{L}\cdot\mathbf{T})_a + \eta_4(\mathbf{L}\cdot\mathbf{T}\cdot\mathbf{L}^2)_a + \eta_5(\mathbf{T}\cdot\mathbf{L}\cdot\mathbf{T}^2)_a, \tag{131}$$

where η_i are the functions of various invariants. To find experimentally five functions is unreal; that is why only the first term in eqn (131) is used. For three and more arguments the representation theorem yields very bulky formulas which cannot be concretized experimentally. Moreover, explicit enumeration of the arguments of a $\mathbf{W}_p$ function is a very strong assumption, because many skew-symmetric tensors like

$$\mathbf{R}\cdot(\dot{\mathbf{U}}\cdot\mathbf{U}^{-1})_a\cdot\mathbf{R}^t; \quad \mathbf{R}\cdot\left(\frac{\partial\varphi}{\partial\mathbf{U}}\cdot\mathbf{U}\right)_a\cdot\mathbf{R}^t$$

can contribute to the plastic spin [see also eqns (103) and (122)].

We do not assume explicitly the arguments of function $\mathbf{W}_p$; the final result is obtained in terms of the dissipation function for arbitrary initial and strain induced anisotropy described by multiple tensorial variables of arbitrary order. For many particular cases the expressions for plastic spin look very simple. For the case with an internal tensorial variable of second order, eqn (109) derived above is equivalent to the first term of eqn (131) obtained with the help of the representation theorem.

Zbib and Aifantis (1988), Dafalias (1992) and Cho and Dafalias (1996) have considered the situation with multiple spin tensors, with a separate one for each variable. In this case our approach allows us to derive explicit equations for each spin tensor as well.

Let us consider some additional aspects of the problem of plastic spin.

Lee and Mallet (1983) propose using in the kinematic hardening rule the objective derivative of tensor $\mathbf{L}$, which is linked with the velocity of rotation of a material fibre which coincides with the principal direction of the tensor $\mathbf{L}$ with the highest principal value. Under the circumstances, oscillations are not present in the simple shearing. This approach cannot be adopted as a general method for the correct allowance of finite rotations, because the application procedure is not clear, when the anisotropy is described using several tensors, fourth order tensors or expressions without an explicit enumeration of tensors.

In some papers directors are determined based on some geometrical or microstructural assumptions. For example, directors coincide with the eigenvectors of plastic stretch tensor for polymers (Boyce *et al.*, 1988; Aravas, 1994) or one of the directors is aligned with the fiber in a unidirectional composite (Aravas, 1994). In this case plastic spin is completely determined, the solution is unique and there is no necessity to apply the stability analysis.

11. CONCLUDING REMARKS

1. The fundamental contradiction in the theory of constitutive equations is revealed for polycrystalline solids: in the general theory *it is impossible to exclude*

rotation relative to some fixed privileged configuration (which for example is isotropic for an initially isotropic material) when the PMFI is applied. The reason is related to the possibility of creating by some thermomechanical process a number of equivalent privileged configurations and it is impossible to get the *same constitutive equation with respect to each of these privileged configurations.* In the classical theory of simple materials there is no way to introduce a fading memory of the "old" privileged configuration in terms of rotations when a new one is created during the deformation process, because there is no constitutive equation for rotation.

2. Two equivalent methods of overcoming the above contradiction are developed. In the first one it is assumed that after excluding the RBR relative to fixed privileged configuration the constitutive equations depend additionally on some rotational internal variable(s), for which some equation(s) will be derived. In the second one we formulate and solve the problem of finding some *variable privileged configuration*, similar to Mandel's isoclinic configuration. Consequently, we arrive at the problem of determination of equations for plastic spin.

3. The key point in our approach is the assumption *that there is no constitutive equation for plastic spin* for a polycrystal. To substantiate the assumption a single crystal is considered. For a single crystal the equation for plastic spin follows from a kinematic constraint that the rate of plastic deformation gradient $\dot{\mathbf{F}}_p$ describes the combination of simple shears and a constitutive equation is not required. In the case of multiplicity of slip systems one or more additional equations is needed, but again this is not a constitutive equation, because all the constitutive equations are already specified. One of the known ways to choose unique slip systems and consequently lattice and plastic spins is to replace the rate-independent plastic model by a viscoplastic one, which is again not related to a constitutive equation for plastic spin.

4. As there is no equation for plastic spin, each point of the deformation process is a bifurcation point. Then it is natural to determine the plastic spin as a *stable homogeneous solution* of the BVP. Based on the postulate of realizability the extremum principle for determination of the stable solution is derived. The obtained *principle of minimum of dissipation rate at time* $t+\Delta t$ is independent of the type of boundary condition (prescribed displacements, stresses or mixed boundary conditions). The same extremum principle for each material point can be obtained when the BVP at arbitrary nonuniform boundary conditions is considered. The unique equations for one or multiple plastic spins for polycrystals and one simple single crystal model are obtained, based on the above extremum principle. For polycrystals one additional scalar constraint equation is necessary which guarantees that the plastic spin is zero when the plastic deformation rate is zero and limits a modulus of the spin tensor. It has been shown that the extremum principle and equation for plastic spin are derived using the same assumption as for the derivation of the associated flow rule, namely *the postulate of realizability.*

5. A number of concrete expressions for plastic spin are derived for polycrystals with initial and strain induced anisotropy, represented by internal variables and material tensors of arbitrary order, with multiple spins, as well as for a simple model of a single crystal. The application of the approach developed for the general model of a single crystal will be considered elsewhere.

We do not consider, but do not exclude the case when the constitutive equation

should be given for some rotational internal variable. In this case stability analysis cannot be applied and micromechanical treatment is necessary. However, perhaps even in such a situation the postulate of realizability and the principle of minimum of rate of dissipation at time $t+\Delta t$ directly or after some reformulation and development will be helpful as a reasonable macroscopic basis. We can repeat the same derivation without reference to stability analysis. The postulate of realizability will concretize the constitutive equation for a rotational internal variable similar to concretization of the flow rule.

It was mentioned at the end of Section 5 that the extremum principles (54) and (60) are valid for an arbitrary material point and deformation process, i.e. as a constitutive equation. Someone might suggest that to avoid confusion and not to violate the tradition the equation for plastic spin should be called constitutive. At the same time the interpretation of this equation as a stable solution of BVP not only corresponds better to the derivation process, but opens up new opportunities for more general situations, e.g. for elastoplastic, viscoplastic or viscoelastic materials. If the stable solution for spin has different forms for different types of boundary condition, then it cannot be a local constitutive equation and the actual spin distribution for non-homogeneous boundary conditions should be found based on the stability analysis for the whole body and each concrete problem. If an analytical expression for spin cannot be found in all situations, then there is no explicit expression for spin, but a numerical procedure only. It may also be that the method of stability analysis suggested here does not give the unique solution (e.g. for single crystals) or cannot be applied at all (e.g. for viscoelastic materials). Then we should use other methods, for example, the method of perturbations, or introduce some imperfection and so on. In the present paper we can directly postulate the principle of the minimum of dissipation rate at time $t+\Delta t$ without application of the postulate of realizability. But the postulate of realizability is a more general assumption which can be applied, for example, to phase transitions and stability problems and does not always result in the principle of the minimum of dissipation rate. In particular, in the framework of stability analysis for elastoplastic materials equations for plastic spins can be derived with the help of the postulate of realizability, but cannot be obtained in the general case using the principle of minimum of dissipation rate.

It is necessary to extend the approach developed for elastoplastic and elastoviscoplastic materials. In this case free energy will be dependent on its own rotation tensors which characterize elastic anisotropy and real time comes into play. We hope that stability analysis will allow us to derive equations for corresponding spin tensors.

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APPENDIX: SOME TENSORIAL DESIGNATIONS AND FORMULAE (LEVITAS, 1996)

All the tensor calculation rules used in the paper may be found in the book by Truesdell and Noll (1965). In addition to them let us introduce the following convenient designation to use the direct form for the contraction of tensors whose ranks are above two. Let $\mathbf{A} = A^{ij}\mathbf{e}_i\mathbf{e}_j$, $\mathbf{B} = B^{ij}\mathbf{e}_i\mathbf{e}_j$, $\mathbf{C} = C^{ij}\mathbf{e}_i\mathbf{e}_j$ and $\mathbf{E} = E^{ijkl}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l$, where A^{ij}, B^{ij}, C^{ij}, and E^{ijkl} are the components of tensors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{E}$; $\mathbf{e}_i$ is the basis vector of the Cartesian coordinate system. Then, let

$$\mathbf{A} \overset{4}{\circ} \mathbf{E} \overset{2}{*} \mathbf{C} \overset{3}{*} \mathbf{B} = (A^{ij}\mathbf{e}_i\mathbf{e}_j) \overset{4}{\circ} (E^{klmn}\mathbf{e}_k\mathbf{e}_l\mathbf{e}_m\mathbf{e}_n) \overset{2}{*} (C^{rt}\mathbf{e}_r\mathbf{e}_t) \overset{3}{*} (B^{cd}\mathbf{e}_c\mathbf{e}_d)$$
$$= A^{ij}E^{klmn}C^{rt}B^{cd}\mathbf{e}_i\mathbf{e}_k(\mathbf{e}_l\cdot\mathbf{e}_c)(\mathbf{e}_m\cdot\mathbf{e}_r)(\mathbf{e}_j\cdot\mathbf{e}_n)\mathbf{e}_t\mathbf{e}_d = A^{ij}E^{kcrj}C^{rt}B^{cd}\mathbf{e}_i\mathbf{e}_k\mathbf{e}_t\mathbf{e}_d, \quad (132)$$

i.e. symbol $\overset{m}{\circ}$ designates the contraction of the nearest basis vector of a second-rank tensor and the m-th basis vector of a tensor above second-rank $\mathbf{E}$ from the left; symbol $\overset{n}{*}$ designates contraction of the nearest basis vector of a second-rank tensor and the n-th basis vector of $\mathbf{E}$ tensor from the right.

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Corrigendum to

"A NEW LOOK AT THE PROBLEM OF PLASTIC SPIN BASED ON STABILITY ANALYSIS"

by VALERY I. LEVITAS

published in J. Mech. Phys. of Solids, 1998, 46, No. 3, 557-590.

It was written before Eq.(60) of the above paper, which is

$$\boldsymbol{W}_p = \eta \boldsymbol{a} \qquad \text{or} \qquad \boldsymbol{W}_p = \frac{\boldsymbol{a}}{|\boldsymbol{a}|} f(\boldsymbol{d}) \qquad \text{at} \quad \boldsymbol{a} \neq \boldsymbol{0}, \tag{1}$$

that the sign of the scalar η has to be determined from the extremum principle (58) of the paper, i.e. from condition

$$\boldsymbol{a}:\boldsymbol{W}_p \to max \qquad \text{at} \qquad q = f(\boldsymbol{d}) - |\boldsymbol{W}_p| = 0\,. \tag{2}$$

Here $\boldsymbol{W}_p$ is the skew-symmetric plastic spin tensor, $\boldsymbol{a}$ is the skew-symmetric tensor defined in the paper, $\boldsymbol{d}$ is the deformation rate, and f is some scalar function.

However, as according to constrain $q = f(\boldsymbol{d}) - |\boldsymbol{W}_p| = 0$ one has $f(\boldsymbol{d}) \geq 0$, then the sign in Eq.$(1)_2$ is chosen. This sign corresponds to $\eta \geq 0$, which was explicitly written in Eq.(100) of the paper. It appears, that this is the wrong sign. Indeed,

$$|\boldsymbol{W}_p|^2 = \boldsymbol{W}_p:\boldsymbol{W}_p^t = \eta\,\boldsymbol{a}:\boldsymbol{W}_p^t = -\eta\,\boldsymbol{a}:\boldsymbol{W}_p \geq 0\,, \tag{3}$$

where superscript t denotes transposition. It is evident that at $\eta \geq 0$ one has $\boldsymbol{a}:\boldsymbol{W}_p \leq 0$ which corresponds to $\boldsymbol{a}:\boldsymbol{W}_p \to min$ and is wrong. Condition $\eta \leq 0$ results in $\boldsymbol{a}:\boldsymbol{W}_p \geq 0$, which agrees with the extremum principle (2).

The easiest way to correct this error is as follows.

1. To consider $\eta \leq 0$ in all equations of the paper.
2. To assume

$$q = f(\boldsymbol{d}) + |\boldsymbol{W}_p| = 0 \tag{4}$$

instead of $q = f(\boldsymbol{d}) - |\boldsymbol{W}_p| = 0$.

3. To use

$$q_i = f_i(\boldsymbol{d}) + |\boldsymbol{W}_{pi}| = 0 \qquad \text{and} \qquad \tilde{q}_j = \tilde{f}_j(\boldsymbol{d}) + |\tilde{\boldsymbol{W}}_{pj}| = 0 \tag{5}$$

instead of Eq.(114) of the paper, i.e. the functions f , f_i , and $\tilde{f}_j$ are negative.

It was mentioned in the paper by Y. F. Dafalias (J. Mech. Phys. of Solids, 2000, 48, 2231-2255) that general approach developed in our paper under consideration cannot accommodate the need for negative η in order to simulate the experimental data by Kim and Yin (1997). After the above corrections, this contradiction disappears.