Lecture # 2: Review of Vector Calculus

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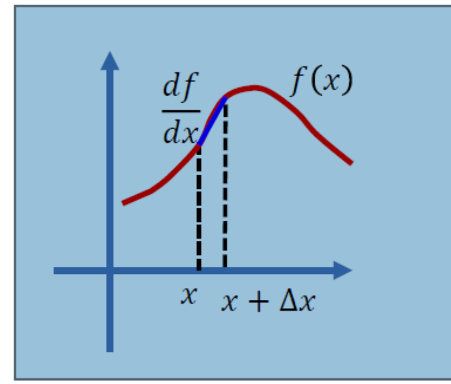
Differential Calculus-derivative

- For a continuous smoothly varying function $f(x)$

$$\frac{df}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$



- Second derivative

$$\frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d^2 f}{dx^2} \equiv f''(x)$$

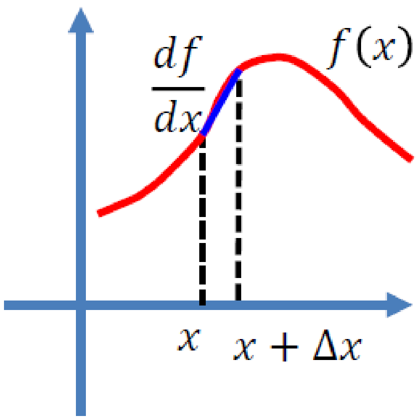
In physics, time derivative is denoted by 'dot'

$$\frac{df}{dt} \equiv \dot{f}$$

❑ Review of Multivariable Calculus

1.1. Review of Partial Differentials and Chain Rule

1.1.1 Definition of Partial Differentials



$$f_x(x_0, y_0) = \lim_{\Delta x \to 0}\left[\frac{f(x_0+\Delta x, y_0) - f(x_0+\Delta x, y_0)}{\Delta x}\right]$$

$$f_y(x_0, y_0) = \lim_{\Delta y \to 0}\left[\frac{f(x_0+\Delta y, y_0) - f(x_0+\Delta x, y_0)}{\Delta y}\right]$$

1.1.2 Properties of Partial Derivatives

$$(f+g)_y = f_y + g_y;$$

$$(f-g)_y = f_y - g_y;$$

$$(f\,g)_y = f_y\,g + f\,g_y;$$

$$(f/g)_y = \frac{f_y\,g - f\,g_y}{g^2}$$

$$(f_x)_x = f_{xx} = \frac{\partial^2 f}{\partial x^2};$$

$$(f_x)_y = f_{xy} = \frac{\partial^2 f}{\partial y \partial x};$$

$$(f_y)_x = f_{yx} = \frac{\partial^2 f}{\partial x \partial y};$$

$$(f_y)_x = (f_x)_y = f_{yx} = f_{xy}$$

Review of Multivariable Calculus

1.1.3 Chain Rule

1). In two- dimensional space:

$$\left.\begin{array}{l} z = f(x,y) \\ x = g_1(t) \\ y = g_2(t) \end{array}\right\} \Rightarrow \quad z = z(x,y) = f(g_1(t), g_2(t)) = z(t)$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\left.\begin{array}{l} w = f(x,y) \\ x = g_1(u,v) \\ y = g_2(u,v) \end{array}\right\} \Rightarrow \quad w = f(x,y) = f(g_1(u,v), g_2(u,v)) = w(u,v)$$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v}$$

2). In three-dimensional space:

$$\left.\begin{array}{l} w = f(x,y,z) \\ x = g_1(t) \\ y = g_2(t) \\ y = g_2(t) \end{array}\right\} \Rightarrow \quad \frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$\left.\begin{array}{l} w = f(x,y,z) \\ x = g_1(u,v) \\ y = g_2(u,v) \\ z = g_3(u,v) \end{array}\right\} \Rightarrow \quad \left\{\begin{array}{l} \dfrac{\partial w}{\partial u} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial u} + \dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial u} \\ \dfrac{\partial w}{\partial v} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial v} + \dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial u} \end{array}\right.$$

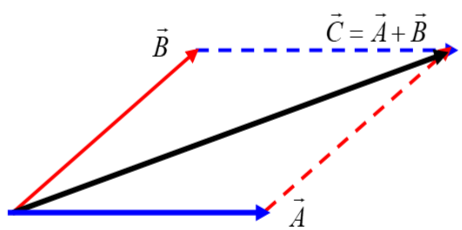
❑ VECTOR ALGEBRA

Definition:

- ***A vector is a quantity that posses both magnitude and direction, and obeys the parallelogram law of addition.***
- ***A scalar is a quantity that possess only magnitude, but no direction.***

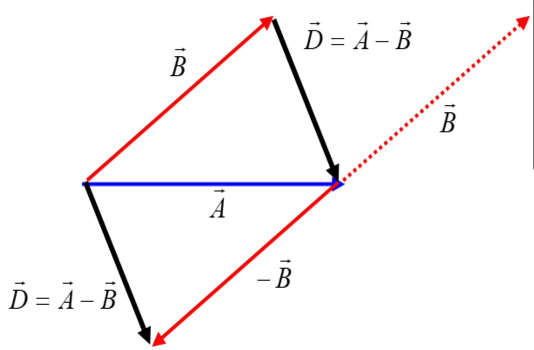
2.2 Vector Addition

$\vec{C} = \vec{A} + \vec{B}$



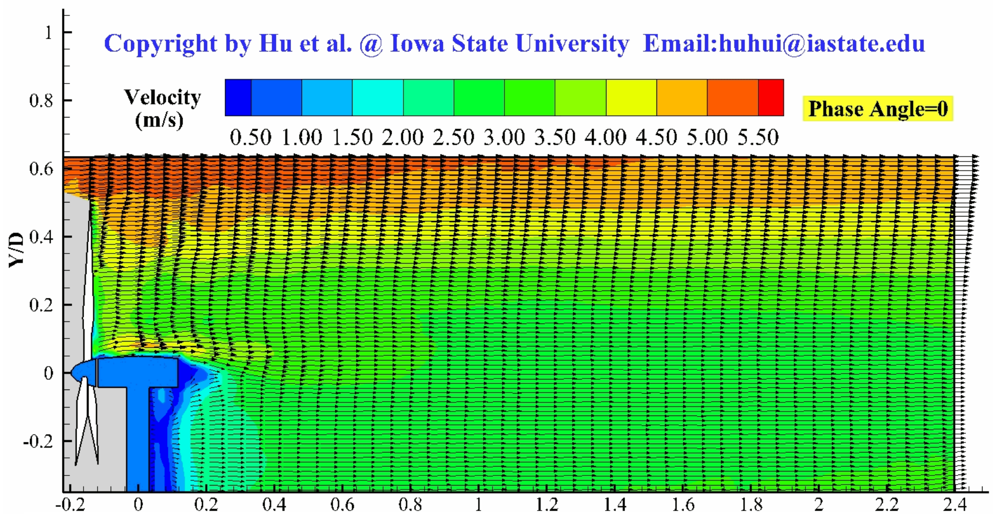
2.3 Vector Subtraction

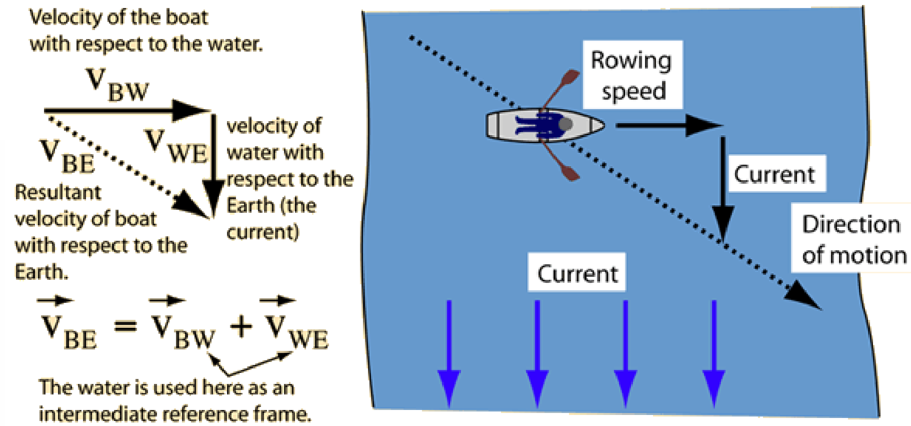
$\vec{D} = \vec{A} - \vec{B}$

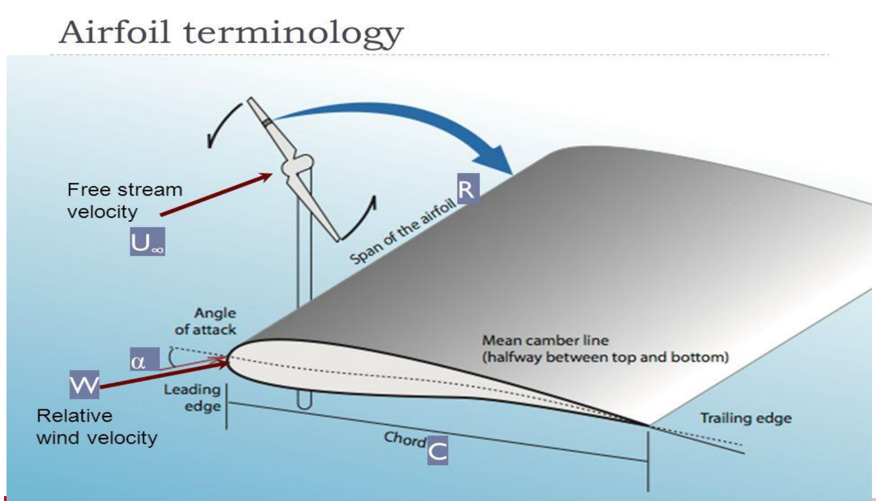


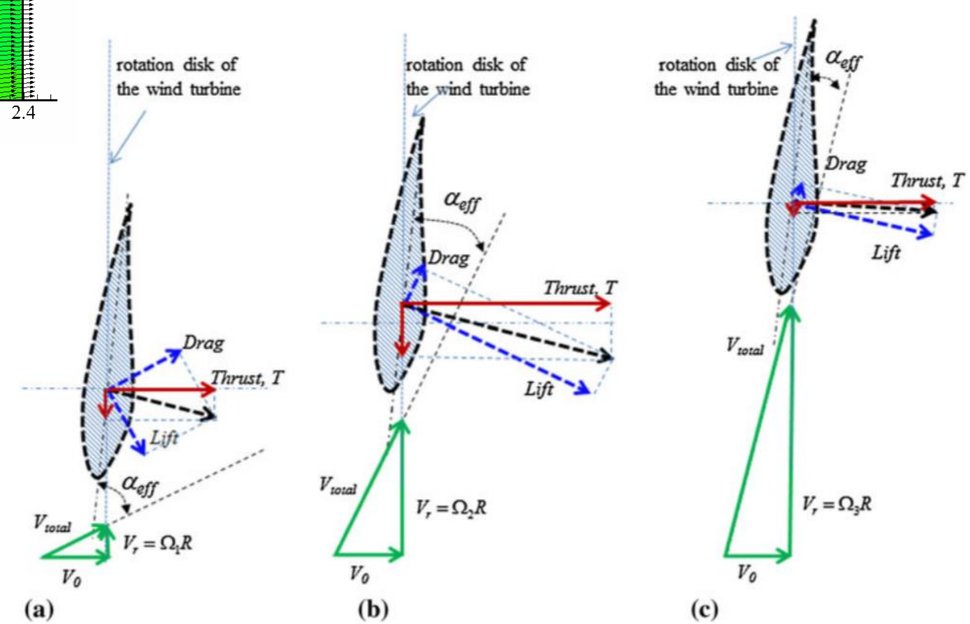
Vector Algebra

- ***Flow particle displacement and flow velocity are vectors***



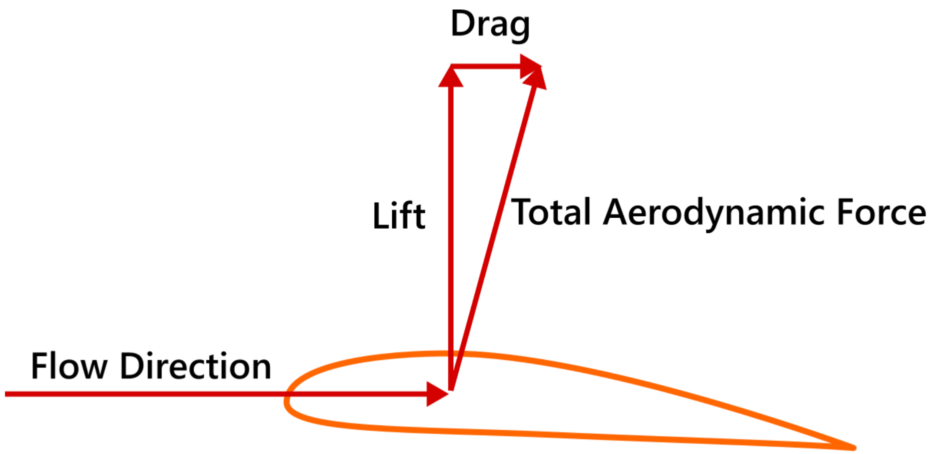


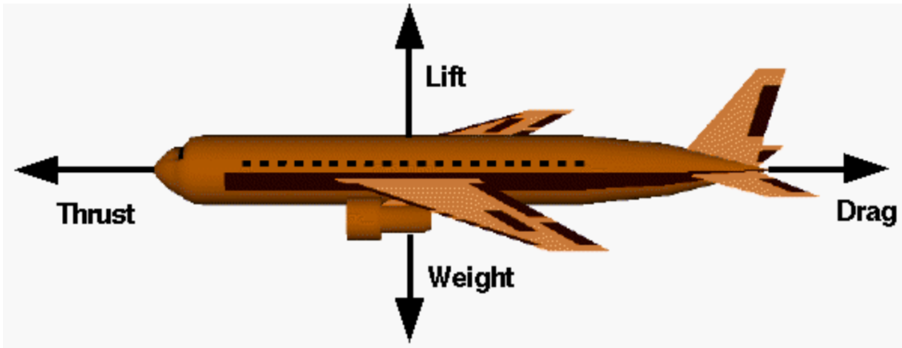




❑ VECTOR ALGEBRA

- ***Aerodynamic forces are vectors***

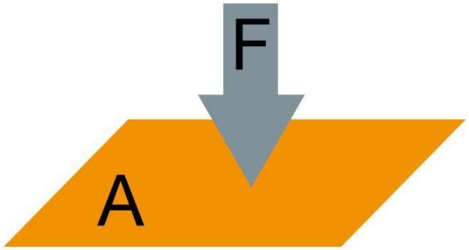


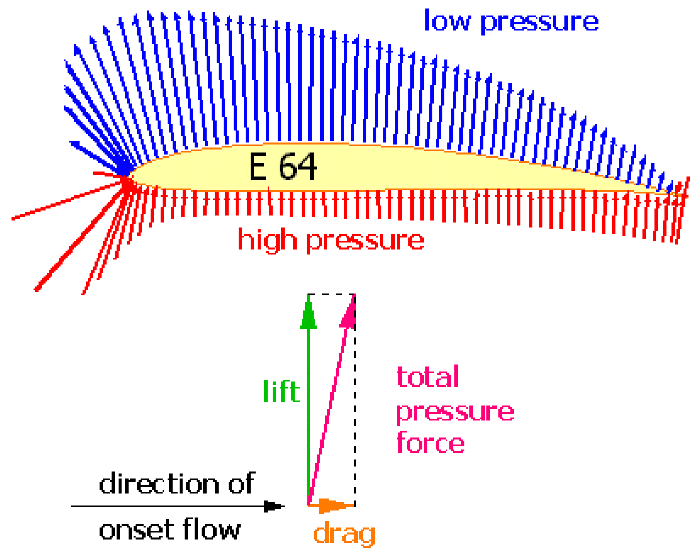


Flight Condition	Effect
Lift > Weight	Plane Rises
Weight > Lift	Plane Falls
Drag > Thrust	Plane Slows
Thrust > Drag	Plane Accelerates

- ***Air pressure is not a vector!***

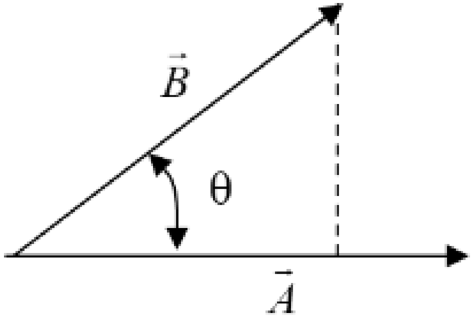
$$Pressure\ (p) = \frac{Force\ (F_n)}{Area(A)}$$





- ***Scalar Product (Dot Product)***

$$\vec{A} \bullet \vec{B} = |\vec{A}||\vec{B}| \cos\theta$$



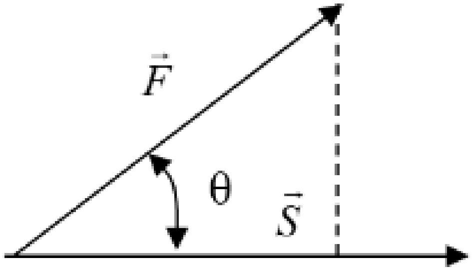
Where $|\vec{A}|, |\vec{B}|$ are the magnitude of the vectors $\vec{A}$ and $\vec{B}$.

θ ($0 \le \theta \le \pi$) is the angle between the vectors $\vec{A}$ and $\vec{B}$ when they are arranged "tail to tail".

- $|\vec{B}| \cos\theta$ is the projection of vector $\vec{B}$ to vector $\vec{A}$.
- If $\theta = \pi/2$, $\vec{A}$ and $\vec{B}$ are orthogonal to each other, and $\vec{A} \bullet \vec{B} = 0$
- Commutative: $\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}$

Example:

Work done by a force $\vec{F}$ during an infinitesimal displacement $\vec{S}$



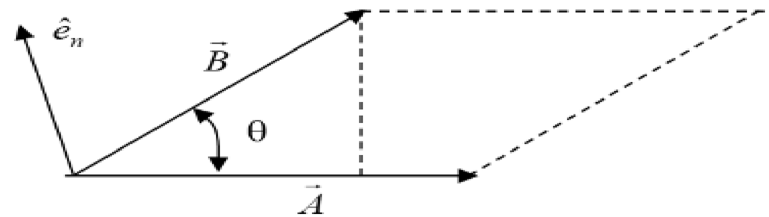
❑ VECTOR ALGEBRA

- ***Vector Product (Cross Product)***

$$\vec{A} \times \vec{B} = \left|\vec{A}\right|\left|\vec{B}\right| \sin\theta \; \hat{e}_n$$

Where $\hat{e}_n$ is the unit vector normal to the plane containing $\vec{A}$ and $\vec{B}$. Direction is determined according to the "right-hand" rule. $0 \le \theta \le \pi$

$\left|\vec{A} \times \vec{B}\right| =$ Area of the parallelogram

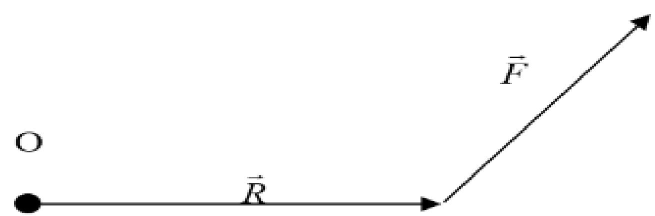


If the two vectors are parallel, that is if $\theta = 0$ or $\theta = \pi$, then $\vec{A} \times \vec{B} = \vec{0}$.

- Vector product is not commutative. i.e., $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$. However, $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

Application example:

Moment about O: $\vec{M}_O = \vec{R} \times \vec{F}$

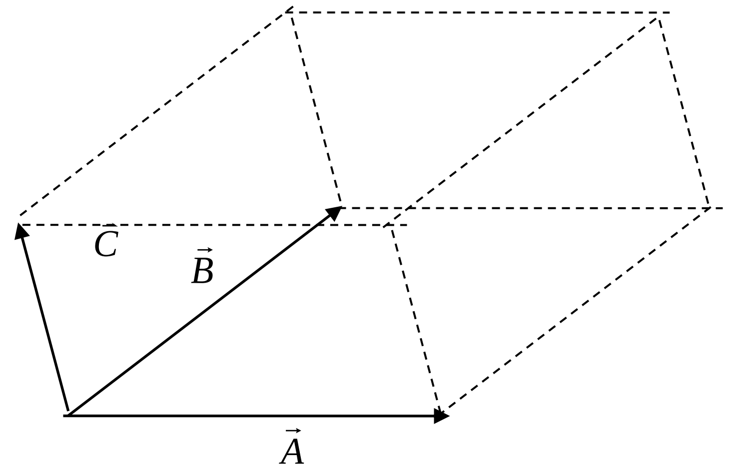


Vector Algebra

- **_Scalar Triple Product:_**

$$\vec{A}\bullet(\vec{B}\times\vec{C})=\vec{C}\bullet(\vec{A}\times\vec{B})=\vec{B}\bullet(\vec{C}\times\vec{A})$$

  - ***is the volume of the parallelepiped formed by the non-coplanar vectors***



- **_Vector Triple Product:_**

$$\vec{A}\times(\vec{B}\times\vec{C})=(\vec{A}\bullet\vec{C})\vec{B}-(\vec{A}\bullet\vec{B})\vec{C}=m\vec{B}-n\vec{C}$$

  - ***Where m, n are scalar parameters.***

$$\vec{A}\times(\vec{B}\times\vec{C})\neq(\vec{A}\times\vec{B})\times\vec{C}$$

$$\begin{aligned}(\vec{A}\times\vec{B})\times\vec{C} &= -\vec{C}\times(\vec{A}\times\vec{B})\\ &= -[(\vec{C}\bullet\vec{B})\vec{A}-(\vec{C}\bullet\vec{A})\vec{B}]\\ &= (\vec{C}\bullet\vec{A})\vec{B}-(\vec{C}\bullet\vec{B})\vec{A}\end{aligned}$$

Thus, vector $(\vec{A}\times\vec{B})\times\vec{C}$ is inside the plane of vectors $\vec{A}$ and $\vec{B}$, while the vector $\vec{A}\times(\vec{B}\times\vec{C})$ is inside the plane of vectors $\vec{B}$ and $\vec{C}$.

Therefore: $\vec{A}\times(\vec{B}\times\vec{C})\neq(\vec{A}\times\vec{B})\times\vec{C}$

❑ VECTOR ALGEBRA

2.8 Unit Vector

A vector whose magnitude is 1 is called a unit vector:

$$\hat{e}_A = \frac{\vec{A}}{\left|\vec{A}\right|}$$

Where $\left|\vec{A}\right|$ is the magnitude of the vector $\vec{A}$, and $\hat{e}_A$ is a unit vector in the direction of $\vec{A}$.

2.9 Vector Differentiation

If $\vec{A}$ and $\vec{B}$ are differentiable vector , α, t are scalars, and $\vec{U} = \vec{A} + \vec{B}$, then,

$$\frac{d\vec{U}}{dt} = \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt}$$

$$\frac{d(\alpha\vec{U})}{dt} = \frac{d\alpha}{dt}\vec{U} + \alpha\frac{d\vec{U}}{dt}$$

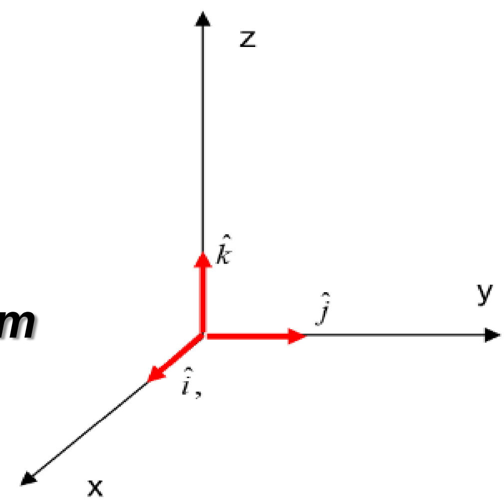
2.10 Product Rules

$$\vec{A} \bullet \vec{A} = (\left|\vec{A}\right|)^2$$

$$\hat{e}_i \bullet \hat{e}_j = \begin{cases} 0 & if\ i \neq j \\ 1 & f\ i = j \end{cases}$$

$$\left|\hat{e}_i \times \hat{e}_j\right| = \begin{cases} 0 & if\ i = j \\ 1 & f\ i \neq j \end{cases}$$

- ***Cartesian system***



❑ VECTOR ALGEBRA

2.11 Components of a Vector

In 3-D, a vector has 3 components. These 3 components are independent of each other. Consider three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$. In component form, these vectors in general can be written as:

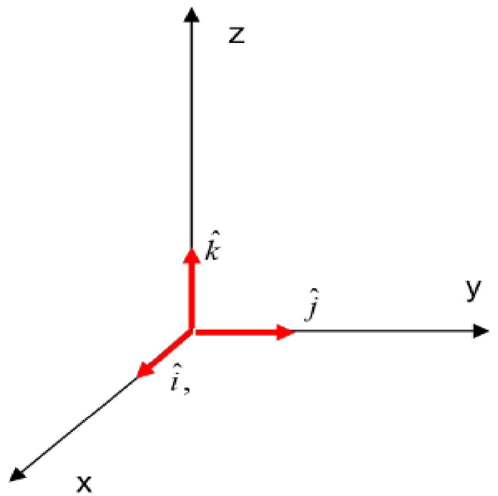
$$\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$$
$$\vec{B} = B_1\hat{e}_1 + B_2\hat{e}_2 + B_3\hat{e}_3$$
$$\vec{C} = C_1\hat{e}_1 + C_2\hat{e}_2 + C_3\hat{e}_3$$

Based on the component form, the following relations can be established:

$$\vec{A} \bullet \vec{B} = A_1B_1 + A_2B_2 + A_3B_3$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$\vec{A} \bullet (\vec{B} \times \vec{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$



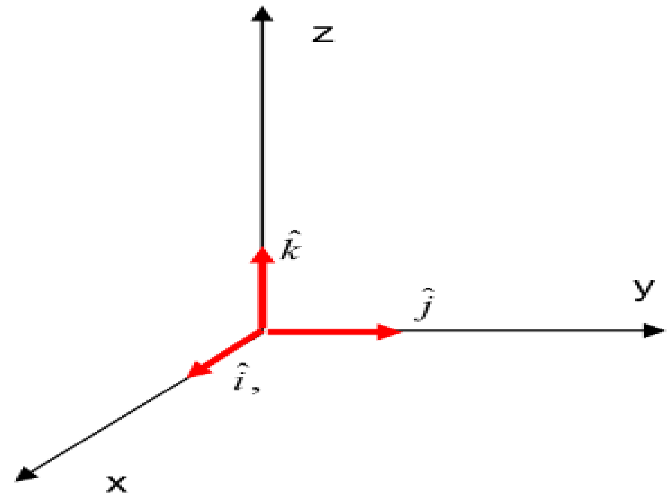
- ***Cartesian system***

$$\vec{A} \times (\vec{B} \times \vec{C}) = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ A_1 & A_2 & A_3 \\ B_2C_3 - B_3C_2 & B_3C_1 - B_1C_3 & B_1C_2 - B_2C_1 \end{vmatrix}$$

❑ Cartesian Coordinate System

Cartesian Coordinate System

Rectangular coordinate system X, Y, Z coordinates and the corresponding unit base vector $\hat{i}$, $\hat{j}$, $\hat{k}$ which are orthonormal.



$$\hat{i} \bullet \hat{j} = \hat{i} \bullet \hat{k} = \hat{j} \bullet \hat{k} = 0$$

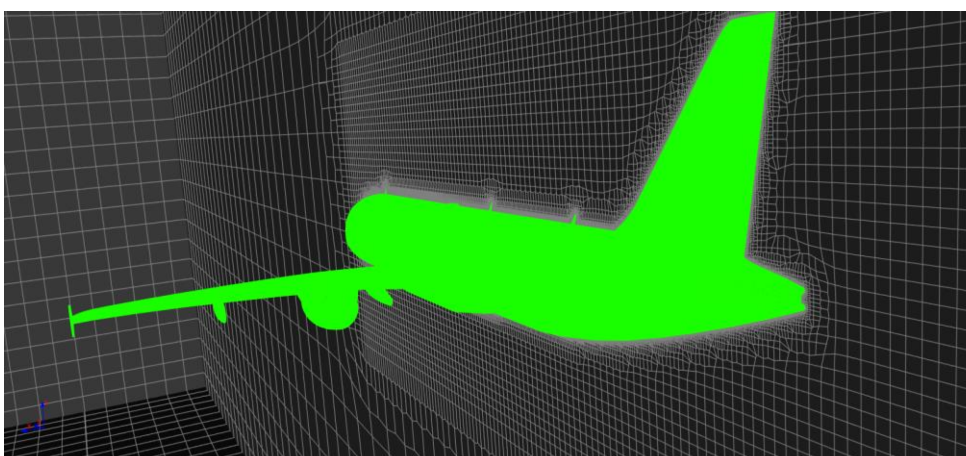
$$\hat{i} \bullet \hat{i} = \hat{j} \bullet \hat{j} = \hat{k} \bullet \hat{k} = 1$$

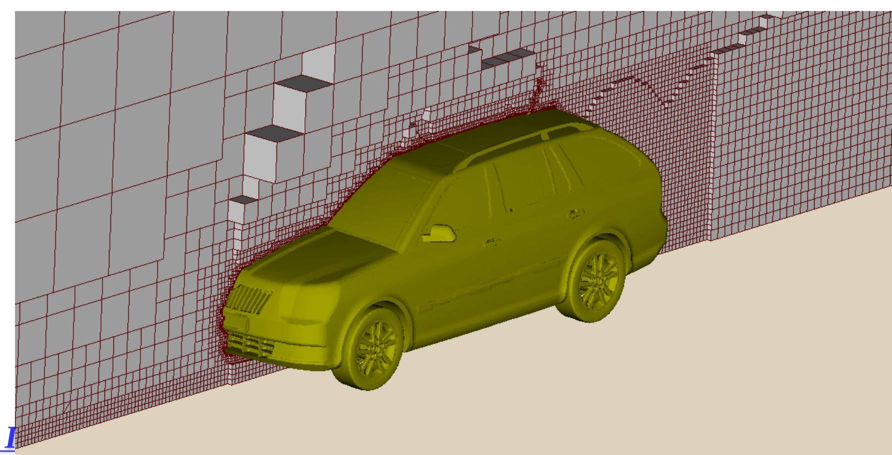
$$\hat{i} \times \hat{j} = \hat{k}; \qquad \hat{j} \times \hat{k} = \hat{i}; \qquad \hat{k} \times \hat{i} = \hat{j}$$

$$\vec{A} = A_x \ \hat{i} + A_Y \ \hat{j} + A_z \ \hat{k}$$

Where $A_x = \vec{A} \bullet \hat{i}$; $\qquad A_y = \vec{A} \bullet \hat{j}$; $\qquad A_z = \vec{A} \bullet \hat{k}$.

In other words, A_x, A_y, A_z are the components of vector $\vec{A}$,
and there are the projections of $\vec{A}$ on X, Y, Z axes respectively.





Coordinate Systems

2.11.2 Cylindrical Coordinate System

Variables in cylindrical coordinate system are (r, θ, z), and the corresponding unit base vectors are $\hat{e}_r, \hat{e}_\theta, \hat{e}_z$

$$\hat{e}_r \bullet \hat{e}_\theta = \hat{e}_r \bullet \hat{e}_z = \hat{e}_\theta \bullet \hat{e}_z = 0$$

$$\hat{e}_r \bullet \hat{e}_r = \hat{e}_\theta \bullet \hat{e}_\theta = \hat{e}_z \bullet \hat{e}_z = 1$$

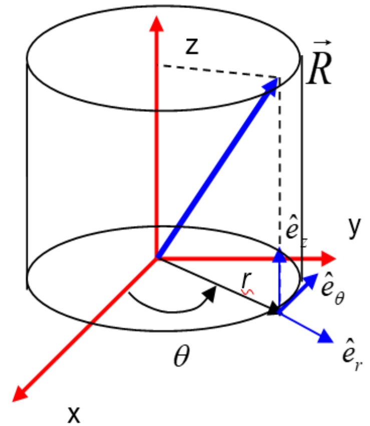
$$\hat{e}_r \times \hat{e}_\theta = \hat{e}_z; \qquad \hat{e}_\theta \times \hat{e}_z = \hat{e}_r; \qquad \hat{e}_z \times \hat{e}_r = \hat{e}_\theta$$

$$\vec{A} = A_r\,\hat{e}_r + A_\theta\,\hat{e}_\theta + A_z\,\hat{e}_z$$

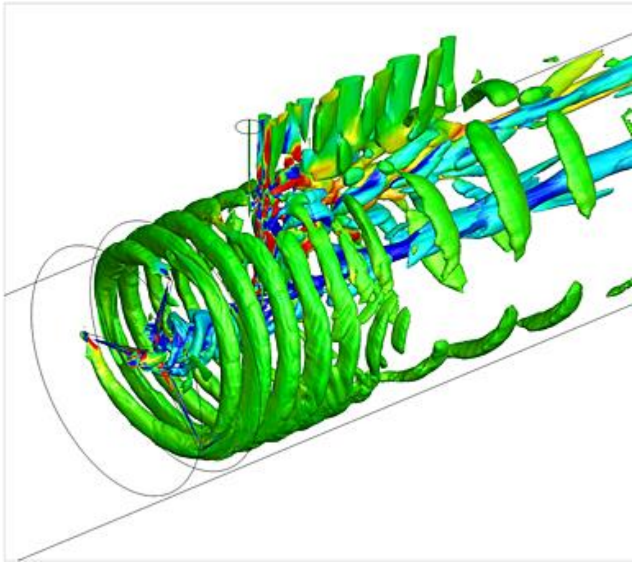
Where:

$$A_r = \vec{A} \bullet \hat{e}_r; \qquad A_\theta = \vec{A} \bullet \hat{e}_\theta; \qquad A_z = \vec{A} \bullet \hat{e}_Z .$$

In other words, A_r, A_θ, A_z are the components of vector $\vec{A}$.

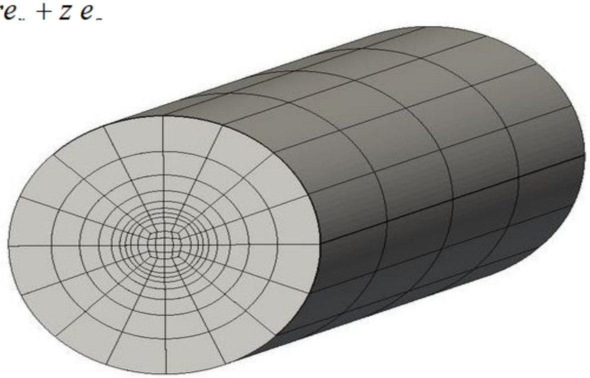


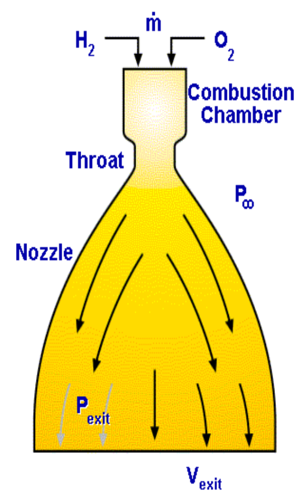
Cylindrical coordinate system (r, θ, z)



The position vector in Cartesian system is given as:

$$\vec{R} = r\hat{e}_r + z\,\hat{e}_z$$





2.11.3 Spherical Coordinate System

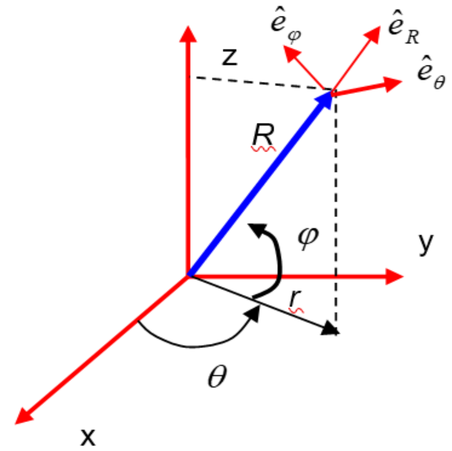
Variables in spherical coordinate system are (R,θ,φ),
and the corresponding unit base vectors are $\hat{e}_R$, $\hat{e}_\theta$, $\hat{e}_\varphi$

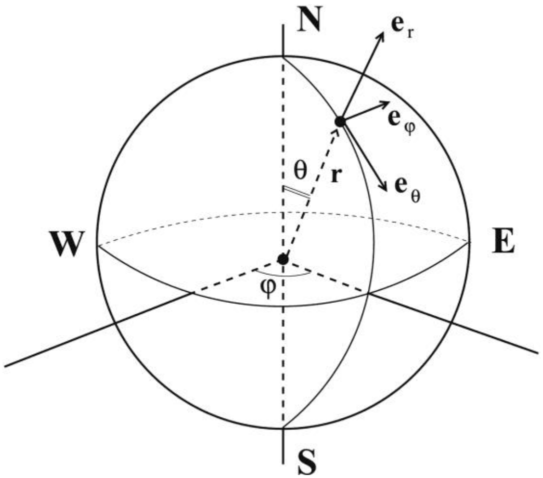
$$\vec{A} = A_R\ \hat{e}_R + A_\theta\ \hat{e}_\theta + A_\varphi\ \hat{e}_\varphi$$

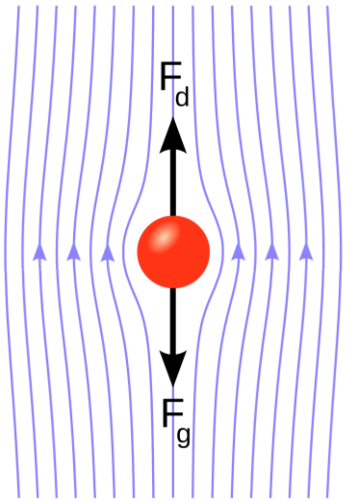
Where:

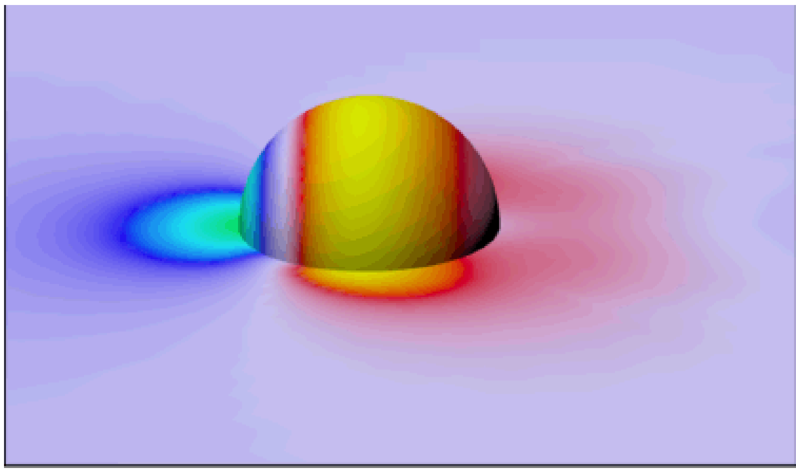
$$A_r = \vec{A} \bullet \hat{e}_r; \qquad A_\theta = \vec{A} \bullet \hat{e}_\theta; \qquad A_z = \vec{A} \bullet\ \hat{e}_Z.$$

In other words, A_r, A_θ, A_φ are the components of vector $\vec{A}$.









❑ Relationship between Different Coordinate Systems

2.12 Relationship between Coordinate Systems

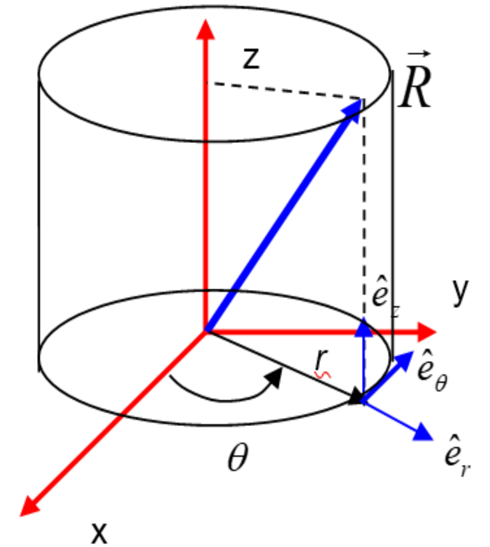
2.12.1 General Transformation

(q_1, q_2, q_3) are the general coordinates of a 3-D coordinate system.

$$
\begin{aligned}
q_1 &= q_1(x, y, z) \\
q_2 &= q_2(x, y, z) \\
q_3 &= q_3(x, y, z)
\end{aligned}
$$

2.12.2 General Inverse Transformation

$$
\begin{aligned}
x &= x(q_1, q_2, q_3) \\
y &= y(q_1, q_2, q_3) \\
z &= z(q_1, q_2, q_3)
\end{aligned}
$$



Cylindrical coordinate system *(r,θ, z)*

<u>For example:</u>
Transformation equations between the Cartesian coordinate and cylindrical coordinate system are:

$$
\begin{aligned}
r &= x^2 + y^2 && (0 \le r \le \infty) \\
\theta &= \arctan(y/x) && (0 \le \theta \le 2\pi) \\
z &= z && (-\infty < z < \infty)
\end{aligned}
$$

The inverse transformation equation will be:

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
z &= z
\end{aligned}
$$

2.13 Scale factors, Unit Vectors and their Derivatives

2.13.1 Scale factors

Scale factor defines the relationship between coordinates and distance along coordinates.

2.13.2 General Coordinate System

A position vector $\vec{R}$ in Cartesian coordinate system is given by:

$$\vec{R} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

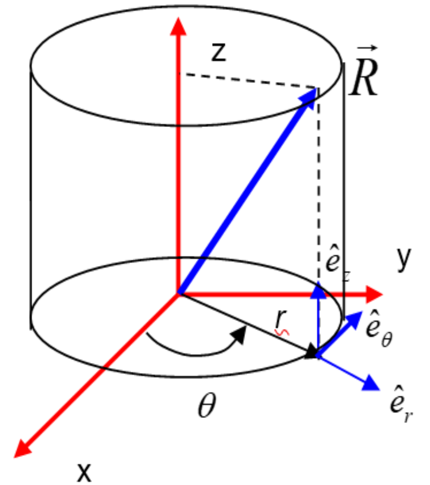
Using the inverse transformation, in a general coordinate system, the position vector can also be written as:

$$\vec{R} = x(q_1, q_2, q_3)\,\hat{i} + y(q_1, q_2, q_3)\,\hat{j} + z(q_1, q_2, q_3)\,\hat{k}$$

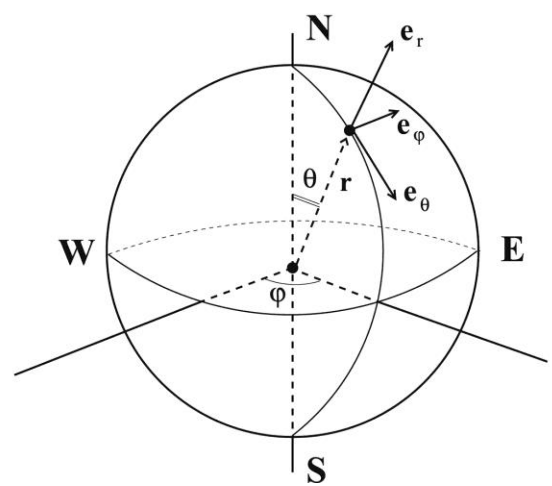
The variation of the position vector along the coordinate direction defines the following relations:

$$\frac{\partial \vec{R}}{\partial q_1} = h_1 \hat{e}_1$$
$$\frac{\partial \vec{R}}{\partial q_2} = h_2 \hat{e}_2$$
$$\frac{\partial \vec{R}}{\partial q_3} = h_3 \hat{e}_3$$

Where h_1, h_2, h_3 are the scale factors and $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are the unit vector in the q_1, q_2, q_3 direction, respectively.



Cylindrical coordinate system (r, θ, z)



❑ Scale Factors and Derivatives of Unit Vectors

2.14.1 Cartesian Coordinate System

In Cartesian coordinate system, unit vectors are:

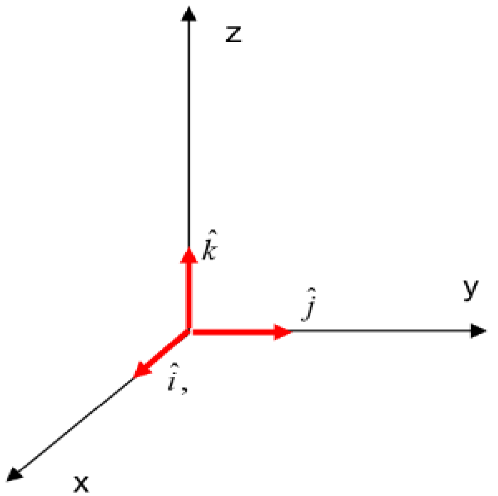
$$\hat{e}_1 = \hat{i}$$
$$\hat{e}_2 = \hat{j}$$
$$\hat{e}_3 = \hat{k}$$

A position vector $\vec{R}$ in Cartesian coordinate system is given by:

$$\vec{R} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

Therefore:

$$\frac{\partial \vec{R}}{\partial q_1} = h_1\,\hat{e}_1 = \hat{i} \quad \Rightarrow \quad h_1 = 1$$
$$\frac{\partial \vec{R}}{\partial q_2} = h_2\,\hat{e}_2 = \hat{j} \quad \Rightarrow \quad h_2 = 1\ ;$$
$$\frac{\partial \vec{R}}{\partial q_3} = h_3\,\hat{e}_3 = \hat{k} \quad \Rightarrow \quad h_3 = 1$$



- ***Cartesian system***

Since the unit vectors are fixed in magnitude and direction in Cartesian coordinate system, therefore:

$$\frac{\partial \hat{i}}{\partial x} = \frac{\partial \hat{j}}{\partial x} = \frac{\partial \hat{k}}{\partial x} = 0$$
$$\frac{\partial \hat{i}}{\partial y} = \frac{\partial \hat{j}}{\partial y} = \frac{\partial \hat{k}}{\partial y} = 0$$
$$\frac{\partial \hat{i}}{\partial z} = \frac{\partial \hat{j}}{\partial z} = \frac{\partial \hat{k}}{\partial z} = 0$$

Transformation between Cartesian System & Cylindrical System

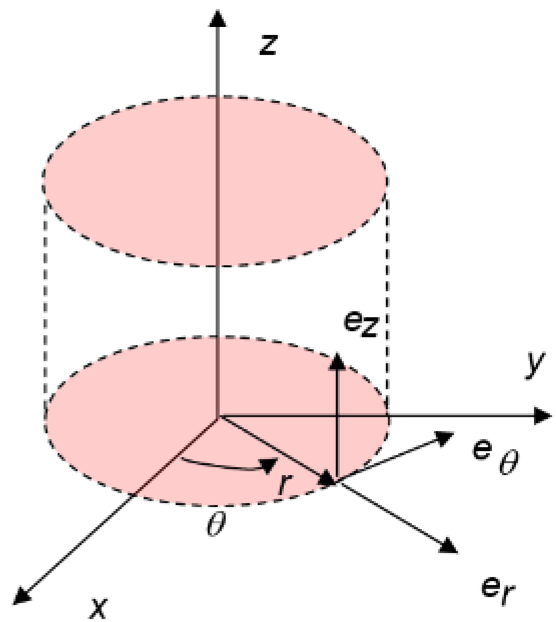
Cylindrical Coordinate System

A point P in space is given by $p(q_1, q_2, q_3)$ or $p(r, \theta, z)$ with base vector $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ given by $(\hat{e}_r, \hat{e}_\theta, \hat{e}_z)$.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = Z \end{cases}$$

$r \geq 0, \quad 0 \leq \theta \leq 2\pi; \quad -\infty \leq Z \leq \infty$

$$dx = dr\cos\theta - r\sin\theta\, d\theta$$
$$dy = dr\sin\theta + r\cos\theta\, d\theta$$
$$dz = dz$$



Cylindrical system
(R,θ,z)

$$(ds)^2 = h_1^{\,2} dq_1^{\,2} + h_2^{\,2} dq_2^{\,2} + h_3^{\,2} dq_3^{\,2}$$
$$= (dx)^2 + (dy)^2 + (dz)^2$$
$$= (\cos\theta\, dr)^2 - 2r\sin\theta\, d\theta\cos\theta\, dr + (r\sin\theta\, d\theta)^2$$
$$+ (\sin\theta\, dr)^2 + 2r\sin\theta\cos\theta\, dr\, d\theta + (r\cos\theta\, d\theta)^2 + (dz)^2$$
$$= (dr)^2 + r^2(d\theta)^2 + (dz)^2$$

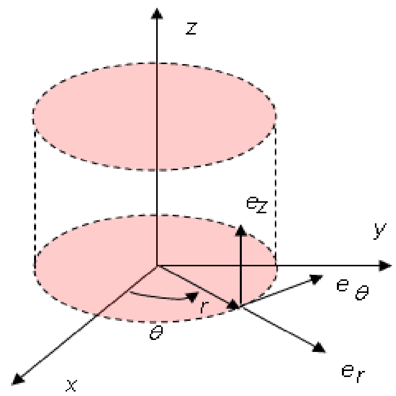
Therefore: $h_1 = 1; \qquad h_2 = r; \qquad h_3 = 1$

Scale Factors and Derivatives of Unit Vectors

r - Direction:

$$h_r\,\hat{e}_r = \frac{\partial \vec{R}}{\partial r} = \frac{\partial x}{\partial r}\,\hat{i} + \frac{\partial y}{\partial r}\,\hat{j} + \frac{\partial z}{\partial r}\,\hat{k} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} + 0\,\hat{k}$$

$$(h_r\,\hat{e}_r)\bullet(h_r\,\hat{e}_r) = (h_r)^2 = \cos^2\theta + \sin^2\theta = 1 \quad\Rightarrow\quad \begin{cases} h_r = 1 \\ \hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \end{cases}$$



θ - Direction:

$$h_\theta\,\hat{e}_\theta = \frac{\partial \vec{R}}{\partial \theta} = \frac{\partial x}{\partial \theta}\,\hat{i} + \frac{\partial y}{\partial \theta}\,\hat{j} + \frac{\partial z}{\partial \theta}\,\hat{k} = -r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j} + 0\,\hat{k}$$

$$(h_\theta\,\hat{e}_\theta)\bullet(h_\theta\,\hat{e}_\theta) = (h_\theta)^2 = r^2(\sin^2\theta + \cos^2\theta) = r^2 \quad\Rightarrow\quad \begin{cases} h_\theta = r \\ \hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} \end{cases}$$

Z - Direction:

$$h_Z\,\hat{e}_Z = \frac{\partial \vec{R}}{\partial Z} = \frac{\partial x}{\partial Z}\,\hat{i} + \frac{\partial y}{\partial Z}\,\hat{j} + \frac{\partial z}{\partial Z}\,\hat{k} = 0\,\hat{i} + 0\,\hat{j} + 1\,\hat{k}$$

$$(h_Z\,\hat{e}_Z)\bullet(h_Z\,\hat{e}_Z) = (h_Z)^2 = 1^2 \quad\Rightarrow\quad \begin{cases} h_Z = 1 \\ \hat{e}_Z = \hat{k} \end{cases}$$

Summarize:

$$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \qquad h_r = 1$$
$$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} \quad ; \quad h_\theta = r$$
$$\hat{e}_Z = \hat{k} \qquad h_z = 1$$

Transformation relationship

$$\begin{bmatrix}\hat{i}\\ \hat{j}\\ \hat{k}\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{e}_r\\ \hat{e}_\theta\\ \hat{e}_Z\end{bmatrix} \quad ;or \quad \begin{bmatrix}\hat{e}_r\\ \hat{e}_\theta\\ \hat{e}_Z\end{bmatrix} = \begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{i}\\ \hat{j}\\ \hat{k}\end{bmatrix}$$

Derivatives of the unit vectors:

$$\frac{\partial \hat{e}_r}{\partial r} = 0 \qquad \frac{\partial \hat{e}_\theta}{\partial r} = 0 \qquad \frac{\partial \hat{e}_Z}{\partial r} = 0$$
$$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta \quad ; \quad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r \quad ; \quad \frac{\partial \hat{e}_Z}{\partial \theta} = 0$$
$$\frac{\partial \hat{e}_r}{\partial z} = 0 \qquad \frac{\partial \hat{e}_\theta}{\partial z} = 0 \qquad \frac{\partial \hat{e}_Z}{\partial z} = 0$$

Example

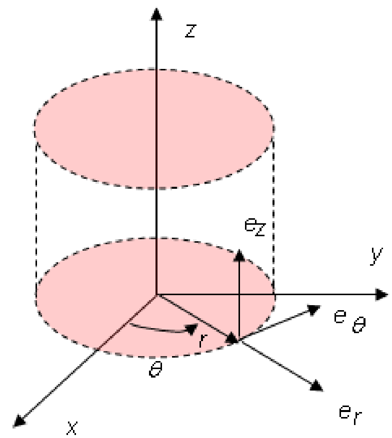
Example: If $\vec{R} = \vec{R}(t) = r\hat{e}_r + z\hat{e}_z$ is the position vector of a particle in cylindrical coordinates, obtain expression for velocity vector, $\vec{V}$, and acceleration vector, $\vec{a}$, at that point.

Since $\hat{e}_r = \hat{e}_r(r,\theta,z)$, then, $d\hat{e}_r = \frac{\partial \hat{e}_r}{\partial \theta}d\theta + \frac{\partial \hat{e}_r}{\partial r}dr + \frac{\partial \hat{e}_r}{\partial z}dz$

Therefore $\frac{d\hat{e}_r}{dt} = \frac{\partial \hat{e}_r}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial \hat{e}_r}{\partial r}\frac{dr}{dt} + \frac{\partial \hat{e}_r}{\partial z}\frac{dz}{dt}$

Similarly, $\frac{d\hat{e}_\theta}{dt} = \frac{\partial \hat{e}_\theta}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial \hat{e}_\theta}{\partial r}\frac{dr}{dt} + \frac{\partial \hat{e}_\theta}{\partial z}\frac{dz}{dt}$

$$\frac{d\hat{e}_z}{dt} = \frac{\partial \hat{e}_z}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial \hat{e}_z}{\partial r}\frac{dr}{dt} + \frac{\partial \hat{e}_z}{\partial z}\frac{dz}{dt}$$



$$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$
$$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$
$$\hat{e}_z = \hat{k}$$

Derivatives of the unit vectors:

$$\frac{\partial \hat{e}_r}{\partial r} = 0 \qquad \frac{\partial \hat{e}_\theta}{\partial r} = 0 \qquad \frac{\partial \hat{e}_z}{\partial r} = 0$$
$$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta \quad ; \quad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r \quad ; \quad \frac{\partial \hat{e}_z}{\partial \theta} = 0$$
$$\frac{\partial \hat{e}_r}{\partial z} = 0 \qquad \frac{\partial \hat{e}_\theta}{\partial z} = 0 \qquad \frac{\partial \hat{e}_z}{\partial z} = 0$$

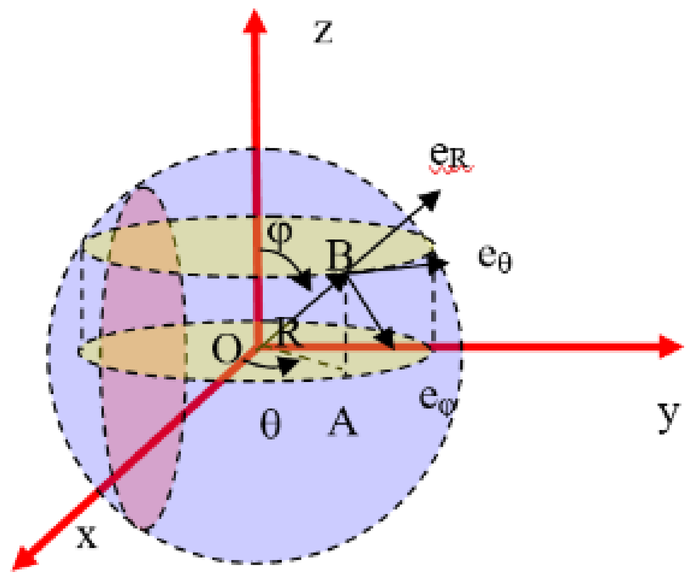
$$\vec{V} = \frac{d\vec{R}}{dt} = r\frac{d\hat{e}_r}{dt} + \frac{dr}{dt}\hat{e}_r + \frac{dz}{dt}\hat{e}_z + z\frac{d\hat{e}_z}{dt}$$
$$= r\left(\frac{\partial \hat{e}_r}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial \hat{e}_r}{\partial r}\frac{dr}{dt} + \frac{\partial \hat{e}_r}{\partial z}\frac{dz}{dt}\right) + \frac{dr}{dt}\hat{e}_r$$
$$+ \frac{dz}{dt}\hat{e}_z + z\left(\frac{\partial \hat{e}_z}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial \hat{e}_z}{\partial r}\frac{dr}{dt} + \frac{\partial \hat{e}_z}{\partial z}\frac{dz}{dt}\right)$$
$$= r\hat{e}_\theta\frac{d\theta}{dt} + \frac{dr}{dt}\hat{e}_r + \frac{dz}{dt}\hat{e}_z$$

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d\left(r\hat{e}_\theta\frac{d\theta}{dt} + \frac{dr}{dt}\hat{e}_r + \frac{dz}{dt}\hat{e}_z\right)}{dt}$$
$$= \frac{dr}{dt}\hat{e}_\theta\frac{d\theta}{dt} + r\frac{d\hat{e}_\theta}{dt}\frac{d\theta}{dt} + r\hat{e}_\theta\frac{d^2\theta}{dt^2} + \frac{d^2r}{dt^2}\hat{e}_r + \frac{dr}{dt}\frac{d\hat{e}_r}{dt} + \frac{d^2z}{dt^2}\hat{e}_z + \frac{dz}{dt}\frac{d\hat{e}_z}{dt}$$
$$= \frac{dr}{dt}\frac{d\theta}{dt}\hat{e}_\theta + r\frac{d\theta}{dt}\left(\frac{\partial \hat{e}_\theta}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial \hat{e}_\theta}{\partial r}\frac{dr}{dt} + \frac{\partial \hat{e}_\theta}{\partial z}\frac{dz}{dt}\right) + r\hat{e}_\theta\frac{d^2\theta}{dt^2} + \frac{d^2r}{dt^2}\hat{e}_r +$$
$$\frac{dr}{dt}\left(\frac{\partial \hat{e}_r}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial \hat{e}_r}{\partial r}\frac{dr}{dt} + \frac{\partial \hat{e}_r}{\partial z}\frac{dz}{dt}\right) + \frac{d^2z}{dt^2}\hat{e}_z + \frac{dz}{dt}\left(\frac{\partial \hat{e}_z}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial \hat{e}_z}{\partial r}\frac{dr}{dt} + \frac{\partial \hat{e}_z}{\partial z}\frac{dz}{dt}\right)$$
$$= \frac{dr}{dt}\frac{d\theta}{dt}\hat{e}_\theta - r\frac{d\theta}{dt}\frac{d\theta}{dt}\hat{e}_r + r\hat{e}_\theta\frac{d^2\theta}{dt^2} + \frac{d^2r}{dt^2}\hat{e}_r + \frac{dr}{dt}\frac{d\theta}{dt}\hat{e}_\theta + \frac{d^2z}{dt^2}\hat{e}_z$$
$$= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{e}_r + \left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right]\hat{e}_\theta + \frac{d^2z}{dt^2}\hat{e}_z$$

❑ TRANSFORMATION BETWEEN CARTESIAN SYSTEM & SPHERIC SYSTEM

Scale factors and unit vectors in Spherical coordinate system (R, φ, θ)

$$\vec{OB} = R\,\hat{e}_R$$
$$OA = R\sin\varphi$$
$$x = R\sin\varphi\cos\theta$$
$$x = r\sin\varphi\sin\theta$$
$$z = r\cos\varphi$$



$(R - \varphi - \theta)$

$$h_R = 1$$
$$\hat{e}_R = \sin\varphi\cos\theta\,\hat{i} + \sin\varphi\sin\theta\,\hat{j} + \cos\varphi\,\hat{k}$$

$$h_\varphi = R$$

$$\hat{e}_\varphi = \cos\varphi\cos\theta\,\hat{i} + \cos\varphi\sin\theta\,\hat{j} - \sin\varphi\,\hat{k}$$

$$h_\theta = R\sin\varphi$$
$$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$