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PRINCIPLE OF MINIMUM DISSIPATION RATE AT TIME $t+\Delta t$ FOR THE PLASTIC SPIN

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1. Introduction

One of the method to satisfy the principle of material frame–indifference (PMFI) for anisotropic polycrystalline plastic materials consists in consideration of constitutive equations in some rotating frame of reference which is equivalent to exclusion of rotation with respect to some configuration. There are two main nonequivalent ways to exclude the rotations. In the first one rotations are excluded relative to the fixed privileged configuration V_0 [1, 2]. In configuration V_0 the particles are in an equilibrium state under zero stresses and deformations and such a state is usually reached after annealing and recrystallization at high temperature. Only under such an assumption can we obtain the classical result $\boldsymbol{T} = \boldsymbol{R}\cdot\hat{\boldsymbol{\Phi}}(\boldsymbol{U}(t'))\cdot\boldsymbol{R}^t$ when we apply the PMFI. Here $\boldsymbol{T}$ is the Cauchy stress, $\hat{\boldsymbol{\Phi}}$ is some functional, $\boldsymbol{R}$ is the orthogonal rotation tensor and $\boldsymbol{U}$ the symmetric right stretch tensor in the polar decomposition of the deformation gradient $\boldsymbol{F} = \boldsymbol{R}\cdot\boldsymbol{U}$ with respect to configuration V_0. An objective corrotational derivative associated with a skew–symmetric spin tensor $\boldsymbol{M} = \dot{\boldsymbol{R}}\cdot\boldsymbol{R}^t$ appears in all evolution equations.

Another approach is related to the introduction of the concept of the plastic spin [3 – 5]. A triad of directors (the analogue of a crystal lattice for single crystal) which characterizes the orientation is ascribed to each point of the medium, and it is postulated that all time derivatives must be taken with respect to variable privileged configuration described by these directors, i.e. rotations are excluded with respect to directors. Configurations obtained when we fix directors (i. e. which rotate together with directors) are called isoclinic configurations. An non–symmetric tensor of plastic deformation gradient appears in the isoclinic configuration and in addition to the flow rule for plastic strain rate a constitutive equation for the plastic spin has to be given.

As plastic spin does not contribute to the rate of dissipation, it is impossible to derive for it some extremum principle or constitutive equation using thermodynamics (as for the plastic deformation rate). The only known macroscopic way is to use the representation theorem [4, 5].

In paper [6] a new approach for determination of plastic spin for rigid–plastic materials is suggested based on stability analysis. Using the postulate of realizability [7] a principle of minimum of dissipation rate at the time $t + \Delta t$ is derived. The aims of the paper are:

- to show that in the general theory we *cannot* exclude rotations with respect to some *fixed* privileged configuration. We should introduce some variable privileged configuration (or additional rotational variable) and exclude rotation with respect to it.

- To apply the principle of *minimum of dissipation rate at the time* $t + \Delta t$ for derivation of the equations for one or multiple plastic spins for arbitrary dissipative material (elastoplastic, viscoplastic and so on).

Direct tensor notations are used, in particular, $(\boldsymbol{A} \cdot \boldsymbol{B})_{ik} = A_{ij} B^{jk}$ and $\boldsymbol{A} : \boldsymbol{B} = A_{ij} B^{ji}$, superscripts t and -1 denote transposition and inverse operation, $\boldsymbol{I}$ the unit tensor, $(\boldsymbol{A})_s = 0.5\left(\boldsymbol{A} + \boldsymbol{A}^t\right)$, $(\boldsymbol{A})_a = 0.5\left(\boldsymbol{A} - \boldsymbol{A}^t\right)$.

2. Contradictions arising under excluding rotation with respect to fixed privileged configuration

We will show that method based on exclusion of rotations relative to fixed privileged configuration V_0 when we apply the PMFI is contradictory, if we do not introduce additional rotational variable. Let us consider the following thermomechanical process as both a thought experiment and simultaneous calculation of the stress–strain state. Let us deform plastically some specimen made from *initially isotropic* materials with anisotropic hardening starting from configuration V_0. Then after removing stresses, annealing and the occurrence of complete recrystallization the material has the *same properties* as in configuration V_0, but occupies another configuration V_{01}. But we do not know about the existence of a new privileged configuration. When we continue a thermomechanical process and deform a material plastically, we should continue to calculate the objective corrotational derivative or memory functional with respect to configuration V_0. Another investigator, obtaining completely annealed isotropic material in a configuration V_{01}, knows nothing about configuration V_0, because it should not influence the material behaviour (we also knew nothing about the eventual existence of completely annealed states before configuration V_0). Producing the same deformation process as we do with respect to V_{01}, he will, in contrast to us, calculate the corrotational derivative or memory functional with respect to configuration V_{01}. Our and his results of the Cauchy stress measurements will be the same (because we both produce the same deformation process starting with the same privileged configuration V_{01}). Our and his results of stress calculation will differ, because the rotation tensor and corrotational derivative are not invariant under a change of reference configuration (see below).
For a model with kinematic hardening, a purely mechanical counterpart of the same contradiction can be shown. Let the back stress tensor $\boldsymbol{L}$ during cyclic loading become equal to zero several times. All the states with the zero back stress are undistinguished and equivalent; with respect to which one should the rotation be excluded? With respect to the first one? But we cannot define experimentally and conceptually which state with $\boldsymbol{L} = 0$ was the first. The last one? Then the spin tensor $\boldsymbol{M}$ will have jumps by passing through the state with $\boldsymbol{L} = 0$ and the results of stress calculation for $\boldsymbol{L}$ history with states with $\boldsymbol{L} = 0$ and with infinitesimal $\boldsymbol{L}$ will have a finite difference.

Let us illustrate the above reasoning with formulae. In some frame of reference κ, let us consider the motion of a small uniformly deformable volume, described by the mapping $\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{r}_0, t)$, where $\boldsymbol{r}$ and $\boldsymbol{r}_0$ are the position vectors of the points of the volume at the time t in the actual configuration V_t and in the reference configuration V_0. A superposition of rigid body rotation (RBR) is described by the equation

$$\boldsymbol{r}^* = \boldsymbol{Q}\cdot\boldsymbol{r}, \tag{1}$$

where $\boldsymbol{Q}$ is the proper orthogonal tensor. Under superposed RBR (1) $\boldsymbol{T}^* = \boldsymbol{Q}\cdot\boldsymbol{T}\cdot\boldsymbol{Q}^t$; $\boldsymbol{d}^* = \boldsymbol{Q}\cdot\boldsymbol{d}\cdot\boldsymbol{Q}^t$; $\boldsymbol{L}^* = \boldsymbol{Q}\cdot\boldsymbol{L}\cdot\boldsymbol{Q}^t$; $\boldsymbol{F}^* = \boldsymbol{Q}\cdot\boldsymbol{F}$, where $\boldsymbol{d}$ is the deformation rate. To exclude rotation with respect to configuration V_0 we consider the rotating frame of reference δ in which $\boldsymbol{r}_\delta = \boldsymbol{R}^t\cdot\boldsymbol{r}$, i.e. this relation is being obtained from Eq. (1) at $\boldsymbol{Q} = \boldsymbol{R}^t$. In the frame δ (i.e. for $\boldsymbol{Q} = \boldsymbol{R}^t$) we obtain $\boldsymbol{F}_\delta = \frac{\partial \boldsymbol{r}_\delta}{\partial \boldsymbol{r}_0} = \boldsymbol{R}^t\cdot\boldsymbol{F} = \boldsymbol{U}$; $\boldsymbol{R}_\delta = \boldsymbol{R}^t\cdot\boldsymbol{R} = \boldsymbol{I}$; $\boldsymbol{T}_\delta = \boldsymbol{R}_\tau^t\cdot\boldsymbol{T}\cdot\boldsymbol{R}_\tau$; $\boldsymbol{d}_\delta = \boldsymbol{R}_\tau^t\cdot\boldsymbol{d}\cdot\boldsymbol{R}_\tau$; $\boldsymbol{L}_\delta = \boldsymbol{R}_\tau^t\cdot\boldsymbol{L}\cdot\boldsymbol{R}_\tau$. Evidently, tensors $\boldsymbol{T}_\delta$, $\boldsymbol{L}_\delta$ and $\boldsymbol{d}_\delta$ are invariant under superposed RBR described by (1).

Let there be two equivalent preferred reference configurations V_0 and V_{01}, in which the material is isotropic and has the same properties, and configuration V_{01} can be obtained after some thermomechanical or mechanical deformation process with the deformation gradient $\tilde{\boldsymbol{\lambda}}(t')$ and the temperature variation $\theta(t')$ with respect to configuration V_0, ending with $\tilde{\boldsymbol{\lambda}} = \boldsymbol{\lambda}$ and $\theta = \theta_0$. Then we continue the deformation process, which will be considered simultaneously with respect to configurations V_0 and V_{01}. The following kinematic relations are valid :

$$\boldsymbol{F}_0 = \boldsymbol{F}_1\cdot\boldsymbol{\lambda}\,; \qquad \boldsymbol{U}_0^2 = \boldsymbol{F}_0^t\cdot\boldsymbol{F}_0 = \boldsymbol{\lambda}\cdot\boldsymbol{F}_1^t\cdot\boldsymbol{F}_1\cdot\boldsymbol{\lambda} = \boldsymbol{\lambda}\cdot\boldsymbol{U}_1^2\cdot\boldsymbol{\lambda}\,; \tag{2}$$

$$\boldsymbol{R}_0\cdot\boldsymbol{U}_0 = \boldsymbol{R}_1\cdot\boldsymbol{U}_1\cdot\boldsymbol{\lambda} \;\Rightarrow\; \qquad \boldsymbol{R}_1 = \boldsymbol{R}_0\cdot\tilde{\boldsymbol{R}}\,; \qquad \tilde{\boldsymbol{R}} := \boldsymbol{U}_0\cdot\boldsymbol{\lambda}^{-1}\cdot\boldsymbol{U}_1^{-1}\,; \tag{3}$$

$$\boldsymbol{M}_1 := \dot{\boldsymbol{R}}_1\cdot\boldsymbol{R}_1^t = \boldsymbol{M}_0 + \boldsymbol{R}_0\cdot\dot{\tilde{\boldsymbol{R}}}\cdot\tilde{\boldsymbol{R}}^t\cdot\boldsymbol{R}_0^t\,; \qquad \boldsymbol{M}_0 := \dot{\boldsymbol{R}}_0\cdot\boldsymbol{R}_0^t\,, \tag{4}$$

where the subscripts 0 and 1 denote the tensors defined with respect to configuration V_0 and V_{01} correspondingly. Let us consider the model with kinematic hardening. If configurations V_0 and V_{01} are equivalent, then at exclusion of rotations relative to each of them the same equations $\dot{\boldsymbol{L}}_{\delta 1} = A\,\boldsymbol{d}_{\delta 1}$ and $\dot{\boldsymbol{L}}_{\delta 2} = A\,\boldsymbol{d}_{\delta 2}$ should be valid i.e.

$$\dot{\overline{\boldsymbol{R}_1^t\cdot\boldsymbol{L}\cdot\boldsymbol{R}_1}} = A\,\boldsymbol{R}_1^t\cdot\boldsymbol{d}\cdot\boldsymbol{R}_1\,; \qquad \dot{\overline{\boldsymbol{R}_0^t\cdot\boldsymbol{L}\cdot\boldsymbol{R}_0}} = A\,\boldsymbol{R}_0^t\cdot\boldsymbol{d}\cdot\boldsymbol{R}_0 \qquad \text{or} \tag{5}$$

$$\dot{\boldsymbol{L}} + \boldsymbol{L}\cdot\boldsymbol{M}_1 + \boldsymbol{M}_1^t\cdot\boldsymbol{L} = A\,\boldsymbol{d}\,; \qquad \dot{\boldsymbol{L}} + \boldsymbol{L}\cdot\boldsymbol{M}_0 + \boldsymbol{M}_0^t\cdot\boldsymbol{L} = A\,\boldsymbol{d}\,. \tag{6}$$

As according to Eq. (4) $\boldsymbol{M}_1 \neq \boldsymbol{M}_0$, then Eqs. $(6)_1$ and $(6)_2$ cannot be equivalent, which is a *contradiction*. Let us assume that the same equation for the simple solid is valid at excluding of rotation with respect to configurations V_0 and V_{01} :

$$\boldsymbol{T}_{\delta 1} := \boldsymbol{R}_1^t\cdot\boldsymbol{T}\cdot\boldsymbol{R}_1 = \hat{\boldsymbol{\Phi}}(\boldsymbol{U}_1(t'))\;; \qquad \boldsymbol{T}_{\delta 0} := \boldsymbol{R}_0^t\cdot\boldsymbol{T}\cdot\boldsymbol{R}_0 = \hat{\boldsymbol{\Phi}}(\boldsymbol{U}_0(t'))\;, \tag{7}$$

where we omit the temperature. We should prove that in general case

$$\boldsymbol{T} = \boldsymbol{R}_1\cdot\hat{\boldsymbol{\Phi}}(\boldsymbol{U}_1(t'))\cdot\boldsymbol{R}_1^t \neq \boldsymbol{T} = \boldsymbol{R}_0\cdot\hat{\boldsymbol{\Phi}}(\boldsymbol{U}_0(t'))\cdot\boldsymbol{R}_0^t\,, \tag{8}$$

despite the fact that Cauchy stress $\boldsymbol{T}$ must be independent of the choice of reference configuration and this is a conceptual contradiction. To prove this, it is sufficient to find one collaborating example and we again can use the same model with kinematic hardening, but in functional representation.

Let us draw a conclusion. If we exclude rotation relative to some fixed privileged configuration V_0, then it is always possible to create by some thermomechanical deformation process a number of completely equivalent privileged configurations V_i. Assuming the same constitutive equations for rotations excluded relative to each of the privileged configurations, we cannot obtain the same equations for the case with rotations. This contradiction shows that in the general theory *we cannot exclude rotations with respect to some fixed privileged configuration.*

To overcome the contradiction found we will use some variable privileged configuration, similar to Mandel's [3] isoclinic configuration, or additional rotational variable.

3. Extremum principle and equation for plastic spin

Assume that in a fixed frame of reference the constitutive equation is as follows [8]

$$\boldsymbol{T} = \boldsymbol{T}\left(\boldsymbol{d}, \boldsymbol{F}, \bar{\boldsymbol{R}}\right) = \lambda \frac{\partial \mathcal{D}\left(\boldsymbol{d}, \boldsymbol{F}, \bar{\boldsymbol{R}}\right)}{\partial \boldsymbol{d}}; \qquad \lambda = \mathcal{D}^{-1}\left(\frac{\partial \mathcal{D}}{\partial \boldsymbol{d}} \cdot \boldsymbol{d}\right), \tag{9}$$

where $\bar{\boldsymbol{R}}$ is some additional rotational variable, $\mathcal{D}\left(\boldsymbol{d}, \boldsymbol{F}, \bar{\boldsymbol{R}}\right) := \boldsymbol{T}\left(\boldsymbol{d}, \boldsymbol{F}, \bar{\boldsymbol{R}}\right) : \boldsymbol{d} \geq 0$ is the dissipative function. For elastoplastic materials $\boldsymbol{d}$ is the dissipative part of deformation rate [2, 7]. Let under a superposed RBR (1) Eq. (9) transforms as

$$\boldsymbol{Q} \cdot \boldsymbol{T} \cdot \boldsymbol{Q}^t = \boldsymbol{T}\left(\boldsymbol{Q} \cdot \boldsymbol{d} \cdot \boldsymbol{Q}^t, \ \boldsymbol{Q} \cdot \boldsymbol{F}, \ \boldsymbol{Q} \cdot \bar{\boldsymbol{R}}\right) = \boldsymbol{Q} \cdot \boldsymbol{T}\left(\boldsymbol{d}, \boldsymbol{F}, \bar{\boldsymbol{R}}\right) \cdot \boldsymbol{Q}^t, \tag{10}$$

i.e. dependence (9) satisfies PMFI. By definition, for rate–independent plastic materials $\boldsymbol{T}(\boldsymbol{d}, ...)$ is a homogeneous function of degree zero in $\boldsymbol{d}$, $\lambda = 1$ and the yield surface $\varphi\left(\boldsymbol{T}, \boldsymbol{F}, \bar{\boldsymbol{R}}\right) = 0$ can be introduced [6 – 8]. Consequently the orthogonal tensor $\bar{\boldsymbol{R}}$ characterizes rotation of a yield surface $\varphi\left(\boldsymbol{T}, \boldsymbol{F}, \bar{\boldsymbol{R}}\right) = 0$ in the space $\boldsymbol{T}$ (as well as rotation of surface $\mathcal{D}\left(\boldsymbol{d}, \boldsymbol{F}, \bar{\boldsymbol{R}}\right) = const$ in the space $\boldsymbol{d}$). This means that tensor $\bar{\boldsymbol{R}}$ can in principle be *measured.*

Let us introduce after Mandel [3] an isoclinic frame of reference χ in which $\bar{\boldsymbol{R}} = \boldsymbol{I}$. To do this we put $\boldsymbol{Q} = \bar{\boldsymbol{R}}^t$ in Eqs. (1) and (10) :

$$\boldsymbol{T}_\chi = \boldsymbol{T}(\boldsymbol{d}_\chi, \boldsymbol{F}_\chi); \qquad \boldsymbol{T}_\chi := \bar{\boldsymbol{R}}^t \cdot \boldsymbol{T} \cdot \bar{\boldsymbol{R}}; \qquad \boldsymbol{d}_\chi = \bar{\boldsymbol{R}}^t \cdot \boldsymbol{d} \cdot \bar{\boldsymbol{R}}; \qquad \boldsymbol{F}_\chi = \bar{\boldsymbol{R}}^t \cdot \boldsymbol{F}. \tag{11}$$

In the frame of reference χ $\mathcal{D} = \mathcal{D}(\boldsymbol{d}_\chi, \boldsymbol{F}_\chi)$ and $\varphi = \varphi(\boldsymbol{T}_\chi, \boldsymbol{F}_\chi) = 0$, i.e. by definition of the frame of reference χ dissipation surface and the yield surface do not rotate in it. Kinematic decompositions have the following form :

$$\boldsymbol{F} = \bar{\boldsymbol{R}} \cdot \boldsymbol{F}_\chi; \qquad \boldsymbol{l} := \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} = \dot{\bar{\boldsymbol{R}}} \cdot \bar{\boldsymbol{R}}^t + \bar{\boldsymbol{R}} \cdot \dot{\boldsymbol{F}}_\chi \cdot \boldsymbol{F}_\chi^{-1} \cdot \bar{\boldsymbol{R}}^t; \tag{12}$$

$$\boldsymbol{W} := \left(\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}\right)_a = \boldsymbol{\Omega} + \boldsymbol{W}_p; \qquad \boldsymbol{\Omega} := \dot{\bar{\boldsymbol{R}}} \cdot \bar{\boldsymbol{R}}^t; \qquad \boldsymbol{W}_p := \bar{\boldsymbol{R}} \cdot \left(\dot{\boldsymbol{F}}_\chi \cdot \boldsymbol{F}_\chi^{-1}\right)_a \cdot \bar{\boldsymbol{R}}^t. \tag{13}$$

Tensors $\boldsymbol{W}_p$ and $\boldsymbol{\Omega}$ are the plastic spin and spin of some privileged orientation in the fixed frame of reference. We should assume the existence of one additional scalar constraint equation $q(\boldsymbol{d}, \boldsymbol{W}_p) = 0$ which guarantees that the plastic spin is zero when the plastic deformation rate is zero and limits a modulus of the spin tensor. Function q can depend on additional parameters e.g. plastic strain. Example of function q for single crystal is given in [6]. As $\bar{\boldsymbol{R}}_\Delta = \bar{\boldsymbol{R}} + (\boldsymbol{W} - \boldsymbol{W}_p) \cdot \bar{\boldsymbol{R}}\, \Delta t$, where subscript Δ means that the parameter is determined at the time $t + \Delta t$, then the principle of minimum of dissipation rate at time $t + \Delta t$ reads as

$$\mathcal{D}\left(\boldsymbol{d}_\Delta , \boldsymbol{F}_\Delta , \bar{\boldsymbol{R}} + (\boldsymbol{W} - \boldsymbol{W}_p) \cdot \bar{\boldsymbol{R}}\, \Delta t\right) < \mathcal{D}\left(\boldsymbol{d}_\Delta , \boldsymbol{F}_\Delta , \bar{\boldsymbol{R}} + \left(\boldsymbol{W} - \boldsymbol{W}_p^0\right) \cdot \bar{\boldsymbol{R}}\, \Delta t\right) \quad \text{at} \quad q\left(\boldsymbol{d}, \boldsymbol{W}_p^0\right) = 0 . \tag{14}$$

The plastic spin $\boldsymbol{W}_p^0$ only varies in the principle (14). The extremum principle (14) can be derived using the same assumption as for the derivation of Eq. (9), namely the postulate of realizability [6, 7]. In one dimensional case the principle (14) means, that we choose the lowest curve at stress–strain diagram. At small Δt the extremum principle (14) can be transformed into

$$\frac{\partial \mathcal{D}}{\partial \bar{\boldsymbol{R}}^t} : \dot{\bar{\boldsymbol{R}}} = \frac{\partial \mathcal{D}}{\partial \bar{\boldsymbol{R}}^t} : (\boldsymbol{W} - \boldsymbol{W}_p) \cdot \bar{\boldsymbol{R}} < \frac{\partial \mathcal{D}}{\partial \bar{\boldsymbol{R}}^t} : \left(\boldsymbol{W} - \boldsymbol{W}_p^0\right) \cdot \bar{\boldsymbol{R}} \quad \text{at} \quad q\left(\boldsymbol{d}, \boldsymbol{W}_p^0\right) = 0 . \tag{15}$$

Additional transformations

$$\frac{\partial \mathcal{D}}{\partial \bar{\boldsymbol{R}}^t} : (\boldsymbol{W} - \boldsymbol{W}_p) \cdot \bar{\boldsymbol{R}} = \bar{\boldsymbol{R}} \cdot \frac{\partial \mathcal{D}}{\partial \bar{\boldsymbol{R}}^t} : (\boldsymbol{W} - \boldsymbol{W}_p) =$$

$$= \left(\bar{\boldsymbol{R}} \cdot \frac{\partial \mathcal{D}}{\partial \bar{\boldsymbol{R}}^t}\right)_a : (\boldsymbol{W} - \boldsymbol{W}_p) = \boldsymbol{a} : (\boldsymbol{W} - \boldsymbol{W}_p) \, ; \qquad \boldsymbol{a} := \left(\bar{\boldsymbol{R}} \cdot \frac{\partial \mathcal{D}}{\partial \bar{\boldsymbol{R}}^t}\right)_a , \tag{16}$$

lead to

$$\boldsymbol{a} : (\boldsymbol{W} - \boldsymbol{W}_p) < \boldsymbol{a} : \left(\boldsymbol{W} - \boldsymbol{W}_p^0\right) \qquad \text{at} \qquad q\left(\boldsymbol{d}, \boldsymbol{W}_p^0\right) = 0 , \tag{17}$$

and

$$\boldsymbol{a} : \boldsymbol{W}_p > \boldsymbol{a} : \boldsymbol{W}_p^0 \qquad \text{or} \qquad \boldsymbol{a} : \boldsymbol{W}_p^0 \to max \qquad \text{at} \qquad q\left(\boldsymbol{d}, \boldsymbol{W}_p^0\right) = 0 . \tag{18}$$

In contrast to extremum principle [8], $\boldsymbol{a}$ and $\boldsymbol{W}_p$ are not conjugate thermodynamic force and rate, because $\boldsymbol{W}_p$ does not contribute to dissipation rate. As the extremum principle (18) is linear in $\boldsymbol{W}_p^0$, it is clear that without an additional constraint the solution cannot be found. That is why we assumed the existence of constraint equation $q\left(\boldsymbol{d}, \boldsymbol{W}_p^0\right) = 0$ from the very beginning. An additional reason to introduce the constraint is a necessity to meet the condition $\boldsymbol{W}_p = 0$ at $\boldsymbol{d} = 0$. Let q be a nonlinear function of $\boldsymbol{W}_p$. In this case from principle (18) the following equation $\boldsymbol{a} = \eta \dfrac{\partial q}{\partial \boldsymbol{W}_p^t}$ is valid, where η is a scalar determined from condition $q = 0$; the sign of η is determined from Eq. $(18)_2$. If we assume as the simplest case that q depends on $\boldsymbol{d}$ and $\boldsymbol{W}_p$ separately and is an isotropic function of $\boldsymbol{W}_p$, i.e.

$$q = f(\boldsymbol{d}) - |\, \boldsymbol{W}_p \,| = 0 , \qquad \text{then} \qquad \boldsymbol{W}_p = \eta\, \boldsymbol{a} = \frac{\boldsymbol{a}}{|\, \boldsymbol{a} \,|} f(\boldsymbol{d}) . \tag{19}$$

In Eq. (19) for $\boldsymbol{W}_p$ the second equality is valid at $\boldsymbol{a} \neq \boldsymbol{0}$ which we assume in the following similar equations as well. We will use this formula for spin, because it is consistent with some known results (see below). In the given case scalar function η completely determines q; example of function η is given e.g. in [9]. But general equation for q can be applied as well. If q is a linear function of $\boldsymbol{W}_p$, then the minimum in the principle (15) is reached on the boundary of the set $\mathcal{C}$ of all admissible $\boldsymbol{W}_p$.

If for some $\boldsymbol{d}$, $\boldsymbol{F}$ and $\boldsymbol{W}_p$ $\dfrac{\partial \mathcal{D}}{\partial \bar{\boldsymbol{R}}^t} = \boldsymbol{a} := 0$ and this point corresponds to the minimum of $\mathcal{D}$, then $\boldsymbol{W}_p = 0$. In the case of maximum $\mathcal{D}$ again $\boldsymbol{W}_p = 0$, but this state is unstable. Arbitrary small or finite deviations from point with $\boldsymbol{a} = 0$ will progress.

4. Initially anisotropic polycrystal

Let us prove the following statement : *when the dissipation function reads as* $\mathcal{D} = \mathcal{D}(\boldsymbol{d}_\chi)$, *then*

$$\boldsymbol{W}_p = f(\boldsymbol{d}) \frac{(\boldsymbol{T}\cdot\boldsymbol{d} - \boldsymbol{d}\cdot\boldsymbol{T})}{|\boldsymbol{T}\cdot\boldsymbol{d} - \boldsymbol{d}\cdot\boldsymbol{T}|} = \eta(\boldsymbol{T}\cdot\boldsymbol{d} - \boldsymbol{d}\cdot\boldsymbol{T}) , \qquad \eta \geq 0 , \tag{20}$$

$$\boldsymbol{\Omega} := \dot{\bar{\boldsymbol{R}}}\cdot\bar{\boldsymbol{R}}^t = \boldsymbol{W} - \eta(\boldsymbol{T}\cdot\boldsymbol{d} - \boldsymbol{d}\cdot\boldsymbol{T}) . \tag{21}$$

Function $\mathcal{D}$ can depend on the number of fixed tensors of arbitrary ranks, characterizing initial anisotropy, as well as on scalar hardening parameters. To calculate tensor $\boldsymbol{a}$ let us determine

$$\frac{\partial \mathcal{D}}{\partial \bar{\boldsymbol{R}}^t} : \dot{\bar{\boldsymbol{R}}} = \frac{\partial \mathcal{D}}{\partial \boldsymbol{d}_\chi} : \frac{d\left(\bar{\boldsymbol{R}}^t\cdot\boldsymbol{d}\cdot\bar{\boldsymbol{R}}\right)}{dt}\Bigg|_{\boldsymbol{d}} = \lambda^{-1}\boldsymbol{T}_\chi : \left(\bar{\boldsymbol{R}}^t\cdot\boldsymbol{d}\cdot\dot{\bar{\boldsymbol{R}}} + \dot{\bar{\boldsymbol{R}}}^t\cdot\boldsymbol{d}\cdot\bar{\boldsymbol{R}}\right) =$$

$$= 2\lambda^{-1}\boldsymbol{T}_\chi : \bar{\boldsymbol{R}}^t\cdot\boldsymbol{d}\cdot\dot{\bar{\boldsymbol{R}}} = 2\lambda^{-1}\boldsymbol{T}_\chi\cdot\bar{\boldsymbol{R}}^t\cdot\boldsymbol{d} : \dot{\bar{\boldsymbol{R}}} , \tag{22}$$

i.e.

$$\frac{\partial \mathcal{D}}{\partial \bar{\boldsymbol{R}}^t} = 2\lambda^{-1}\boldsymbol{T}_\chi\cdot\bar{\boldsymbol{R}}^t\cdot\boldsymbol{d} , \qquad \bar{\boldsymbol{R}}\cdot\frac{\partial \mathcal{D}}{\partial \bar{\boldsymbol{R}}^t} = 2\lambda^{-1}\bar{\boldsymbol{R}}\cdot\boldsymbol{T}_\chi\cdot\bar{\boldsymbol{R}}^t\cdot\boldsymbol{d} = 2\lambda^{-1}\boldsymbol{T}\cdot\boldsymbol{d} \tag{23}$$

and according to Eq. (19), Eq. (20) is valid. For isotropic material tensors $\boldsymbol{T}$ and $\boldsymbol{d}$ are coaxial and according to Eq. (20) $\boldsymbol{W}_p \equiv 0$.
Let us have as an example in the isoclinic frame χ $\mathcal{D} = (\boldsymbol{d}_\chi : \boldsymbol{E}_0 : \boldsymbol{d}_\chi)^{1/2}$, where $\boldsymbol{E}_0$ is the constant fourth order tensor. Then $\boldsymbol{T}_\chi = \frac{\partial \mathcal{D}}{\partial \boldsymbol{d}_\chi} = \mathcal{D}^{-1}\,\boldsymbol{E}_0 : \boldsymbol{d}_\chi$ and for the yield conditions it follows $\varphi = \boldsymbol{T}_\chi : \boldsymbol{E}_0 : \boldsymbol{T}_\chi - 1 = 0$. Alternative expressions in the fixed frame of reference are

$$\mathcal{D} = (\boldsymbol{d} : \boldsymbol{E} : \boldsymbol{d})^{1/2} ; \qquad \boldsymbol{T} = \mathcal{D}^{-1}\,\boldsymbol{E} : \boldsymbol{d} ; \qquad \varphi = \boldsymbol{T} : \boldsymbol{E} : \boldsymbol{T} - 1 = 0 , \tag{24}$$

where $\boldsymbol{E} := \bar{\boldsymbol{R}} \times \boldsymbol{E}_0 := E^{klmn}\left(\bar{\boldsymbol{R}}\cdot\boldsymbol{e}_k\right)\left(\bar{\boldsymbol{R}}\cdot\boldsymbol{e}_e\right)\left(\bar{\boldsymbol{R}}\cdot\boldsymbol{e}_m\right)\left(\bar{\boldsymbol{R}}\cdot\boldsymbol{e}_n\right)$ is rotated with the tensor $\bar{\boldsymbol{R}}$ tensor $\boldsymbol{E}_0 := E^{klmn}\,\boldsymbol{e}_k\,\boldsymbol{e}_l\,\boldsymbol{e}_m\,\boldsymbol{e}_n$. Consequently tensor $\bar{\boldsymbol{R}}$ characterizes a rotation of the ellipsoidal yield surface in a stress space $\boldsymbol{T}$. Eq. (21) determines such a rotation. When vector $\boldsymbol{T}$ is directed along the symmetry axes of ellipsoid, then vectors $\boldsymbol{T}$ and $\boldsymbol{d}$ are collinear and again $\boldsymbol{W}_p \equiv 0$. If vector $\boldsymbol{T}$ is directed along the shortest axis of the yield surface, then state with $\boldsymbol{W}_p = 0$ is stable; in other cases small or finite perturbation will lead to a deviation from state $\boldsymbol{W}_p = 0$.
If in isoclinic configuration χ $\mathcal{D} = \mathcal{D}(\boldsymbol{d}_\chi , \boldsymbol{F}_\chi)$, then the calculation of tensor $\boldsymbol{a}$ results in

$$\boldsymbol{W}_p = \eta\left((\boldsymbol{T}\cdot\boldsymbol{d} - \boldsymbol{T}\cdot\boldsymbol{d}) + 0.5\,\lambda\left(\frac{\partial \mathcal{D}}{\partial \boldsymbol{F}}\cdot\boldsymbol{F}^t - \boldsymbol{F}\cdot\frac{\partial \mathcal{D}}{\partial \boldsymbol{F}^t}\right)\right) . \tag{25}$$

When $\mathcal{D}$ depends on $\boldsymbol{F}$ in terms $\boldsymbol{U}$, then $\dfrac{\partial \mathcal{D}}{\partial \boldsymbol{F}}\cdot\boldsymbol{F}^t = \dfrac{\partial \mathcal{D}}{\partial \boldsymbol{U}}\cdot\boldsymbol{U}$. If $\mathcal{D}$ is an isotropic function of $\boldsymbol{U}$, tensors $\boldsymbol{U}$ and $\dfrac{\partial \mathcal{D}}{\partial \boldsymbol{U}}$ are coaxial and Eq. (25) reduces to Eq. (20).

5. Theory with internal variables

Assume that history dependence of the $\mathcal{D}$ is taken into account with the help of internal variable $\boldsymbol{L}$, which is for example the back stress tensor $(\boldsymbol{L} = \boldsymbol{L}^t) : \mathcal{D} = \mathcal{D}\left(\boldsymbol{d}, \boldsymbol{L}, \bar{\boldsymbol{R}}\right) = \mathcal{D}(\boldsymbol{d}_\chi, \boldsymbol{L}_\chi)$. In the frame of reference χ we assume the same equation as at small strains $\dot{\boldsymbol{L}}_\chi = A\,\boldsymbol{d}_\chi$, then in the fixed frame of reference

$$\dot{\boldsymbol{L}} + \boldsymbol{L}\cdot\boldsymbol{\Omega} + \boldsymbol{\Omega}^t\cdot\boldsymbol{L} = A\,\boldsymbol{d}, \qquad \boldsymbol{\Omega} = \dot{\bar{\boldsymbol{R}}}\cdot\bar{\boldsymbol{R}}^t . \tag{26}$$

To determine $\dfrac{\partial \mathcal{D}}{\partial \bar{\boldsymbol{R}}^t}$ let us find the terms proportional to $\dot{\bar{\boldsymbol{R}}}$ in expression for $\dot{\mathcal{D}}$:

$$\dot{\mathcal{D}} = \frac{\partial \mathcal{D}}{\partial \boldsymbol{d}_\chi}:\dot{\boldsymbol{d}}_\chi + \frac{\partial \mathcal{D}}{\partial \boldsymbol{L}_\chi}:\dot{\boldsymbol{L}}_\chi = \lambda^{-1}\boldsymbol{T}_\chi:\left(\bar{\boldsymbol{R}}^t\cdot\dot{\boldsymbol{d}}\cdot\bar{\boldsymbol{R}} + \left.\frac{d\left(\bar{\boldsymbol{R}}^t\cdot\boldsymbol{d}\cdot\bar{\boldsymbol{R}}\right)}{dt}\right|_{\boldsymbol{d}}\right) + \frac{\partial \mathcal{D}}{\partial \boldsymbol{L}_\chi}:A\,\boldsymbol{d}_\chi =$$

$$= \lambda^{-1}\boldsymbol{T}:\dot{\boldsymbol{d}} + 2\lambda^{-1}\boldsymbol{T}_\chi\cdot\bar{\boldsymbol{R}}^t\cdot\boldsymbol{d}\cdot\dot{\bar{\boldsymbol{R}}} + \frac{\partial \mathcal{D}}{\partial \boldsymbol{L}}:A\,\boldsymbol{d}. \tag{27}$$

Eq. (22) was used. As the term related to $\boldsymbol{L}$ does not produce in Eq. (27) the terms proportional to $\dot{\bar{\boldsymbol{R}}}$, then equations for determination $\boldsymbol{W}_p$ will be the same as for the case with $\mathcal{D} = \mathcal{D}(\boldsymbol{d}_\chi)$, i.e. Eqs. (21)–(23). The same equations are valid in the case of several internal variables with the evolution equations of type Eq. (26). If a material is initially isotropic and

$$\mathcal{D} = F(\boldsymbol{d}) + \boldsymbol{L}:\boldsymbol{d} = F(\boldsymbol{d}_\chi) + \boldsymbol{L}_\chi:\boldsymbol{d}_\chi , \tag{28}$$

where F is the isotropic and homogeneous degree one function of $\boldsymbol{d}$, then

$$\boldsymbol{T} = \frac{\partial F}{\partial \boldsymbol{d}} + \boldsymbol{L}, \qquad \boldsymbol{T}\cdot\boldsymbol{d} - \boldsymbol{d}\cdot\boldsymbol{T} = \left(\frac{\partial F}{\partial \boldsymbol{d}}\cdot\boldsymbol{d} - \boldsymbol{d}\cdot\frac{\partial F}{\partial \boldsymbol{d}}\right) + (\boldsymbol{L}\cdot\boldsymbol{d} - \boldsymbol{d}\cdot\boldsymbol{L}) . \tag{29}$$

For isotropic F tensors $\dfrac{\partial F}{\partial \boldsymbol{d}}$ and $\boldsymbol{d}$ are coaxial and can be transposed, so consequently the first bracket disappears. Thus for materials of type (28) we arrive at the well-known and well-investigated [4, 5, 9] equation :

$$\boldsymbol{W}_p = \eta\,(\boldsymbol{L}\cdot\boldsymbol{d} - \boldsymbol{d}\cdot\boldsymbol{L}) , \qquad \boldsymbol{\Omega} = \boldsymbol{W} - \eta\,(\boldsymbol{L}\cdot\boldsymbol{d} - \boldsymbol{d}\cdot\boldsymbol{L}) . \tag{30}$$

The results of works [5, 9] show that with proper choice of a scalar-valued function η Eqs. (26), (29) and (30) at $\dfrac{\partial F}{\partial \boldsymbol{d}} = k\,\dfrac{\boldsymbol{d}}{|\boldsymbol{d}|}$ allow us to describe some model situations.

6. Theory with multiple spins

The idea of multiple spins is presented e.g. in papers [10, 11]. Let us assume that the dissipation function depends on several orthogonal tensors $\bar{\boldsymbol{R}}_i$ and $\tilde{\boldsymbol{R}}_j$ in the following form

$$\mathcal{D}\left(\boldsymbol{d}, \bar{\boldsymbol{R}}_i \times \boldsymbol{E}_{0i}, \tilde{\boldsymbol{R}}_j\cdot\boldsymbol{L}_{\chi j}\cdot\tilde{\boldsymbol{R}}_j^t\right) = \mathcal{D}(\boldsymbol{d}, \boldsymbol{E}_i, \boldsymbol{L}_j) , \tag{31}$$

where $\boldsymbol{L}_j = \tilde{\boldsymbol{R}}_j \cdot \boldsymbol{L}_{\chi j} \cdot \tilde{\boldsymbol{R}}_j^t$, $\boldsymbol{E}_i := \bar{\boldsymbol{R}}_i \times \boldsymbol{E}_{0i}$, $\boldsymbol{E}_{0i} = \boldsymbol{const}_i$. The fixed tensors of arbitrary order $\boldsymbol{E}_{0i}$ characterize initial anisotropy, the internal variables $\boldsymbol{L}_j$ describe the strain induce anisotropy. Consequently each tensor $\boldsymbol{E}_{0i}$ or $\boldsymbol{L}_{\chi j}$ rotates with its own rotation tensor $\bar{\boldsymbol{R}}_i$ or $\tilde{\boldsymbol{R}}_j$. For tensors $\boldsymbol{L}_j$ the following evolution equations are given

$$\dot{\boldsymbol{L}}_{\chi j} = A_j \boldsymbol{d}_{\chi j}\,; \qquad \boldsymbol{d}_{\chi j} := \tilde{\boldsymbol{R}}_j^t \cdot \boldsymbol{d} \cdot \tilde{\boldsymbol{R}}_j \qquad \text{or}$$
$$\dot{\boldsymbol{L}}_j + \boldsymbol{L}_j \cdot \boldsymbol{\omega}_j + \boldsymbol{\omega}_j^t \cdot \boldsymbol{L}_j = A_j \boldsymbol{d}\,, \qquad \boldsymbol{\omega}_j := \dot{\tilde{\boldsymbol{R}}}_j \cdot \tilde{\boldsymbol{R}}_j^t. \tag{32}$$

For each $\boldsymbol{\Omega}_i = \dot{\bar{\boldsymbol{R}}}_i \cdot \bar{\boldsymbol{R}}_i^t$ and $\boldsymbol{\omega}_j$ we can define a corresponding plastic spin by formulas

$$\boldsymbol{W} = \boldsymbol{\Omega}_i + \boldsymbol{W}_{pi}\,; \qquad \boldsymbol{W} = \boldsymbol{\omega}_j + \tilde{\boldsymbol{W}}_{pj}\,. \tag{33}$$

To determine spins $\boldsymbol{W}_{pi}$ and $\tilde{\boldsymbol{W}}_{pj}$ we need the existence of scalar constraint equations $q_i\,(\boldsymbol{d},\,\boldsymbol{W}_{pi}) = 0$ and $\tilde{q}_j\left(\boldsymbol{d},\,\tilde{\boldsymbol{W}}_{pj}\right) = 0$ or their particular form

$$q_i = f_i\,(\boldsymbol{d}) - |\,\boldsymbol{W}_{pi}\,| = 0\,; \qquad \tilde{q}_j = \tilde{f}_j\,(\boldsymbol{d}) - |\,\tilde{\boldsymbol{W}}_{pj}\,| = 0\,. \tag{34}$$

The principle of minimum of dissipation rate at time $t + \Delta t$ reads :

$$\mathcal{D}\left(\boldsymbol{d}_\Delta,\ \bar{\boldsymbol{R}}_{i\Delta} \times \boldsymbol{E}_{0i},\ \tilde{\boldsymbol{R}}_{j\Delta} \cdot \boldsymbol{L}_{\chi\Delta} \cdot \tilde{\boldsymbol{R}}_{j\Delta}^t\right) < \mathcal{D}\left(\boldsymbol{d}_\Delta,\ \bar{\boldsymbol{R}}_{i\Delta}^0 \times \boldsymbol{E}_{0i},\ \tilde{\boldsymbol{R}}_{j\Delta}^0 \cdot \boldsymbol{L}_{\chi\Delta} \cdot \tilde{\boldsymbol{R}}_{j\Delta}^{t0}\right)\,; \tag{35}$$

$$\bar{\boldsymbol{R}}_{i\Delta}^0 := \bar{\boldsymbol{R}}_i + \left(\boldsymbol{W} - \boldsymbol{W}_{pi}^0\right) \cdot \bar{\boldsymbol{R}}_i\,\Delta t\,; \qquad \tilde{\boldsymbol{R}}_{j\Delta}^0 := \tilde{\boldsymbol{R}}_j + \left(\boldsymbol{W} - \tilde{\boldsymbol{W}}_{pj}^0\right) \cdot \tilde{\boldsymbol{R}}_j\,\Delta t \tag{36}$$

under constraints (34). For infinitesimal Δt for each i and j we obtain extremum principles:

$$-\frac{\partial \mathcal{D}}{\partial \boldsymbol{L}_j} : \left(\boldsymbol{L}_j \cdot \left(\boldsymbol{W} - \tilde{\boldsymbol{W}}_{pj}^0\right) + \left(\boldsymbol{W}^t - \tilde{\boldsymbol{W}}_{pj}^{t0}\right) \cdot \boldsymbol{L}_j\right) \to min\,; \tag{37}$$

$$\frac{\partial \mathcal{D}}{\partial \boldsymbol{E}_i^t} \cdots \dot{\boldsymbol{E}}_i^0 \to min\,; \qquad \text{where} \qquad \boldsymbol{E}_i^t = E_i^{\cdot\cdot nmlk}\, \boldsymbol{e}_k\, \boldsymbol{e}_l\, \boldsymbol{e}_m\, \boldsymbol{e}_n \ldots \tag{38}$$

is the transposed tensor $\boldsymbol{E}_i = E_i^{klmn\ldots}\, \boldsymbol{e}_k\, \boldsymbol{e}_l\, \boldsymbol{e}_m\, \boldsymbol{e}_n \ldots$. From the principles (37) in the same way as in Eqs. (22) and (23) we obtain

$$\left(\frac{\partial \mathcal{D}}{\partial \boldsymbol{L}_j} \cdot \boldsymbol{L}_j\right)_a : \tilde{\boldsymbol{W}}_{pj}^0 \to min\,; \qquad \tilde{\boldsymbol{W}}_{pj} = -\tilde{f}_j\,(\boldsymbol{d})\, \frac{\left(\frac{\partial \mathcal{D}}{\partial \boldsymbol{L}_j} \cdot \boldsymbol{L}_j\right)_a}{\left|\left(\frac{\partial \mathcal{D}}{\partial \boldsymbol{L}_j} \cdot \boldsymbol{L}_j\right)_a\right|}\,. \tag{39}$$

As an example we consider an initially anisotropic polycrystal with kinematic hardening, i.e.

$$\mathcal{D} = (\boldsymbol{d} : \boldsymbol{E} : \boldsymbol{d})^{1/2} + \boldsymbol{L} : \boldsymbol{d} = (\boldsymbol{d}_\chi : \boldsymbol{E}_0 : \boldsymbol{d}_\chi)^{1/2} + \boldsymbol{L} : \boldsymbol{d}\,; \tag{40}$$

$$\dot{\boldsymbol{L}} + \boldsymbol{L} \cdot \boldsymbol{\omega} + \boldsymbol{\omega}^t \cdot \boldsymbol{L} = A\,\boldsymbol{d}\,; \qquad \boldsymbol{d}_\chi = \bar{\boldsymbol{R}}^t \cdot \boldsymbol{d} \cdot \bar{\boldsymbol{R}}\,, \tag{41}$$

i.e. tensor $\boldsymbol{E}_0$ characterizing initial anisotropy and the back stress tensor $\boldsymbol{L}$ rotate with the spins $\dot{\bar{\boldsymbol{R}}} \cdot \bar{\boldsymbol{R}}^t$ and $\boldsymbol{\omega}$ respectively. For plastic spin $\tilde{\boldsymbol{W}}_p$ related to $\boldsymbol{L}$ we can use directly Eq. $(39)_2$

$$\tilde{\boldsymbol{W}}_p = \tilde{f}\,(\boldsymbol{d})\, \frac{\boldsymbol{L} \cdot \boldsymbol{d} - \boldsymbol{d} \cdot \boldsymbol{L}}{|\boldsymbol{L} \cdot \boldsymbol{d} - \boldsymbol{d} \cdot \boldsymbol{L}|}\,. \tag{42}$$

For the spin $\boldsymbol{W}_p$ connected to $\boldsymbol{E}_0$ we can also directly apply Eq. (38) at $\dfrac{\partial \mathcal{D}}{\partial \boldsymbol{E}^t} = \boldsymbol{d}\boldsymbol{d}$, but a more simple way can be used. According to Eq. (9)

$$\boldsymbol{T} = \frac{\boldsymbol{E}:\boldsymbol{d}}{(\boldsymbol{d}:\boldsymbol{E}:\boldsymbol{d})^{1/2}} + \boldsymbol{L} ; \qquad \boldsymbol{T}_\chi = \frac{\boldsymbol{E}_0:\boldsymbol{d}_\chi}{(\boldsymbol{d}_\chi:\boldsymbol{E}_0:\boldsymbol{d}_\chi)^{1/2}} + \boldsymbol{L}_\chi ; \tag{43}$$

$$\boldsymbol{T}_\chi - \boldsymbol{L}_\chi = \frac{\partial \mathcal{D}_1}{\partial \boldsymbol{d}_\chi} ; \qquad \mathcal{D}_1 := (\boldsymbol{d}_\chi:\boldsymbol{E}_0:\boldsymbol{d}_\chi)^{1/2} ; \tag{44}$$

Then

$$\frac{\partial \mathcal{D}}{\partial \bar{\boldsymbol{R}}^t} : \dot{\bar{\boldsymbol{R}}} = \frac{\partial \mathcal{D}_1}{\partial \bar{\boldsymbol{R}}^t} : \dot{\bar{\boldsymbol{R}}} = \frac{\partial \mathcal{D}_1}{\partial \boldsymbol{d}_\chi} : \left. \frac{d\left(\bar{\boldsymbol{R}}^t \cdot \boldsymbol{d} \cdot \bar{\boldsymbol{R}}\right)}{d\,t} \right|_{\boldsymbol{d}} =$$
$$= (\boldsymbol{T}_\chi - \boldsymbol{L}_\chi) : \left(\bar{\boldsymbol{R}}^t \cdot \boldsymbol{d} \cdot \dot{\bar{\boldsymbol{R}}} + \dot{\bar{\boldsymbol{R}}}^t \cdot \boldsymbol{d} \cdot \bar{\boldsymbol{R}}\right) = 2\,(\boldsymbol{T}_\chi - \boldsymbol{L}_\chi) \cdot \bar{\boldsymbol{R}}^t \cdot \boldsymbol{d} : \dot{\bar{\boldsymbol{R}}} ; \tag{45}$$

$$\bar{\boldsymbol{R}} \cdot \frac{\partial \mathcal{D}}{\partial \bar{\boldsymbol{R}}^t} = 2\,\bar{\boldsymbol{R}} \cdot (\boldsymbol{T}_\chi - \boldsymbol{L}_\chi) \cdot \bar{\boldsymbol{R}}^t \cdot \boldsymbol{d} = 2\,(\boldsymbol{T} - \boldsymbol{L}) \cdot \boldsymbol{d} ; \tag{46}$$

$$\boldsymbol{W}_p = f\,(\boldsymbol{d})\, \frac{(\boldsymbol{T} - \boldsymbol{L}) \cdot \boldsymbol{d} - \boldsymbol{d} \cdot (\boldsymbol{T} - \boldsymbol{L})}{|(\boldsymbol{T} - \boldsymbol{L}) \cdot \boldsymbol{d} - \boldsymbol{d} \cdot (\boldsymbol{T} - \boldsymbol{L})|} . \tag{47}$$

As all the tensors in Eqs. (21), (26), (30), (41), (42) and (47) are independent of the reference configuration, the contradiction mentioned in Section 2 and related to the appearance of several equivalent fixed preferred configurations (e.g. with $\boldsymbol{L} = 0$) does not arise. The deformation gradient in Eq. (25) depends on the choice of reference configuration. The question of whether it is possible in this case to avoid the contradiction revealed in Section 2 will be treated elsewhere.

7. Concluding remarks

1. The fundamental contradiction in the theory of constitutive equations is revealed for polycrystalline solids: in the general theory *it is impossible to exclude rotation relative to some fixed privileged configuration* (which e.g. isotropic for an initially isotropic material) when the PMFI is applied. The reason is related to the possibility of creating by some thermomechanical process a number of equivalent privileged configurations and it is impossible to get the *same constitutive equation with respect to each of these privileged configurations.* In the classical theory of simple materials there is no way to introduce a fading memory in terms of rotations about the "old " privileged configuration when a new one is created during the deformation process, because there is no constitutive equation for rotation.

2. To overcome the above contradiction we assume that the constitutive equations depend additionally on some rotational variable, for which some equation will be derived. This is equivalent to the problem of finding some *variable privileged configuration,* similar to Mandel's isoclinic configuration. Consequently, we arrive at the problem of determination of equation for plastic spin.

3. The *principle of minimum of dissipation rate at the time* $t + \Delta t$ is formulated and applied to derive the unique equations for one or multiple plastic spins for polycrystals. This principle can

be derived using the same assumption as for the derivation of Eq. (9), namely the postulate of realizability [6, 7]. One additional scalar constraint equation is necessary which guarantees that the plastic spin is zero when the plastic deformation rate is zero and limits a modulus of the spin tensor. A number of concrete expressions for plastic spin are derived for polycrystals with initial and strain induced anisotropy, represented by internal variables and material tensors of arbitrary order, with multiple spins.

Our approach allows us to derive much more concrete results, than the method based on the representation theorem [4, 5]. For example, if $\boldsymbol{W}_p = \boldsymbol{W}_p(\boldsymbol{T}, \boldsymbol{L})$, where $\boldsymbol{L}$ is the symmetrical second order tensor (an internal variable), then from the representation theorem it follows [4, 5]

$$\boldsymbol{W}_p = \eta_1 (\boldsymbol{L}\cdot\boldsymbol{T})_a + \eta_2 \left(\boldsymbol{L}^2\cdot\boldsymbol{T}\right)_a + \eta_3 \left(\boldsymbol{L}\cdot\boldsymbol{T}^2\right)_a + \eta_4 \left(\boldsymbol{L}\cdot\boldsymbol{T}\cdot\boldsymbol{L}^2\right)_a + \eta_5 \left(\boldsymbol{T}\cdot\boldsymbol{L}\cdot\boldsymbol{T}^2\right)_a , \qquad (48)$$

where η_i are the functions of various invariants. To find experimentally five functions is unreal, that is why only the first term in Eq. (48) is used. For three and more arguments the representation theorem gives very bulky formulas which cannot be concretized experimentally. Moreover, explicit enumeration of the arguments of a $\boldsymbol{W}_p$ function is a very strong assumption, because many skew-symmetric tensors like $\boldsymbol{R}\cdot\left(\dot{\boldsymbol{U}}\cdot\boldsymbol{U}^{-1}\right)_a\cdot\boldsymbol{R}^t$; $\boldsymbol{R}\cdot\left(\frac{\partial\varphi}{\partial\boldsymbol{U}}\cdot\boldsymbol{U}\right)_a\cdot\boldsymbol{R}^t$ can contribute to the plastic spin.

We do not assume explicitly the arguments of function $\boldsymbol{W}_p$; the final result is obtained in terms of the dissipation function for arbitrary initial and strain induced anisotropy described by multiple tensorial variables of arbitrary order. For many particular cases the expressions for plastic spin look very simple. For the case with an internal tensorial variable of second order Eq. (30) derived above is equivalent to the first term of Eq. (48) obtained with the help of the representation theorem.

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References

[1] Casey J. and Naghdi P.M., *Trans. ASME : J. Appl. Mech.* **47**, No. 3 (1980) 672–675.

[2] Levitas V.I., Large Deformation of Materials with Complex Rheological Properties at Normal and High Pressure (Nova Science Publishers, New York, 1996).

[3] Mandel J., *Int. J. Solids and Struct.* **9** (1973) 725–740.

[4] Loret B., *Mech. Mater.* **2** (1983) 278–304.

[5] Dafalias Y.F., Plasticity Today: Modelling, Methods and Applications (eds. A. Sawczuk and G. Bianchi), pp. 135–151. Elsevier Appl. Sci. Publ., London, New York, 1985.

[6] Levitas V.I., Dynamic Plasticity and Structural Behaviours. Proceedings of "Plasticity'95" (eds. S. Tanimura and A. Khan), pp. 261–264. Gordon and Breach Publishers, 1995.

[7] Levitas V.I., *Int. J. Eng. Sci.* **33** (1995) 921–971.

[8] Ziegler H., An Introduction to Thermomechanics (North Holland, Amsterdam, 1977).

[9] Paulun J. E. and Pecherski R. B., *Int. J. Plasticity* **3** (1987), 303-314.

[10] Zbib H.M. and Aifantis E.C., *Acta Mechanica* **75** (1988) 15–56.

[11] Cho H.W. and Dafalias Y.F., *Int. J. Plasticity* **12**, No. 7 (1996) 903–925.

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Corrigendum to

"A NEW LOOK AT THE PROBLEM OF PLASTIC SPIN BASED ON STABILITY ANALYSIS"

by VALERY I. LEVITAS

published in J. Mech. Phys. of Solids, 1998, 46, No. 3, 557-590.

It was written before Eq.(60) of the above paper, which is

$$\boldsymbol{W}_p = \eta\, \boldsymbol{a} \qquad \text{or} \qquad \boldsymbol{W}_p = \frac{\boldsymbol{a}}{|\,\boldsymbol{a}\,|} f(\boldsymbol{d}) \qquad \text{at} \quad \boldsymbol{a} \neq \boldsymbol{0}\,, \tag{1}$$

that the sign of the scalar η has to be determined from the extremum principle (58) of the paper, i.e. from condition

$$\boldsymbol{a}\!:\!\boldsymbol{W}_p \rightarrow max \qquad \text{at} \qquad q = f(\boldsymbol{d}) - |\,\boldsymbol{W}_p\,| = 0\,. \tag{2}$$

Here $\boldsymbol{W}_p$ is the skew-symmetric plastic spin tensor, $\boldsymbol{a}$ is the skew-symmetric tensor defined in the paper, $\boldsymbol{d}$ is the deformation rate, and f is some scalar function.
However, as according to constrain $q = f(\boldsymbol{d}) - |\,\boldsymbol{W}_p\,| = 0$ one has $f(\boldsymbol{d}) \geq 0$, then the sign in Eq.$(1)_2$ is chosen. This sign corresponds to $\eta \geq 0$, which was explicitly written in Eq.(100) of the paper. It appears, that this is the wrong sign. Indeed,

$$|\,\boldsymbol{W}_p\,|^2 = \boldsymbol{W}_p\!:\!\boldsymbol{W}_p^t = \eta\,\boldsymbol{a}\!:\!\boldsymbol{W}_p^t = -\eta\,\boldsymbol{a}\!:\!\boldsymbol{W}_p \geq 0\,, \tag{3}$$

where superscript t denotes transposition. It is evident that at $\eta \geq 0$ one has $\boldsymbol{a}\!:\!\boldsymbol{W}_p \leq 0$ which corresponds to $\boldsymbol{a}\!:\!\boldsymbol{W}_p \rightarrow min$ and is wrong. Condition $\eta \leq 0$ results in $\boldsymbol{a}\!:\!\boldsymbol{W}_p \geq 0$, which agrees with the extremum principle (2).
The easiest way to correct this error is as follows.

1. To consider $\eta \leq 0$ in all equations of the paper.
2. To assume

$$q = f(\boldsymbol{d}) + |\,\boldsymbol{W}_p\,| = 0 \tag{4}$$

instead of $q = f(\boldsymbol{d}) - |\,\boldsymbol{W}_p\,| = 0$.

3. To use

$$q_i = f_i(\boldsymbol{d}) + |\,\boldsymbol{W}_{pi}\,| = 0 \qquad \text{and} \qquad \tilde{q}_j = \tilde{f}_j(\boldsymbol{d}) + |\,\tilde{\boldsymbol{W}}_{pj}\,| = 0 \tag{5}$$

instead of Eq.(114) of the paper, i.e. the functions f, f_i, and $\tilde{f}_j$ are negative.

It was mentioned in the paper by Y. F. Dafalias (J. Mech. Phys. of Solids, 2000, 48, 2231-2255) that general approach developed in our paper under consideration cannot accommodate the need for negative η in order to simulate the experimental data by Kim and Yin (1997). After the above corrections, this contradiction disappears.