Lecture # 12: Shock Waves and De Laval Nozzle

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Sources/ Further reading:
Anderson, “Fundamentals of Aerodynamics” Part 3 Chs 7 & 8
Subsonic, Transonic, Supersonic and Hypersonic Flows

Subsonic flows: $M < 1.0$
Transonic flows: $M \approx 1.0$
Supersonic flows: $M > 1.0$
Hypersonic flows: $M > 5.0$

Sonic boom
Subsonic and Supersonic Flow

- a. Stationary sound source
- b. Source moving with $V_{\text{source}} < V_{\text{sound}}$
- c. Source moving with $V_{\text{source}} = V_{\text{sound}}$ (Mach 1 - breaking the sound barrier)
- d. Source moving with $V_{\text{source}} > V_{\text{sound}}$ (Mach 1.4 - supersonic)
**Shock Waves**

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**Normal Shock Wave**
(The airstream slows to subsonic)

**Oblique Shock Wave**
(The airstream slows down, but remains supersonic)

**Expansion Wave**
(The airstream accelerates, and the air behind the shock wave is higher supersonic)
Review of Quasi-1D Nozzle Flow

**Liquid Rocket Engine**

- **Fuel**
- **Oxidizer**
- **Combustion Chamber**
- **Nozzle**
- **Exhaust**
- **Throat**
- **Exit**

- \( V \) = Velocity
- \( \dot{m} \) = mass flow rate
- \( p \) = pressure

**Thrust** = \[ F = \dot{m} V_e + (p_e - p_0) A_e \]
Review of Quasi-1D Nozzle Flow

Assumptions:
- Steady-state
- Inviscid
- No body forces

Quasi-1D:
- Area is allowed to vary but flow variables are considered a function of $x$ only

Mass conservation
\[ \iiint_V \frac{\partial \rho}{\partial t} \, dV + \iint_S \rho \vec{U} \cdot \vec{n} \, dS = 0 \]

Momentum conservation
\[ \frac{\partial}{\partial t} \iiint_V \rho \vec{U} \, dV + \iint_S \rho (\vec{U} \cdot \vec{n}) \vec{U} \, dS = -\iint_S p \vec{n} \cdot dS + \iiint_V \rho \vec{f} \, dV + F_{viscous} \]

Energy conservation
\[ \frac{\partial}{\partial t} \iiint_V \rho \left( e + \frac{U^2}{2} \right) \, dV + \iint_S \rho \left( e + \frac{U^2}{2} \right) \vec{U} \cdot \vec{n} \, dS = -\iint_S p \vec{U} \cdot \vec{n} \, dS + \iiint_V \rho \frac{\partial q}{\partial t} \, dV + \iiint_V \rho (\vec{f} \cdot \vec{U}) \, dV \]
Review of Quasi-1D Nozzle Flow

Assumptions:
• Steady-state
• Inviscid
• No body forces

Quasi-1D:
• Area is allowed to vary but flow variables are considered a function of x only

\[
\int_0^\infty \int_0^\infty \frac{\partial \rho}{\partial t} dV + \int_S \rho \vec{U} \cdot \vec{n} dS = 0
\]

Mass conservation
\[
- \rho u A + (u + du)(\rho + d\rho)(A + dA) = 0
\]
\[
- \rho u A + \rho u A + \rho u dA + \rho d\rho A + d\rho u A + \text{higher order terms} = 0
\]
\[
\rho Au = \text{Const.}
\]
\[
\frac{dA}{A} + \frac{d\rho}{\rho} + \frac{du}{u} = 0
\]
Review of Quasi-1D Nozzle Flow

Assumptions: Steady-state, Inviscid, No body forces

Quasi-1D: Area is allowed to vary but flow variables are considered a function of x only

Momentum conservation

\[
\frac{\partial}{\partial t} \iiint_V \rho \vec{U} dV + \iiint_S \rho (\vec{U} \cdot \vec{n}) \vec{U} dS = -\iiint_S p dS + \iiint_V \rho \vec{f} dV + F_{viscous}
\]

\[
-\rho u^2 A + (\rho + d \rho)(u + du)(u + du)(A + dA) = PA - (P + dP)(A + dA) + 2 \left( \frac{PdA}{2} \right)
\]

Since:

\[
\frac{dA}{A} + \frac{d\rho}{\rho} + \frac{du}{u} = 0 \quad \Rightarrow \quad \frac{dA}{A} + \frac{du}{u} - \frac{u}{a^2} du = 0
\]

\[
\frac{dA}{A} + \frac{du}{u} \left(1 - \frac{u^2}{a^2} \right) = \frac{dA}{A} + \frac{du}{u} \left(1 - M^2 \right) = 0
\]

\[
\frac{dA}{A} = (M^2 - 1) \frac{du}{u}
\]
Review of Quasi-1D Nozzle Flow

\[ \frac{dA}{A} = (M^2 - 1) \frac{du}{u} \]

Subsonic flow (Ma < 1)
- \( dA > 0 \)
- \( dV < 0 \)

Supersonic flow (Ma > 1)
- \( dA > 0 \)
- \( dV > 0 \)

Throat
- M=1

Flow
- u increasing
- M<1 → M>1

(a)

Flow
- u decreasing
- M>1 → M<1

(b)

(a)
Review of Quasi-1D Nozzle Flow

Isentropic relations (Thermodynamics):

\[
\frac{P_2}{P_1} = \left(\frac{\rho_2}{\rho_1}\right)^\gamma = \left(\frac{T_2}{T_1}\right)^\frac{\gamma}{\gamma-1}
\]

Energy Equation:

\[
C_pT_0 = C_pT + \frac{V^2}{2} \Rightarrow \frac{T_0}{T} = 1 + \frac{V^2}{2C_pT} = 1 + \frac{\gamma-1}{2} \frac{V^2}{\gamma RT}
\]

\[
\frac{T_0}{T} = 1 + \frac{\gamma-1}{2} M^2
\]

\[
\frac{P_0}{P} = (1 + \frac{\gamma-1}{2} M^2)^\frac{\gamma}{\gamma-1}
\]

\[
\frac{\rho_0}{\rho} = (1 + \frac{\gamma-1}{2} M^2)^{\frac{1}{\gamma-1}}
\]
Review of Quasi-1D Nozzle Flow

\[ \rho^* u^* A^* = \rho u A \]

at \( A^* \) section: \( M = \frac{u^*}{a^*} = 1 \Rightarrow u^* = a^* \)

isentropic relation

\[
\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2
\]

\[
\frac{P_0}{P} = (1 + \frac{\gamma - 1}{2} M^2)^{\frac{\gamma}{\gamma - 1}}
\]

\[
\frac{\rho_0}{\rho} = (1 + \frac{\gamma - 1}{2} M^2)^{\frac{1}{\gamma - 1}}
\]

\[
\left( \frac{A}{A^*} \right)^2 = \frac{1}{M^2} \left[ \frac{2}{\gamma - 1} \left( 1 + \frac{\gamma - 1}{2} M^2 \right) \right]^{\frac{\gamma + 1}{\gamma - 1}}
\]

\[
\frac{A}{A^*} = \frac{2}{\gamma - 1} \left( 1 + \frac{\gamma - 1}{2} M^2 \right)^{\frac{\gamma + 1}{\gamma - 1}}
\]

Flow

\( M < 1 \)

\( M > 1 \)

Throat

\( A^* \)

\( P^*, T^*, \rho^* \)

\( M^* = 1 \)

\( u^* = a^* = (KRT^*)^{1/2} \)
Review of Quasi-1D Nozzle Flow

A de Laval nozzle (or convergent-divergent nozzle, CD nozzle) is a tube that is pinched in the middle, making an hourglass-shape. It is used as a means of accelerating the flow of a gas passing through it to a supersonic speed.
Test Section

Tank with compressed air

Test section
## AerE344 Lab: Pressure Measurements in a de Laval Nozzle

**Tank with compressed air**

<table>
<thead>
<tr>
<th>Tap No.</th>
<th>Distance downstream of throat (inches)</th>
<th>Area (Sq. inches)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-4.00</td>
<td>0.800</td>
</tr>
<tr>
<td>2</td>
<td>-1.50</td>
<td>0.529</td>
</tr>
<tr>
<td>3</td>
<td>-0.30</td>
<td>0.480</td>
</tr>
<tr>
<td>4</td>
<td>-0.18</td>
<td>0.478</td>
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<tr>
<td>5</td>
<td>0.00</td>
<td>0.476</td>
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<tr>
<td>6</td>
<td>0.15</td>
<td>0.497</td>
</tr>
<tr>
<td>7</td>
<td>0.30</td>
<td>0.518</td>
</tr>
<tr>
<td>8</td>
<td>0.45</td>
<td>0.539</td>
</tr>
<tr>
<td>9</td>
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<td>0.560</td>
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<tr>
<td>11</td>
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<td>0.599</td>
</tr>
<tr>
<td>12</td>
<td>1.05</td>
<td>0.616</td>
</tr>
<tr>
<td>13</td>
<td>1.20</td>
<td>0.627</td>
</tr>
<tr>
<td>14</td>
<td>1.35</td>
<td>0.632</td>
</tr>
<tr>
<td>15</td>
<td>1.45</td>
<td>0.634</td>
</tr>
</tbody>
</table>

**Test section**

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1st, 2nd, and 3rd critical conditions

- **1st critical condition**: Subsonic flow
- **2nd critical condition**: Shock at exit
- **3rd critical condition**: Overexpanded

- **Design condition**: M>1 with waves
- **Underexpanded**: M>1 no waves

Flow parameters:
- $P_0$: Increasing
- $M$: Increasing
1. Under-expanded flow
2. 3rd critical
3. Over-expanded flow with oblique shocks
4. 2nd critical
5. Normal shock existing inside the nozzle
6. 1st critical
1st, 2nd, and 3rd critical conditions

**Under-expanded flow**

Flow close to 3rd critical

**Over-expanded flow**

2nd critical – shock is at nozzle exit

Over-expanded flow with shock between nozzle exit and throat

1st critical – shock is almost at the nozzle throat.
Pressure Distribution Prediction within a De Laval Nozzle by using Numerical Approach

• Using the area ratio, the Mach number at any point up to the shock can be determined:

\[
\left( \frac{A}{A^*} \right)^2 = \frac{1}{M^2} \left[ \frac{2}{\gamma+1} \left(1 + \frac{\gamma - 1}{2} M^2 \right) \right]^{\frac{\gamma+1}{\gamma-1}}
\]

• After finding Mach number at front of shock, calculate Mach number after shock using:

\[
M_2^2 = \frac{1 + \frac{\gamma - 1}{2} M_1^2}{\gamma M_1^2 - \frac{\gamma - 1}{2}}
\]

• Then, calculate the \( A_2^* \)

\[
\left( A_2^* \right)^2 = \frac{M_2^2}{\gamma+1} \left[ \frac{2}{\gamma+1} \left(1 + \frac{\gamma - 1}{2} M_2^2 \right) \right]^{\frac{\gamma+1}{\gamma-1}}
\]

which allows us calculate the remaining Mach number distribution

\[
\left( \frac{A}{A_2^*} \right)^2 = \frac{1}{M^2} \left[ \frac{2}{\gamma+1} \left(1 + \frac{\gamma - 1}{2} M^2 \right) \right]^{\frac{\gamma+1}{\gamma-1}}
\]
2. Find pressure distribution
   a. Pressure at exit is same as atmospheric pressure for shock inside nozzle ($P_e = P_{atm}$). For shock after lip of nozzle, total pressure is constant throughout the interior of the nozzle ($P_{t2} = P_{t1}$).

   b. Find total pressure behind the shock:

   $$P_{t2} = \frac{P_2}{P_e} P_e$$
   where
   $$\frac{P_{t2}}{P_e} = \left(1 + \frac{\gamma - 1}{2} M_e^2\right)^{\frac{\gamma}{\gamma - 1}}$$  \hspace{1cm} (2.4)

   c. Any pressure behind the shock is therefore:

   $$P = P_{t2} \left(1 + \frac{\gamma - 1}{2} M_e^2\right)^{\frac{2}{\gamma - 1}}$$  \hspace{1cm} (2.5)

   d. Calculate $P_{t1}$ ahead of shock:

   $$P_{t1} = \frac{P_1 P_2 P_3}{P_1 P_2 P_{t2}}$$  \hspace{1cm} (2.6)

   where you can use Total-Static relation for 1st and 3rd ratios, and for the middle ratio:

   $$\frac{P_1}{P_2} = \frac{1 + \gamma M_2^2}{1 + \gamma M_1^2}$$
   or
   $$\frac{P_2}{P_1} = 1 + \frac{\gamma}{\gamma + 1} \left(M_1^2 - 1\right)$$  \hspace{1cm} (2.7)

   e. Now that you have the total pressure upstream of the shock, as well as the Mach number calculated earlier you can calculate the pressure upstream of the shock.
Pressure Distribution Prediction within a De Laval Nozzle by using Numerical Approach

![Diagram of De Laval Nozzle with Shockwave]

- Method #1, by using equations:
  \[
  \left(\frac{A}{A'}\right)^2 = \frac{1}{M^2} \left[\frac{2}{\gamma + 1} \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma+1}\right]^{\gamma+1}
  \]

- Method #2: by using Isentropic Flow properties table (Appendix-A of Anderson’s textbook)

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<td>15</td>
<td>1.45</td>
<td>0.634</td>
</tr>
</tbody>
</table>

- If the shockwave is located at position of tap#12:

<table>
<thead>
<tr>
<th>Tap No.</th>
<th>A/A*</th>
<th>Mach #</th>
<th>P/P1</th>
<th>P</th>
<th>Pp</th>
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<td>0.67</td>
<td>0.7401</td>
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<tr>
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<td>1.008</td>
<td>0.97</td>
<td>0.5469</td>
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<td>1.258</td>
<td>1.61</td>
<td>0.2318</td>
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<tr>
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<td>1.352</td>
<td>1.64</td>
<td>0.2217</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Pressure Distribution Prediction within a De Laval Nozzle by using Table Method

By using the normal shock tables with $M_1 = 1.64$ we find that $M_2 = 0.686$. (Appendix-B of Anderson’s textbook)

Next, we find the sonic reference area behind the shock using the area-Mach relation. i.e., $M_2 = 0.686$ (Appendix-A of Anderson’s textbook)

Find sonic reference behind the shock using the area-Mach relationship:

$$\left( \frac{A_2^*}{A_5^*} \right)^2 = A_5^2 M_2^2 \left[ \frac{2}{\gamma + 1} \left( 1 + \frac{\gamma - 1}{2} M_2^2 \right) \right]^{\frac{\gamma - 1}{\gamma + 1}}$$

i.e., $A_2^* = 0.557$ sq. Inches

If the shockwave is located at position of tab#12:

<table>
<thead>
<tr>
<th>Tap No.</th>
<th>$A/A^*$</th>
<th>Mach #</th>
<th>$P/P_1$</th>
<th>$P$</th>
<th>$P_g$</th>
</tr>
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<tbody>
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<td>1.681</td>
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<td>0.9098</td>
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<td></td>
</tr>
</tbody>
</table>
Pressure Distribution Prediction within a De Laval Nozzle by using Table Method

With the exit pressure to be sea-level standard pressure. We now calculate the total pressure behind the shock using this value of exit pressure and the pressure ratio at the exit:

\[ P_{t2} = \frac{P_t}{P} = \left( \frac{1}{0.7528} \right) 14.7 = 19.53 \]

Our last major task is to find the total pressure ahead of the shock, \( P_{t1} \):

\[ P_{t1} = \frac{P_{t1}}{P_1} \frac{P_1}{P_2} \frac{P_2}{P_{t2}} \]

<table>
<thead>
<tr>
<th>Tap No.</th>
<th>A/A*</th>
<th>Mach #</th>
<th>P/P_t</th>
<th>P</th>
<th>P_t</th>
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<td>post-shock</td>
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<td>0.7528</td>
<td>14.7</td>
<td>0</td>
</tr>
</tbody>
</table>
Examples of the previous lab reports

Theoretical Data - Gauge Pressure vs. Position

Experimental Results - Pressure vs. Nozzle Location

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AerE344 Lab#10: Set up Schlieren and Shadowgraph Systems to Visualize a Thermal Plume Flow of a Burning Candle

Before the candle is on

After the candle is on

Schlieren image of a the thermal plume of a burning candle
AerE344 Lab#10: Set up Schlieren and Shadowgraph Systems to Visualize a Thermal Plume Flow of a Burning Candle

- Set up Schlieren and Shadowgraph Systems to visualize a thermal plume flow.
- Sign-in sheet signature.
- No lab report.