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PLASTICITY THEORY OF MICROINHOMOGENEOUS MATERIALS AT LARGE STRAIN GRADIENT

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Introduction

The complicated behaviour of multiphase and microheterogeneous materials occurs in many processes. Thus large strain and stress gradients appear near contact surfaces (due to friction), in localization regions (slip bands), and in compression of thin layers (Prandtl problem). Non-uniformly distributed inertia and body forces are observed in dynamic and magnetic material processing. Couple stress theories give only a phenomenological approach to the above pecularities, but micromechanics neglects these phenomena. In fact, body and inertia forces are not included in micromechanical description and boundary conditions are macroscopically uniform.There are several definitions of the concept of macroscopically uniform boundary data (Hill [1,2], Havner [3], for a detailed analysis see Havner [4]), which include some types of non-uniformity of stress or strain fields, e.g. "wavelike" fluctuations [1], but we need to take into account the macroscopic non-uniformity. The simplest case of non-uniformity is the linearly distributed fields. The objective of this paper is the development of the general micromechanical approach to describe the heterogeneous media under macroscopically linear varied boundary data and for arbitrary body and inertia forces. Thus Hill's formulas for the representative volume of the microheterogeneous materials under macroscopically uniform boundary conditions are generalized for boundary data corresponding to linear distributed macroscopic stress or strain tensors and the presence of non-uniformly distributed inertia and body forces. Relations for macroscopic stress, hyperstress, distortion and distortion gradient tensors are obtained. Energy identities are proved. Expressions for macroscopic inelastic strain and strain gradients, internal energy and elastic moduli are derived. The macroscopic normality principle for time-dependent and time-independent materials in stress-hyperstress space is substantiated. It is shown that the effective elastic

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constants are independent of the inelastic strain distribution.
Let $\mathbf{I}$ and $\mathbf{C}^6$ be the second-order and sixth-order unit tensors respectively, $\mathbf{C}$ and $\mathbf{C}_t$ the forth-order tensors ($\mathbf{C} : \mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^t), \mathbf{C}_t : \mathbf{A} = \mathbf{A}^t$). The superscript t denotes transposition, dots mean contractions of the tensors and ∇ the gradient operator.

Macroscopic Variables

Consider the representative volume v of the microheterogeneous material bounded by a surface S with the unit normal vector $\mathbf{n}$. Let $\mathbf{r}$ be the particle position, $\mathbf{u}(\mathbf{r})$ the continuous displacement field, $\tilde{\boldsymbol{\sigma}}$ the stress tensor and $\tilde{\boldsymbol{\beta}} = \nabla\mathbf{u}$ the distortion tensor, where $\tilde{}$ means the local value of parameters as opposed to the macroscopic counterpart. Introduce the following macroscopic variables

$$\boldsymbol{\sigma}^t = v^{-1}\int \mathbf{r}\tilde{\boldsymbol{\sigma}} \cdot \mathbf{n}dS; \quad \mathbf{M} = \frac{v^{-1}}{2}\int \mathbf{r}\mathbf{r}\tilde{\boldsymbol{\sigma}} \cdot \mathbf{n}dS; \tag{1}$$

$$\boldsymbol{\beta} = v^{-1}\int \mathbf{u}\mathbf{n}dS; \quad \mathbf{B} = v^{-1}\int \tilde{\boldsymbol{\beta}}\mathbf{n}dS, \tag{2}$$

where the third-order tensors of the hyperstress $\mathbf{M}$ and the macroscopic distortion gradient $\mathbf{B}$ characterize the heterogenity of the $\tilde{\boldsymbol{\sigma}}$ and $\tilde{\boldsymbol{\beta}}$ distributions. Using the Gauss theorem and the equations of motion

$$\nabla \cdot \tilde{\boldsymbol{\sigma}} = \tilde{\varrho}(\ddot{\mathbf{u}} - \tilde{\mathbf{f}}),$$

where $\tilde{\varrho}$ is the mass density and $\tilde{\mathbf{f}}$ the body forces, we obtain from Eqs. (1) and (2)

$$\boldsymbol{\sigma}^t = \langle\tilde{\boldsymbol{\sigma}}^t\rangle + \langle\tilde{\varrho}\mathbf{r}(\ddot{\mathbf{u}} - \tilde{\mathbf{f}})\rangle; \quad \mathbf{M} = \mathbf{C} : \langle\mathbf{r}\tilde{\boldsymbol{\sigma}}^t\rangle + \frac{1}{2}\langle\tilde{\varrho}\mathbf{r}\mathbf{r}(\ddot{\mathbf{u}} - \tilde{\mathbf{f}})\rangle; \tag{3}$$

$$\boldsymbol{\beta} = \langle\tilde{\boldsymbol{\beta}}\rangle; \quad \mathbf{B} = \langle\nabla\nabla\mathbf{u}\rangle = \langle\nabla\tilde{\boldsymbol{\beta}}\rangle, \tag{4}$$

where $\langle ... \rangle = v^{-1}\int(...)dv$, $\langle\mathbf{r}\rangle = 0$. Evidently that tensors $\mathbf{M}$ and $\mathbf{B}$ are symmetric over the indices 1,2 and 2,3 respectively and uniform inertia and body forces do not contribute to $\boldsymbol{\sigma}$, but contribute to $\mathbf{M}$. In this formulation $\boldsymbol{\sigma}$ is nonsymmetric. The special case of $\tilde{\boldsymbol{\sigma}}$ symmetry is treated later. Direct calculations prove the following identities

$$v^{-1}\int(\mathbf{u} - \mathbf{u}_0 - \boldsymbol{\beta}\cdot\mathbf{r} - \frac{1}{2}\mathbf{B} : \mathbf{r}\mathbf{r})(\tilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma} - \mathbf{r}\cdot\mathbf{P})\cdot\mathbf{n}dS = \tag{5}$$
$$= \langle\tilde{\boldsymbol{\beta}}\cdot\tilde{\boldsymbol{\sigma}}^t\rangle - \boldsymbol{\beta}\cdot\boldsymbol{\sigma}^t - \mathbf{B} : \mathbf{M} - \langle\tilde{\boldsymbol{\beta}}\mathbf{r}\rangle : \bar{\mathbf{P}} + \mathbf{B} : \boldsymbol{\Gamma} : \bar{\mathbf{P}} + \langle\tilde{\varrho}\mathbf{u}(\ddot{\mathbf{u}} - \tilde{\mathbf{f}})\rangle - \langle\mathbf{u}\rangle\langle\tilde{\varrho}(\ddot{\mathbf{u}} - \tilde{\mathbf{f}})\rangle;$$

$$v^{-1}\int(\mathbf{u} - \mathbf{u}_0 - \boldsymbol{\beta}\cdot\mathbf{r} - \frac{1}{2}\mathbf{B} : \mathbf{r}\mathbf{r})\cdot(\tilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma} - \mathbf{r}\cdot\mathbf{P})\cdot\mathbf{n}dS = \tag{6}$$
$$= \langle\tilde{\boldsymbol{\beta}} : \tilde{\boldsymbol{\sigma}}^t\rangle - \boldsymbol{\beta} : \boldsymbol{\sigma}^t - \mathbf{B} \vdots \mathbf{M} - \langle\tilde{\boldsymbol{\beta}}\mathbf{r}\rangle \vdots \bar{\mathbf{P}} + \mathbf{B} : \boldsymbol{\Gamma} \vdots \bar{\mathbf{P}} + \langle\tilde{\varrho}\mathbf{u}\cdot(\ddot{\mathbf{u}} - \tilde{\mathbf{f}})\rangle - \langle\mathbf{u}\rangle\cdot\langle\tilde{\varrho}(\ddot{\mathbf{u}} - \tilde{\mathbf{f}})\rangle,$$

where the constant third-order tensor $\mathbf{P}$ characterizes the heterogenity of the $\tilde{\boldsymbol{\sigma}}$ distribution on S; $\bar{\mathbf{P}} = \mathbf{P} : \mathbf{C}_t$; $\boldsymbol{\Gamma} = (2v)^{-1} \int \mathbf{rrnr} dS$; $\mathbf{u}_0$ is some constant displacement vector. Eqs. (5) and (6) remain true when the time derivative is taken for all of the characteristics of the strained state $(\mathbf{u}, \mathbf{u}_0, \tilde{\boldsymbol{\beta}}, \boldsymbol{\beta}, \mathbf{B})$ or (and) stressed state $(\tilde{\boldsymbol{\sigma}}, \boldsymbol{\sigma}, \mathbf{P}, \mathbf{M})$. When we have on S $\mathbf{u} = \mathbf{u}_0 + \boldsymbol{\beta} \cdot \mathbf{r} + \frac{1}{2} \mathbf{B} : \mathbf{rr}$ then

$$\langle \tilde{\boldsymbol{\beta}} \mathbf{r} \rangle = \mathbf{B} : \boldsymbol{\Gamma} + (\mathbf{u}_0 - \langle \mathbf{u} \rangle) \mathbf{I}; \tag{7}$$

$$\langle \tilde{\boldsymbol{\beta}} \cdot \tilde{\boldsymbol{\sigma}}^t \rangle = \boldsymbol{\beta} \cdot \boldsymbol{\sigma}^t + \mathbf{B} : \mathbf{M} + \langle \tilde{\varrho} (\mathbf{u}_0 - \langle \mathbf{u} \rangle)(\ddot{\mathbf{u}} - \tilde{\mathbf{f}}) \rangle; \tag{8}$$

$$\langle \tilde{\boldsymbol{\beta}} : \tilde{\boldsymbol{\sigma}}^t \rangle = \boldsymbol{\beta} : \boldsymbol{\sigma}^t + \mathbf{B} \,\vdots\, \mathbf{M} + \langle \tilde{\varrho} (\mathbf{u}_0 - \langle \mathbf{u} \rangle) \cdot (\ddot{\mathbf{u}} - \tilde{\mathbf{f}}) \rangle. \tag{9}$$

Eq.(9) is the expression for the averaged stress work. When we have on S $\tilde{\boldsymbol{\sigma}} \cdot \mathbf{n} = = \boldsymbol{\sigma} \cdot \mathbf{n} + \mathbf{r} \cdot \mathbf{P} \cdot \mathbf{n}$ then $\mathbf{M} = \boldsymbol{\Gamma} : \bar{\mathbf{P}}$, $\mathbf{I} : \bar{\mathbf{P}} = \langle \tilde{\varrho} (\ddot{\mathbf{u}} - \tilde{\mathbf{f}}) \rangle = \langle \nabla \cdot \tilde{\boldsymbol{\sigma}} \rangle$ (from the equations of motion) and

$$\langle \tilde{\boldsymbol{\beta}} \cdot \tilde{\boldsymbol{\sigma}}^t \rangle = \boldsymbol{\beta} \cdot \boldsymbol{\sigma}^t + \langle \tilde{\boldsymbol{\beta}} \mathbf{r} \rangle : \bar{\mathbf{P}} - \langle \tilde{\varrho} (\mathbf{u} - \langle \mathbf{u} \rangle)(\ddot{\mathbf{u}} - \tilde{\mathbf{f}}) \rangle, \tag{10}$$

$$\langle \tilde{\boldsymbol{\beta}} : \tilde{\boldsymbol{\sigma}}^t \rangle = \boldsymbol{\beta} : \boldsymbol{\sigma}^t + \langle \tilde{\boldsymbol{\beta}} \mathbf{r} \rangle \,\vdots\, \bar{\mathbf{P}} - \langle \tilde{\varrho} (\mathbf{u} - \langle \mathbf{u} \rangle) \cdot (\ddot{\mathbf{u}} - \tilde{\mathbf{f}}) \rangle. \tag{11}$$

Note that condition $\langle \tilde{\boldsymbol{\beta}} \cdot \tilde{\boldsymbol{\sigma}}^t \rangle = \boldsymbol{\beta} \cdot \boldsymbol{\sigma}^t$ may be taken as the definition of macroscopically homogeneous fields (Havner [3]). Then extra terms in Eqs.(8) and (10) characterise the effect of inertia and body forces and non-uniformity of stress and strain fields.

Consider some particular cases. If we choose $\mathbf{u}_0 = \langle \mathbf{u} \rangle$ then the last term in Eqs.(7)-(9) disappears and inertia and body forces do not influence the energy balance explicitly, but as the components of $\boldsymbol{\sigma}$ and $\mathbf{M}$ only (see Eq.(3)).

For $\tilde{\mathbf{f}} = \ddot{\mathbf{u}} = 0$ we have

$$\boldsymbol{\sigma} = \langle \tilde{\boldsymbol{\sigma}} \rangle; \quad \mathbf{M} = \mathbf{C} : \langle \mathbf{r} \tilde{\boldsymbol{\sigma}}^t \rangle; \quad \mathbf{I} : \bar{\mathbf{P}} = 0. \tag{12}$$

If $\tilde{\boldsymbol{\sigma}} = \tilde{\boldsymbol{\sigma}}^t$ than $\boldsymbol{\sigma} = \boldsymbol{\sigma}^t$ and nonsymmetry of the tensor $\boldsymbol{\sigma}$ can appear only for nonuniformly distributed inertia and body forces. From Eq.(12) the symmetry of the $\mathbf{M}$ tensor on indices 2 and 3 follows. Consequently, in Eq.(9) tensors $\tilde{\boldsymbol{\beta}}$ and $\boldsymbol{\beta}$ may be substituted by the strain tensors $\tilde{\boldsymbol{\epsilon}} = \mathbf{C} : \tilde{\boldsymbol{\beta}}$ and $\boldsymbol{\epsilon} = \langle \tilde{\boldsymbol{\epsilon}} \rangle$ and $\mathbf{B}$–by the strain gradient tensor $\langle \nabla \tilde{\boldsymbol{\epsilon}} \rangle$, because the skew-symmetric part of these tensors does not contribute to work.

The left side of Eqs.(5) and (6) will be zero when the above displacements are prescribed on the one part of S and the above stresses on the rest part of S. But resulting equations will not have the same simple form as Eqs.(8)-(11) and derivation of the constitutive equations will not be so simple.

Inelastic behaviour

Consider inelastic material under a prescribed displacement on S and $\tilde{\mathbf{f}} = \ddot{\mathbf{u}} = 0$. Let

$$\tilde{\boldsymbol{\epsilon}} = \tilde{\boldsymbol{\epsilon}}^e + \tilde{\boldsymbol{\epsilon}}^i; \quad \boldsymbol{\epsilon} = \boldsymbol{\epsilon}^e + \boldsymbol{\epsilon}^i; \quad \mathbf{B} = \mathbf{B}^e + \mathbf{B}^i, \tag{13}$$

where the superscripts e and i stand for "elastic" and "inelastic". Inelastic strain may include plastic, viscous and thermal parts. Assume a linear elasticity law

$$\tilde{\boldsymbol{\sigma}} = \tilde{\mathbf{E}} : \tilde{\boldsymbol{\epsilon}}^e; \quad \boldsymbol{\sigma} = \mathbf{E} : \boldsymbol{\epsilon}^e + \mathbf{B}^e \,\vdots\, \boldsymbol{\pi}; \quad \mathbf{M} = \boldsymbol{\nu} \,\vdots\, \mathbf{B}^e + \boldsymbol{\xi} : \boldsymbol{\epsilon}^e, \tag{14}$$

where $\tilde{\mathbf{E}}$ is the local elastic moduli tensor, $\mathbf{E}, \boldsymbol{\pi}, \boldsymbol{\nu}$ and $\boldsymbol{\xi}$ – the effective elastic moduli of the volume v. Assume also that due to linear elastic behaviour

$$\tilde{\boldsymbol{\sigma}} = \tilde{\mathbf{A}} : \boldsymbol{\sigma} + \tilde{\mathbf{L}} \,\vdots\, \mathbf{M} + \tilde{\boldsymbol{\chi}}, \tag{15}$$

where $\tilde{\boldsymbol{\chi}}$ are the residual stresses (at $\boldsymbol{\sigma} = \mathbf{M} = 0$) and $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{L}}$ are the forth- and fifth-order elastic localization tensors, respectively. Condition (12) gives

$$\langle \tilde{\mathbf{A}} \rangle = \mathbf{C}; \;\; \langle \tilde{\mathbf{L}} \rangle = 0; \;\; \mathbf{C} : \langle \mathbf{r}\tilde{\mathbf{A}} \rangle = 0; \;\; \mathbf{C} : \langle \mathbf{r}\tilde{\mathbf{L}} \rangle = \mathbf{C}^6; \;\; \langle \tilde{\boldsymbol{\chi}} \rangle = \mathbf{C} : \langle \mathbf{r}\tilde{\boldsymbol{\chi}} \rangle = 0. \tag{16}$$

Define the local internal energy in the form $\tilde{U} = \frac{1}{2}\tilde{\boldsymbol{\epsilon}}^e : \tilde{\mathbf{E}} : \tilde{\boldsymbol{\epsilon}}^e$ and for the volume v assume $U = \langle \tilde{U} \rangle$. From equation

$$\langle \tilde{\boldsymbol{\sigma}} : \tilde{\boldsymbol{\epsilon}} \rangle = \langle \tilde{\boldsymbol{\epsilon}}^e : \tilde{\mathbf{E}} : \tilde{\boldsymbol{\epsilon}}^e \rangle + \langle \tilde{\boldsymbol{\sigma}} : \tilde{\boldsymbol{\epsilon}}^i \rangle = \boldsymbol{\sigma} : \boldsymbol{\epsilon} + \mathbf{B} \,\vdots\, \mathbf{M} = \boldsymbol{\sigma} : \boldsymbol{\epsilon}^e + \boldsymbol{\sigma} : \boldsymbol{\epsilon}^i + \mathbf{B}^e \,\vdots\, \mathbf{M} + \mathbf{B}^i \,\vdots\, \mathbf{M} \tag{17}$$

we determine

$$2U = \langle \tilde{\boldsymbol{\epsilon}}^e : \tilde{\mathbf{E}} : \tilde{\boldsymbol{\epsilon}}^e \rangle = \boldsymbol{\sigma} : \boldsymbol{\epsilon}^e + \boldsymbol{\sigma} : \boldsymbol{\epsilon}^i + \mathbf{B}^e \,\vdots\, \mathbf{M} + \mathbf{B}^i \,\vdots\, \mathbf{M} - \langle \tilde{\boldsymbol{\sigma}} : \tilde{\boldsymbol{\epsilon}}^i \rangle.$$

Using Eqs. (14) and (15) we obtain

$$\begin{aligned} 2U = {} & \boldsymbol{\epsilon}^e : \mathbf{E} : \boldsymbol{\epsilon}^e + \mathbf{B}^e \,\vdots\, \boldsymbol{\nu} \,\vdots\, \mathbf{B}^e + \mathbf{B}^e \,\vdots\, (\boldsymbol{\pi} + \boldsymbol{\xi}) : \boldsymbol{\epsilon}^e - \\ & -\langle \tilde{\boldsymbol{\chi}} : \tilde{\boldsymbol{\epsilon}}^i \rangle + (\boldsymbol{\epsilon}^e : \mathbf{E} + \mathbf{B}^e \,\vdots\, \boldsymbol{\pi}) : (\boldsymbol{\epsilon}^i - \langle \tilde{\boldsymbol{\epsilon}}^i : \tilde{\mathbf{A}} \rangle) + (\mathbf{B}^i - \langle \tilde{\boldsymbol{\epsilon}}^i : \tilde{\mathbf{L}} \rangle) : (\boldsymbol{\nu} \,\vdots\, \mathbf{B}^e + \boldsymbol{\xi} : \boldsymbol{\epsilon}^e). \end{aligned} \tag{18}$$

From thermodynamics

$$\boldsymbol{\sigma} = \frac{\partial U}{\partial \boldsymbol{\epsilon}^e} = \mathbf{E} : \boldsymbol{\epsilon}^e + \mathbf{B}^e \,\vdots\, \bar{\boldsymbol{\pi}} + \frac{1}{2}\mathbf{E} : (\boldsymbol{\epsilon}^i - \langle \tilde{\boldsymbol{\epsilon}}^i : \tilde{\mathbf{A}} \rangle) + \frac{1}{2}(\mathbf{B}^i - \langle \tilde{\boldsymbol{\epsilon}}^i : \tilde{\mathbf{L}} \rangle) \,\vdots\, \boldsymbol{\xi}; \tag{19}$$

$$\mathbf{M} = \frac{\partial U}{\partial \mathbf{B}^e} = \boldsymbol{\nu} \,\vdots\, \mathbf{B}^e + \bar{\boldsymbol{\pi}} : \boldsymbol{\epsilon}^e + \frac{1}{2}\boldsymbol{\pi} : (\boldsymbol{\epsilon}^i - \langle \tilde{\boldsymbol{\epsilon}}^i : \tilde{\mathbf{A}} \rangle) + \frac{1}{2}(\mathbf{B}^i - \langle \tilde{\boldsymbol{\epsilon}}^i : \tilde{\mathbf{L}} \rangle) \,\vdots\, \boldsymbol{\nu}. \tag{20}$$

where $\bar{\boldsymbol{\pi}} = 0.5\,(\boldsymbol{\pi} + \boldsymbol{\xi})$. From the conditions $\boldsymbol{\epsilon}^e = \mathbf{B}^e = 0$ when $\boldsymbol{\sigma} = \mathbf{M} = 0$ (by definition $\boldsymbol{\epsilon}^i$ and $\mathbf{B}^i$) or from comparison of Eqs. (14) and (19), (20) it follows that

$$\boldsymbol{\epsilon}^i = \langle \tilde{\boldsymbol{\epsilon}}^i : \tilde{\mathbf{A}} \rangle; \quad \mathbf{B}^i = \langle \tilde{\boldsymbol{\epsilon}}^i \; : \; \tilde{\mathbf{L}} \rangle \tag{21}$$

as well as Eqs.(14) at $\bar{\boldsymbol{\pi}} = \boldsymbol{\pi} = \boldsymbol{\xi}$. First Eq.(21) is well known Mandell [5] equation, second Eq.(21) is a new one. To transform the term $\tilde{\boldsymbol{\chi}} : \tilde{\boldsymbol{\epsilon}}^i$ we take into account that $\tilde{\boldsymbol{\chi}} = \tilde{\mathbf{E}} : (\tilde{\boldsymbol{\epsilon}}^0 - \tilde{\boldsymbol{\epsilon}}^i)$, where $\tilde{\boldsymbol{\epsilon}}^0$ is the compatible strain field at $\boldsymbol{\sigma} = \mathbf{M} = 0$. Then

$$\tilde{\boldsymbol{\chi}} : \tilde{\boldsymbol{\epsilon}}^i = \tilde{\boldsymbol{\chi}} : \tilde{\boldsymbol{\epsilon}}^0 - (\tilde{\boldsymbol{\epsilon}}^0 - \tilde{\boldsymbol{\epsilon}}^i) : \tilde{\mathbf{E}} : (\tilde{\boldsymbol{\epsilon}}^0 - \tilde{\boldsymbol{\epsilon}}^i). \tag{22}$$

As $\tilde{\boldsymbol{\chi}}$ and $\tilde{\boldsymbol{\epsilon}}^0$ are statically and kinematically admissible fields at $\boldsymbol{\sigma} = \mathbf{M} = 0$, from Eq. (9) we have

$$\langle \tilde{\boldsymbol{\chi}} : \tilde{\boldsymbol{\epsilon}}^0 \rangle = \boldsymbol{\sigma} : \langle \tilde{\boldsymbol{\epsilon}}^0 \rangle + \mathbf{M} \vdots \langle \nabla \tilde{\boldsymbol{\epsilon}}^0 \rangle = 0. \tag{23}$$

Thus

$$U = \frac{1}{2}\boldsymbol{\epsilon}^e : \mathbf{E} : \boldsymbol{\epsilon}^e + \frac{1}{2}\mathbf{B}^e \vdots \boldsymbol{\nu} \vdots \mathbf{B}^e + \mathbf{B}^e \vdots \boldsymbol{\pi} : \boldsymbol{\epsilon}^e + \frac{1}{2}\langle (\tilde{\boldsymbol{\epsilon}}^0 - \tilde{\boldsymbol{\epsilon}}^i) : \tilde{\mathbf{E}} : (\tilde{\boldsymbol{\epsilon}}^0 - \tilde{\boldsymbol{\epsilon}}^i) \rangle. \tag{24}$$

Let us assume that for each point of time-independent or time dependent media the normality principle is valid

$$\dot{\tilde{\boldsymbol{\epsilon}}}^i = \tilde{h}\frac{\partial \tilde{f}}{\partial \tilde{\boldsymbol{\sigma}}} \quad \text{or} \quad \dot{\tilde{\boldsymbol{\epsilon}}}^i = \tilde{z}\frac{\partial \tilde{D}}{\partial \tilde{\boldsymbol{\sigma}}}, \quad \tilde{z} = \tilde{D}\left(\frac{\partial \tilde{D}}{\partial \tilde{\boldsymbol{\sigma}}} : \tilde{\boldsymbol{\sigma}}\right)^{-1}, \tag{25}$$

where $\tilde{f}(\tilde{\boldsymbol{\sigma}}) = 0$ is the yield condition, $\tilde{D}(\tilde{\boldsymbol{\sigma}})$ dissipative function. It follows from Eqs.(15) that in $(\boldsymbol{\sigma}, \mathbf{M})$-space

$$\tilde{f}(\tilde{\boldsymbol{\sigma}}) = \tilde{f}(\tilde{\mathbf{A}} : \boldsymbol{\sigma} + \tilde{\mathbf{L}} \vdots \mathbf{M} + \tilde{\boldsymbol{\chi}}) = 0; \quad \tilde{D}(\tilde{\boldsymbol{\sigma}}) = \tilde{D}(\tilde{\mathbf{A}} : \boldsymbol{\sigma} + \tilde{\mathbf{L}} \vdots \mathbf{M} + \tilde{\boldsymbol{\chi}}); \tag{26}$$

$$\frac{\partial \tilde{f}}{\partial \boldsymbol{\sigma}} = \frac{\partial \tilde{f}}{\partial \tilde{\boldsymbol{\sigma}}} : \tilde{\mathbf{A}}; \quad \frac{\partial \tilde{f}}{\partial \mathbf{M}} = \frac{\partial \tilde{f}}{\partial \tilde{\boldsymbol{\sigma}}} : \tilde{\mathbf{L}} \tag{27}$$

and for the time-independent and time-dependent behaviour we have respectively

$$\dot{\boldsymbol{\epsilon}}^i = \langle \dot{\tilde{\boldsymbol{\epsilon}}}^i : \tilde{\mathbf{A}} \rangle = \langle \tilde{h}\frac{\partial \tilde{f}}{\partial \tilde{\boldsymbol{\sigma}}} : \tilde{\mathbf{A}} \rangle = \langle \tilde{h}\frac{\partial \tilde{f}}{\partial \boldsymbol{\sigma}} \rangle; \tag{28}$$

$$\dot{\mathbf{B}}^i = \langle \dot{\tilde{\boldsymbol{\epsilon}}}^i : \tilde{\mathbf{L}} \rangle = \langle \tilde{h}\frac{\partial \tilde{f}}{\partial \tilde{\boldsymbol{\sigma}}} : \tilde{\mathbf{L}} \rangle = \langle \tilde{h}\frac{\partial \tilde{f}}{\partial \mathbf{M}} \rangle; \tag{29}$$

$$\dot{\boldsymbol{\epsilon}}^i = \langle \tilde{z}\frac{\partial \tilde{D}}{\partial \tilde{\boldsymbol{\sigma}}} : \tilde{\mathbf{A}} \rangle = \langle \tilde{z}\frac{\partial \tilde{D}}{\partial \boldsymbol{\sigma}} \rangle; \tag{30}$$

$$\dot{\mathbf{B}}^i = \langle \tilde{z}\frac{\partial \tilde{D}}{\partial \tilde{\boldsymbol{\sigma}}} : \tilde{\mathbf{L}} \rangle = \langle \tilde{z}\frac{\partial \tilde{D}}{\partial \mathbf{M}} \rangle. \tag{31}$$

Thus the associative flow rule and the normality principle are valid for smooth or vertex-like macroscopic potential for the finite volume v. Let the inverted Eqs.(14) have the form

$$\tilde{\boldsymbol{\epsilon}}^e = \tilde{\boldsymbol{\lambda}} : \tilde{\boldsymbol{\sigma}}; \quad \boldsymbol{\epsilon}^e = \boldsymbol{\lambda} : \boldsymbol{\sigma} + \mathbf{Y} \vdots \mathbf{M}; \quad \mathbf{B}^e = \mathbf{M} \vdots \mathbf{Z} + \boldsymbol{\sigma} : \mathbf{Y}, \tag{32}$$

where $\tilde{\boldsymbol\lambda} = \tilde{\mathbf{E}}^{-1}$. Then

$$\boldsymbol\epsilon = \langle\tilde{\boldsymbol\epsilon}\rangle = \langle\tilde{\boldsymbol\epsilon}^e + \tilde{\boldsymbol\epsilon}^i\rangle = \langle\tilde{\boldsymbol\lambda} : \tilde{\boldsymbol\sigma} + \tilde{\boldsymbol\epsilon}^i\rangle = \tag{33}$$

$$= \langle\tilde{\boldsymbol\lambda} : \tilde{\mathbf{A}}\rangle : \boldsymbol\sigma + \langle\tilde{\boldsymbol\lambda} : \tilde{\mathbf{L}}\rangle \,\vdots\, \mathbf{M} + \langle\tilde{\boldsymbol\epsilon}^i + \tilde{\boldsymbol\lambda} : \tilde{\boldsymbol\chi}\rangle = \boldsymbol\lambda : \boldsymbol\sigma + \mathbf{Y} \,\vdots\, \mathbf{M} + \langle\tilde{\boldsymbol\epsilon}^0\rangle = \boldsymbol\lambda : \boldsymbol\sigma + \mathbf{Y} \,\vdots\, \mathbf{M} + \boldsymbol\epsilon^i;$$

$$\mathbf{B} = \langle\nabla\tilde{\boldsymbol\epsilon}\rangle = \langle\nabla(\tilde{\boldsymbol\lambda} : \tilde{\boldsymbol\sigma} + \tilde{\boldsymbol\epsilon}^i)\rangle = \tag{34}$$

$$= \boldsymbol\sigma : \langle\nabla(\tilde{\mathbf{A}}^t : \tilde{\boldsymbol\lambda})\rangle + \mathbf{M} \,\vdots\, \langle\nabla(\tilde{\mathbf{L}}^0 : \tilde{\boldsymbol\lambda})\rangle + \langle\nabla\tilde{\boldsymbol\epsilon}^0\rangle = \mathbf{M} \,\vdots\, \mathbf{Z} + \boldsymbol\sigma : \mathbf{Y} + \mathbf{B}^i,$$

where $\tilde{\mathbf{L}}^0 = \tilde{L}_{ijklm}\mathbf{g}^k\mathbf{g}^l\mathbf{g}^m\mathbf{g}^i\mathbf{g}^j$, $\mathbf{g}^a$ are the basis vectors, and we have $\boldsymbol\epsilon^i = \langle\tilde{\boldsymbol\epsilon}^0\rangle$, $\mathbf{B}^i = \langle\nabla\tilde{\boldsymbol\epsilon}^0\rangle$ and

$$\boldsymbol\lambda = \langle\tilde{\boldsymbol\lambda} : \tilde{\mathbf{A}}\rangle; \quad \mathbf{Y} = \langle\tilde{\boldsymbol\lambda} : \tilde{\mathbf{L}}\rangle = \langle\nabla(\tilde{\mathbf{A}}^t : \tilde{\boldsymbol\lambda})\rangle; \quad \mathbf{Z} = \langle\nabla(\tilde{\mathbf{L}}^0 : \tilde{\boldsymbol\lambda})\rangle. \tag{35}$$

Evidently, if $\tilde{\boldsymbol\lambda}$ is independent of $\tilde{\boldsymbol\epsilon}^i$ then the effective elastic constants do not depend on the $\tilde{\boldsymbol\epsilon}^i$ distribution and are determined by the same formulas as in elasticity theory.

Conclusions

The relations proposed generalize Hill's equations for the case of large strain and stress gradient and the presence of inertia and body forces. Using the assumption of a parabolic distribution of displacement in each phase of an n-phase composite it is possible to determine explicit formulas for all fields in phases using a method which is analogeous to that proposed by Levitas for the large [6,8] and small [7,8] strains. The results obtained and their generalization may serve as the micromechanical basis for various theories of materials with microstructure, having characteristic length. It is known that these theories give a finite width of the shear band. Evidently this is not a result of fundamental material properties, but rather of the prescribed displacement distribution at the finite volume surface. More realistic models have to include the possibility of $\mathbf{u}(\mathbf{r})$ discontinuities, for instance, using two different values of $\mathbf{u}_0, \boldsymbol\beta$ and $\mathbf{B}$ on two parts of the S surface. Jumps of $\mathbf{u}(\mathbf{r})$ in v are also necessary to describe granular materials with rotated grains.

Generalization of the results of paragraph 2 for finite strain is trivial, while the description of inelastic behaviour requires additional work. If we introduce besides reference V_0 and current V configurations some intermidiate one V_p at $\boldsymbol\sigma = \mathbf{M} = 0$, then mapping of S surface from V_0 to V and to V_p will be parabolic, but from V_p to V will not. That is why it is unclear how to choose the proper measures of elastic strain and strain gradient , which met the similar requirements as in work of Levitas [9,10] for macrouniform fields.

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