Lecture # 3: Review of Vector Calculus

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2. 15 Vector Calculus

2.15.1 Del, the Vector Differential Operator:

$$\nabla = \hat{e}_1 \frac{\partial}{h_1 \partial q_1} + \hat{e}_2 \frac{\partial}{h_2 \partial q_2} + \hat{e}_3 \frac{\partial}{h_3 \partial q_3}$$

Where $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are three orthogonal unit vectors, h_1, h_2, h_3 are the scale factors along the coordinate axes q_1, q_2, q_3.

2.15.2. Cartesian Coordinate System

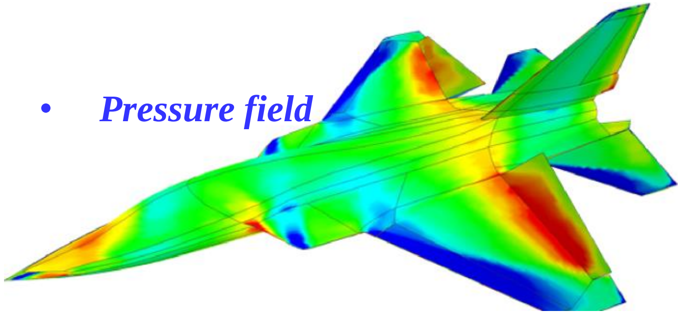
$$\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$

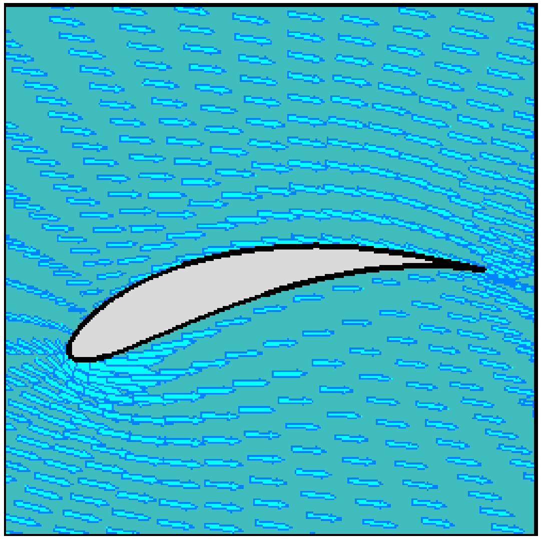
2.15.3. Cylindrical Coordinate System

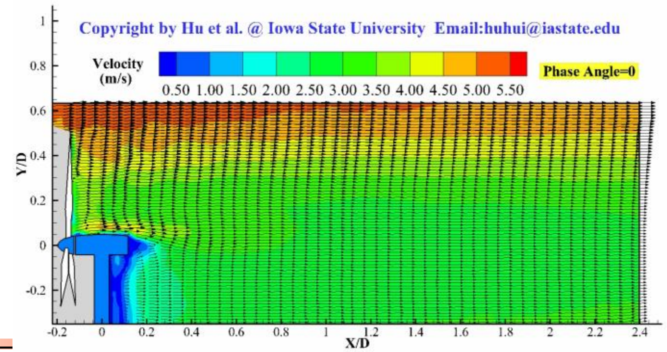
$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}$$

2.16 Scalar and Vector Field to Describe Physical Problems

- *Pressure field*







Type of functions

- A scalar as a function of a scalar, for example: $\mu = \mu(T)$
- A vector as a function of a scalar, for example: $\vec{R} = \vec{R}(t)$
- A scalar as a function of a vector, for example: $T = T(\vec{R})$
- A vector as a function of a vector, for example: $\vec{V} = \vec{V}(\vec{R})$

General description: $\phi = \phi(\vec{R},t)$ and $\vec{A} = \vec{A}(\vec{R},t)$

Scalar field: A scalar quantity given as a function of coordinate space and time, *t*, is called scalar field.

For examples: $p = p(x,y,z,t) = p(\vec{R},t)$ and $T = T(x,y,z,t) = T(\vec{R},t)$

Vector field: A vector quantity given as a function of coordinate space and time, *t*, is called vector field.

For examples: $\vec{V} = \vec{V}(x,y,z,t) = \vec{V}(\vec{R},t)$ and $\vec{M} = \vec{M}(x,y,z,t) = \vec{M}(\vec{R},t)$

- In general, a field denotes a region throughout which a quantity is defined as a function of location within the region and time.

- If the quantity is independent of time, the field is steady or stationary.

❑ Gradient

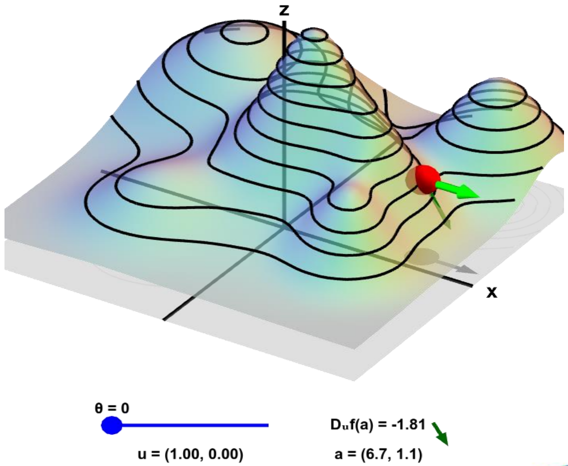
2.17 Gradient

Gradient is a vector generated by the differentiation of a scalar function

Let $\phi = \phi(\vec{R}) = \phi(q_1, q_2, q_3)$

Since ϕ is a function of a vector $\vec{R}$, there are infinite number of directions in which to take the increment $\Delta\vec{R}$. The total change in ϕ, $d\phi$, would in general be different in different directions.

Spatial derivative of ϕ at a point is expressed as derivatives of ϕ in three independent directions. Gradient of a scalar is a vector.



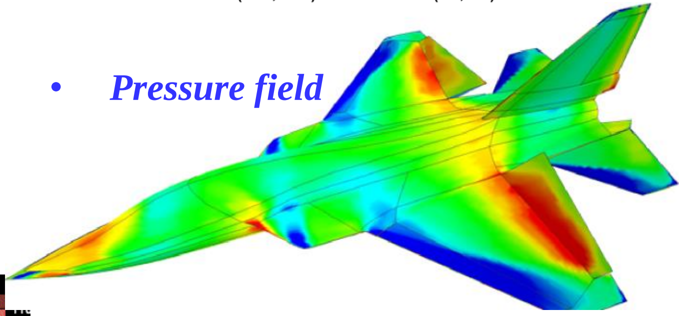
2.17.1 Concept of Gradient

At any point, the gradient of a scalar function ϕ is equal in magnitude and direction to the greatest derivative of ϕ with respect to distance at the point.

Rate of change of scalar ϕ along two paths are of special importance:

1. Path along which the scalar is constant. (Isolines)
2. Path along which the rate of change of the scalar is the maximum (gradient line)

- *Pressure field*



2.17.2 General Coordinate System:

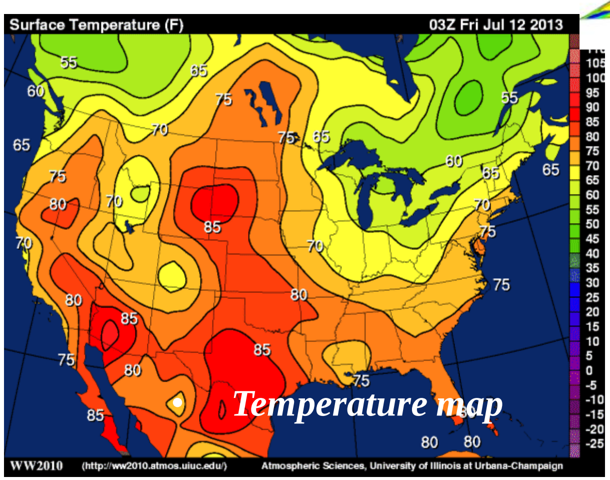
$$\nabla\phi = \hat{e}_1 \frac{\partial\phi}{h_1 \partial q_1} + \hat{e}_2 \frac{\partial\phi}{h_2 \partial q_2} + \hat{e}_3 \frac{\partial\phi}{h_3 \partial q_3}$$

2.17.2 Cartesian Coordinate System:

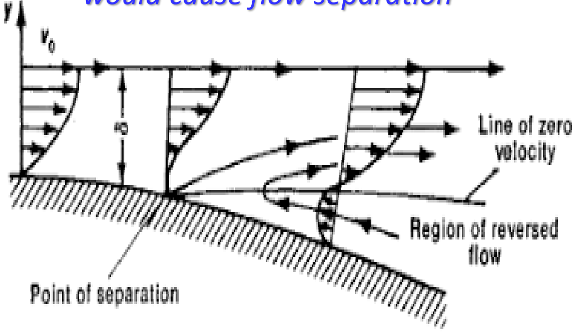
$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

2.17. 3 Cylindrical Coordinate System

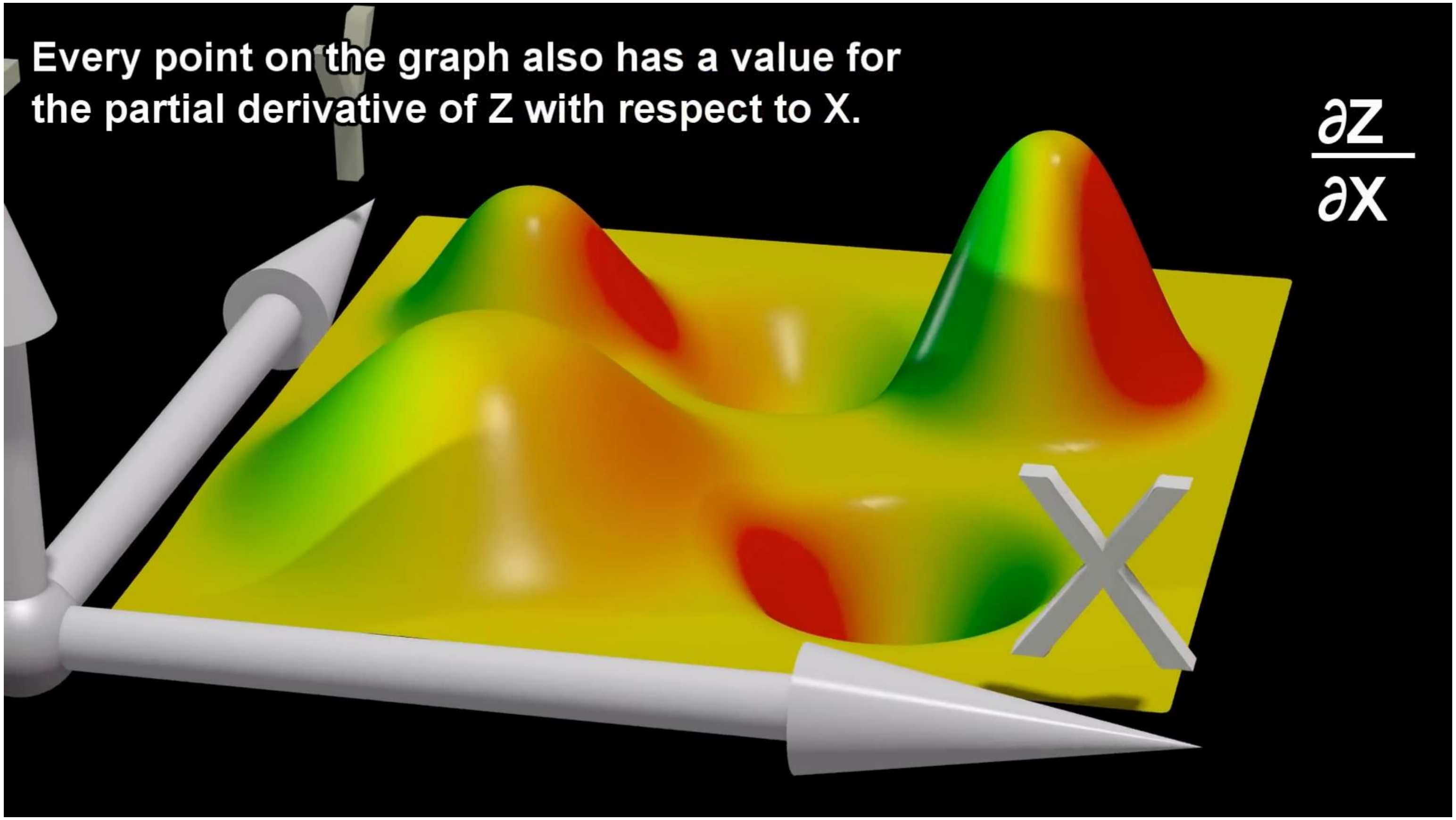
$$\nabla\phi = \hat{e}_r \frac{\partial\phi}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial\phi}{\partial\theta} + \hat{e}_z \frac{\partial\phi}{\partial z}$$



- *Great negative pressure gradient would cause flow separation*



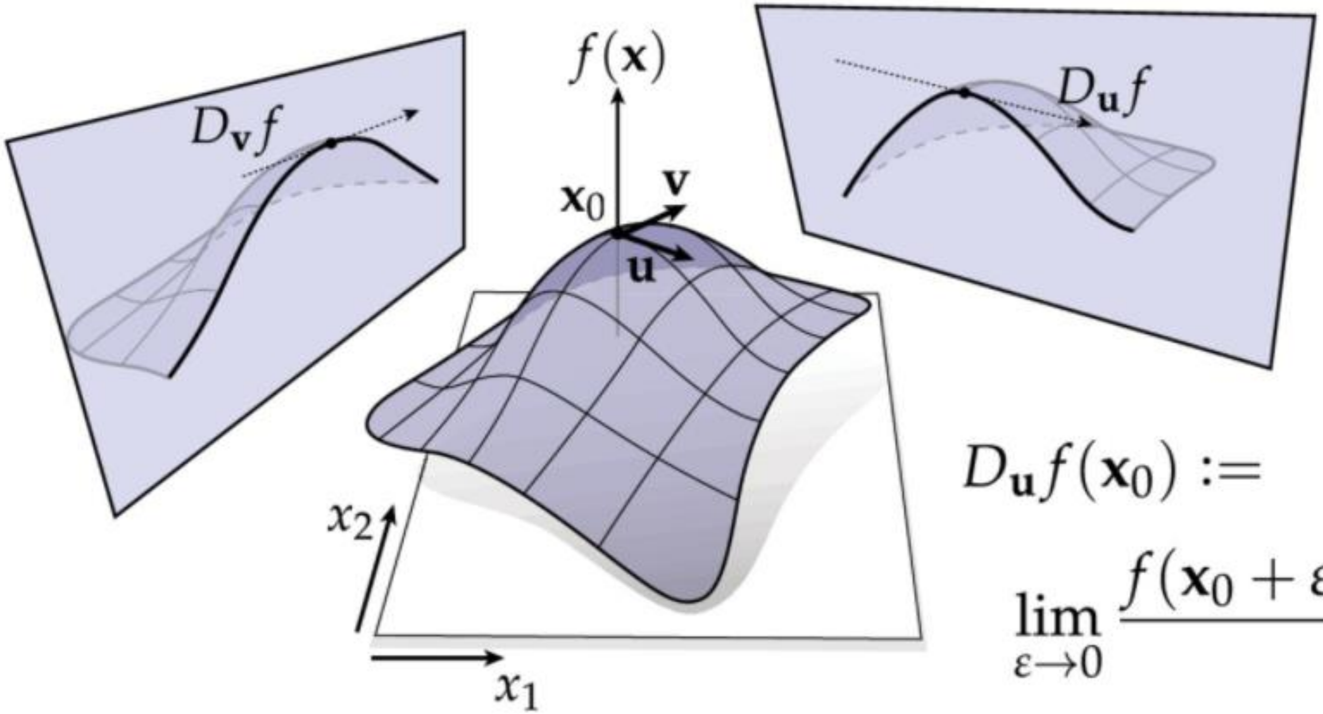
❑ Gradient


Every point on the graph also has a value for
the partial derivative of Z with respect to X.
∂Z
∂X
X

❑ Gradient

Directional Derivative

- One way: suppose we have a function f(x1,x2)
  - Take a "slice" through the function along some line
  - Then just apply the usual derivative!
  - Called the directional derivative



take a small step along u

$$D_{\mathbf{u}} f(\mathbf{x}_0) := \lim_{\varepsilon \to 0} \frac{f(\mathbf{x}_0 + \varepsilon \mathbf{u}) - f(\mathbf{x}_0)}{\varepsilon}$$

❑ Gradient

2.18 Concept of directional derivative

Consider the change of ϕ over the directed distance $d\vec{R}$ (i, e., $\vec{R} \to \vec{R} + \Delta\vec{R}$), find $d\phi = \lim_{\Delta\vec{R}\to 0}[\phi(\vec{R} + \Delta\vec{R}) - \phi(\vec{R})] = ?$

From the total differential formula of the calculus, the first order differential in ϕ will be

$$d\phi = \frac{\partial\phi}{\partial q_1}dq_1 + \frac{\partial\phi}{\partial q_2}dq_2 + \frac{\partial\phi}{\partial q_3}dq_3 + high\ \ Orders\ \ terms$$

$$\approx \frac{\partial\phi}{\partial q_1}dq_1 + \frac{\partial\phi}{\partial q_2}dq_2 + \frac{\partial\phi}{\partial q_3}dq_3 = \frac{1}{h_1}\frac{\partial\phi}{\partial q_1}h_1 dq_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial q_2}h_2 dq_2 + \frac{1}{h_3}\frac{\partial\phi}{\partial q_3}h_3 dq_3$$

$$= \frac{1}{h_1}\frac{\partial\phi}{\partial q_1}ds_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial q_2}ds_2 + \frac{1}{h_3}\frac{\partial\phi}{\partial q_3}ds_3$$

Since $d\vec{R} = d\vec{S} = ds_1\,\hat{e}_1 + ds_2\,\hat{e}_2 + ds_3\,\hat{e}_3$

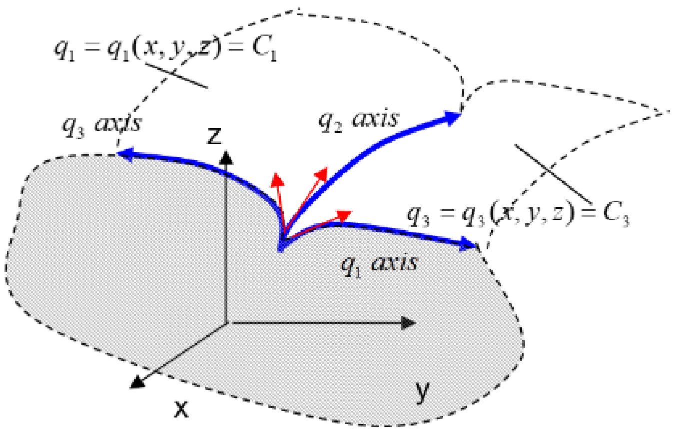
Now introduce a vector $[\frac{1}{h_1}\frac{\partial\phi}{\partial q_1}, \frac{1}{h_2}\frac{\partial\phi}{\partial q_2}, \frac{1}{h_3}\frac{\partial\phi}{\partial q_3}]$ denoted by $\nabla\phi$ in the curvilinear orthogonal coordinate system with unit vector $(\hat{e}_1, \hat{e}_2. \hat{e}_3)$, then,

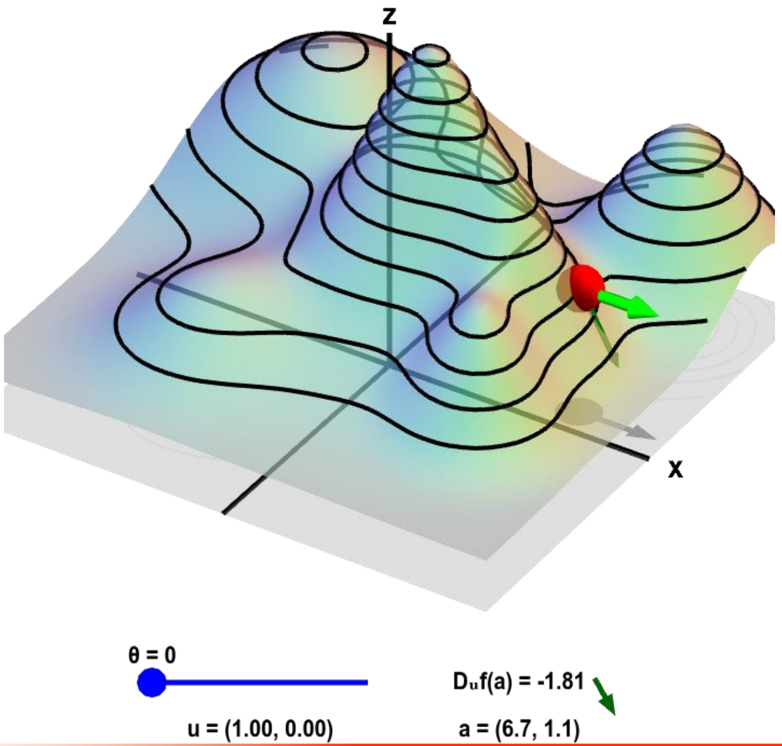
$$d\phi = [\frac{1}{h_1}\frac{\partial\phi}{\partial q_1}, \frac{1}{h_2}\frac{\partial\phi}{\partial q_2}, \frac{1}{h_3}\frac{\partial\phi}{\partial q_3}] \bullet [ds_1, ds_2, ds_3]$$

$$= \nabla\phi \bullet d\vec{R} = \nabla\phi \bullet d\vec{S}$$

Since $d\vec{S} = dS \cdot \hat{e}_S$ therefore, $\frac{d\phi}{dS} = \nabla\phi \bullet \hat{e}_s$

- Directional derivative of $\phi(\vec{R})$ in any chosen direction is equal to the component of the gradient vector in that direction.
- $\frac{d\phi}{dS} = \nabla\phi \bullet \hat{e}_s$ is a maximum when $\nabla\phi \bullet \hat{e}_s$ is a maximum. i.e., when $\nabla\phi$ and $\hat{e}_s$ are in the same direction. In other words, $\nabla\phi$ is the direction of maximum changes of ϕ and $|\nabla\phi|$ is the magnitude of the change.
- The greatest rate of change of ϕ with respect to coordinate space at a point take place in the direction of $\nabla\phi$ and has the magnitude of the vector $\nabla\phi$.



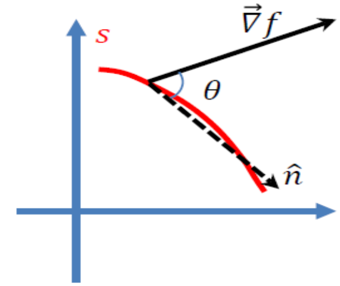


Gradient

Gradient and level contours

To maximize $\frac{df}{ds} \rightarrow \theta = 0$

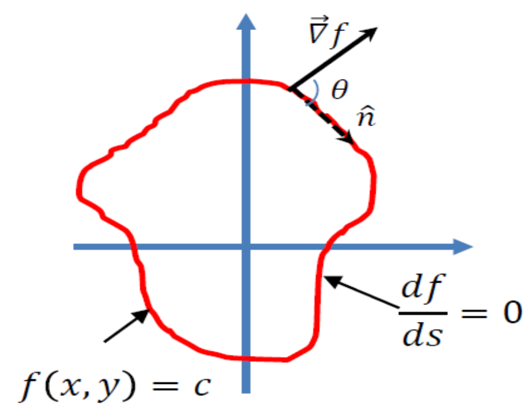
$\vec{\nabla} f$ is in direction of max rate of increase in f

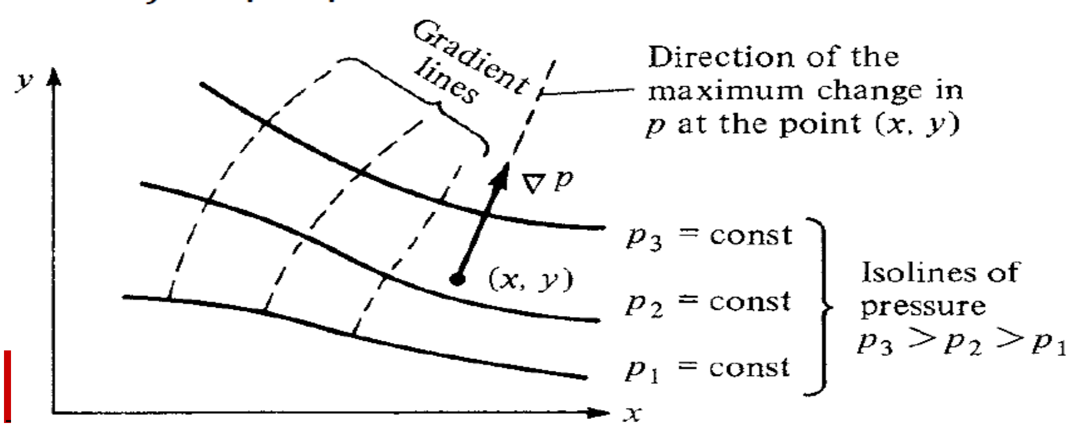


On a level curve

$$\frac{df}{ds} = 0 \rightarrow \hat{n}.\vec{\nabla} f = 0 \rightarrow \cos\theta = 0 \rightarrow \theta = 90°$$

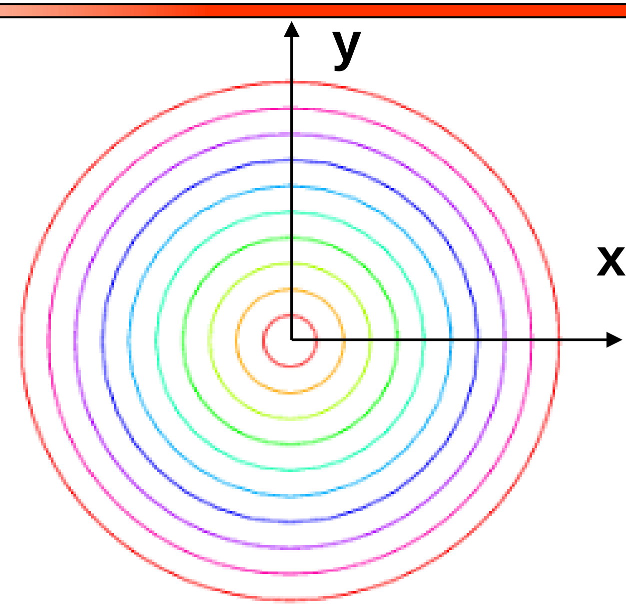
$\vec{\nabla} f$ is perpendicular to the level curve





❑ Gradient - Example

- For $f(x,y) = x^2 + y^2$
  - Find and plot level curves
  - Find the equation of the gradient and plot the gradient vectors.



Level curves: $f(x,y) = c \rightarrow x^2 + y^2 = c$ (equation of a circle with radius $\sqrt{c}$)

- Gradient vector

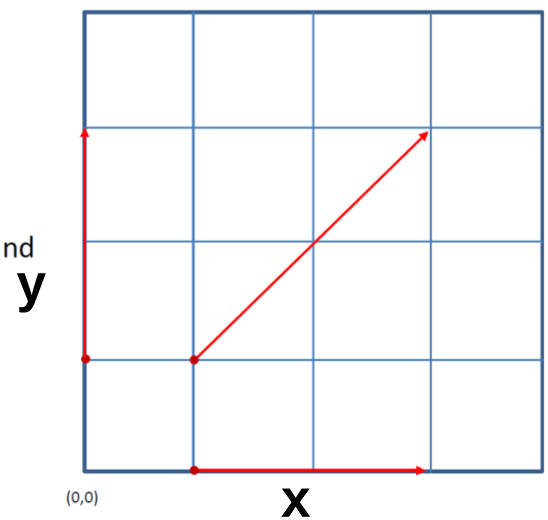
$$\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

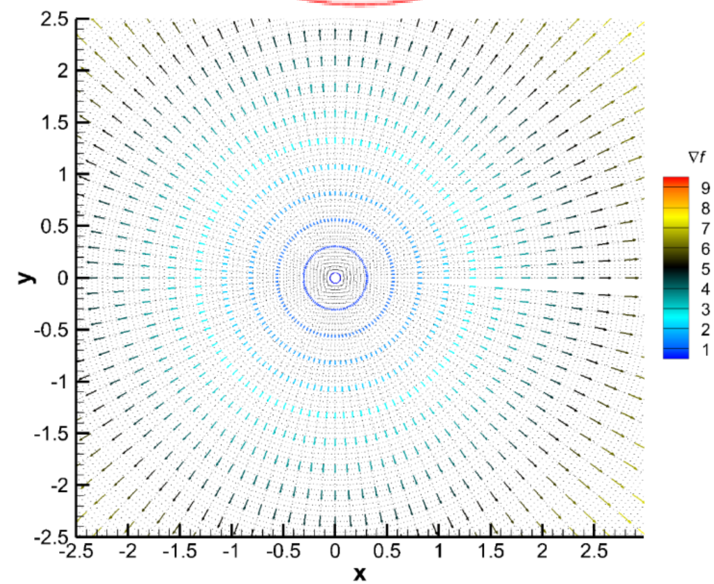
$$\frac{\partial f}{\partial x} = 2x\,, \frac{\partial f}{\partial y} = 2y$$

$$\vec{\nabla} f = (2x, 2y)$$

Plug in some arbitrary x, y values and
Draw the gradient lines (vectors)

x	y	$\vec{\nabla} f$
0	0	(0,0)
0	1	(0,2)
1	0	(2,0)
1	1	(2,2)
..	..	...





Gradient lines are straight lines going through origin

❑ Divergence of a Vector Field

2.19 Divergence of a Vector Field

Definition: The divergence of a vector ($\nabla \bullet \vec{B}$) at a point is the net outflow (efflux) of the vector field per unit volume enclosing the point.

$$\vec{V} = V_1\ \hat{e}_1 + V_2\ \hat{e}_2 + V_3\ \hat{e}_3$$

$$\nabla = \frac{\hat{e}_1}{h_1}\frac{\partial}{\partial q_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial q_2} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial q_3}$$

$$\nabla \bullet \vec{V} = \left(\frac{\hat{e}_1}{h_1}\frac{\partial}{\partial q_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial q_2} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial q_3}\right) \bullet (V_1\ \hat{e}_1 + V_2\ \hat{e}_2 + V_3\ \hat{e}_3)$$

Cartesian system:

$$\nabla \bullet \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \bullet (\hat{i}V_x + \hat{j}V_y + \hat{k}V_z)$$

$$= \hat{i} \bullet \frac{\partial(\hat{i}V_x + \hat{j}V_y + \hat{k}V_z)}{\partial x} + \hat{j} \bullet \frac{\partial(\hat{i}V_x + \hat{j}V_y + \hat{k}V_z)}{\partial y} + \hat{k} \bullet \frac{\partial(\hat{i}V_x + \hat{j}V_y + \hat{k}V_z)}{\partial z}$$

$$= \hat{i} \bullet \left[V_x\frac{\partial \hat{i}}{\partial x} + \hat{i}\frac{\partial V_x}{\partial x} + V_y\frac{\partial \hat{j}}{\partial x} + \hat{j}\frac{\partial V_y}{\partial x} + \frac{\partial \hat{k}}{\partial x}V_z + \hat{k}\frac{\partial V_z}{\partial x}\right] + \hat{j} \bullet \frac{\partial(\hat{i}V_x + \hat{j}V_y + \hat{k}V_z)}{\partial y} + \hat{k} \bullet \frac{\partial(\hat{i}V_x + \hat{j}V_y + \hat{k}V_z)}{\partial z}$$

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

Divergence of a Vector Field

<u>Cylindrical system:</u>

$$\nabla \bullet \vec{V} = (\hat{e}_r \frac{\partial}{\partial r} + \frac{\hat{e}_\theta}{r}\frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}) \bullet (V_r\,\hat{e}_r + V_\theta\,\hat{e}_\theta + V_z\,\hat{e}_z)$$

$$= \hat{e}_r \bullet \frac{\partial(V_r\,\hat{e}_r + V_\theta\,\hat{e}_\theta + V_z\,\hat{e}_z)}{\partial r} + \frac{\hat{e}_\theta}{r} \bullet \frac{\partial(V_r\,\hat{e}_r + V_\theta\,\hat{e}_\theta + V_z\,\hat{e}_z)}{\partial \theta} + \hat{e}_z \bullet \frac{\partial(V_r\,\hat{e}_r + V_\theta\,\hat{e}_\theta + V_z\,\hat{e}_z)}{\partial z}$$

$$Term\,1 = \hat{e}_r \bullet \frac{\partial(V_r\,\hat{e}_r + V_\theta\,\hat{e}_\theta + V_z\,\hat{e}_z)}{\partial r}$$

$$= \hat{e}_r \bullet (\frac{\partial V_r}{\partial r}\hat{e}_r + V_r\frac{\partial \hat{e}_r}{\partial r} + \frac{\partial V_r}{\partial r}\hat{e}_\theta + V_\theta\frac{\partial \hat{e}_\theta}{\partial r} + \frac{\partial V_z}{\partial r}\hat{e}_z + V_z\frac{\partial \hat{e}_z}{\partial r})$$

$$= \frac{\partial V_r}{\partial r}$$

$$Term\,2 = \frac{\hat{e}_\theta}{r} \bullet \frac{\partial(V_r\,\hat{e}_r + V_\theta\,\hat{e}_\theta + V_z\,\hat{e}_z)}{\partial \theta}$$

$$= \frac{\hat{e}_\theta}{r} \bullet (\frac{\partial V_r}{\partial \theta}\hat{e}_r + V_r\frac{\partial \hat{e}_r}{\partial \theta} + \frac{\partial V_r}{\partial \theta}\hat{e}_\theta + V_\theta\frac{\partial \hat{e}_\theta}{\partial \theta} + \frac{\partial V_z}{\partial \theta}\hat{e}_z + V_z\frac{\partial \hat{e}_z}{\partial \theta})$$

$$= \frac{\hat{e}_\theta}{r} \bullet (\frac{\partial V_r}{\partial \theta}\hat{e}_r + V_r\hat{e}_\theta + \frac{\partial V_r}{\partial \theta}\hat{e}_\theta - V_\theta\hat{e}_r + \frac{\partial V_z}{\partial \theta}\hat{e}_z)$$

$$= \frac{V_r}{r} + \frac{1}{r}\frac{\partial V_r}{\partial \theta}$$

$$Term\,3 = \hat{e}_z \bullet \frac{\partial(V_r\,\hat{e}_r + V_\theta\,\hat{e}_\theta + V_z\,\hat{e}_z)}{\partial z}$$

$$= \hat{e}_z \bullet (\frac{\partial V_r}{\partial z}\hat{e}_r + V_r\frac{\partial \hat{e}_r}{\partial z} + \frac{\partial V_r}{\partial z}\hat{e}_\theta + V_\theta\frac{\partial \hat{e}_\theta}{\partial z} + \frac{\partial V_z}{\partial z}\hat{e}_z + V_z\frac{\partial \hat{e}_z}{\partial z})$$

$$= \frac{\partial V_z}{\partial z}$$

Therefore,

$$\nabla \bullet \vec{V} = (\hat{e}_r \frac{\partial}{\partial r} + \frac{\hat{e}_\theta}{r}\frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}) \bullet (V_r\,\hat{e}_r + V_\theta\,\hat{e}_\theta + V_z\,\hat{e}_z)$$

$$= \frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z}$$

$$= \frac{1}{r}[\frac{\partial(rV_r)}{\partial r} + \frac{\partial V_\theta}{\partial \theta} + \frac{\partial(rV_z)}{\partial z}]$$

Summarize:

$$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \qquad h_r = 1$$

$$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} \quad ; \quad h_\theta = r$$

$$\hat{e}_Z = \hat{k} \qquad h_z = 1$$

<u>Derivatives of the unit vectors:</u>

$$\frac{\partial \hat{e}_r}{\partial r} = 0 \qquad \frac{\partial \hat{e}_\theta}{\partial r} = 0 \qquad \frac{\partial \hat{e}_Z}{\partial r} = 0$$

$$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta \quad ; \quad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r \quad ; \quad \frac{\partial \hat{e}_Z}{\partial \theta} = 0$$

$$\frac{\partial \hat{e}_r}{\partial z} = 0 \qquad \frac{\partial \hat{e}_\theta}{\partial z} = 0 \qquad \frac{\partial \hat{e}_Z}{\partial z} = 0$$

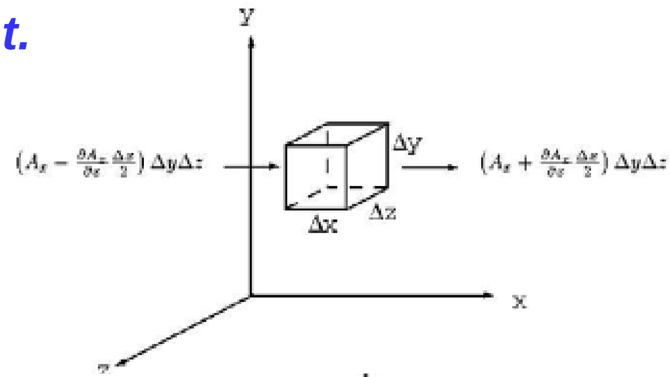
- ***In general form:***

$$\nabla \bullet \vec{V} = \frac{1}{h_1 h_2 h_3}[\frac{\partial(h_2 h_3 V_1)}{\partial q_1} + \frac{\partial(h_1 h_3 V_2)}{\partial q_2} + \frac{\partial(h_1 h_2 V_3)}{\partial q_3}]$$

❑ Physical Meaning of Divergence of a Vector Field

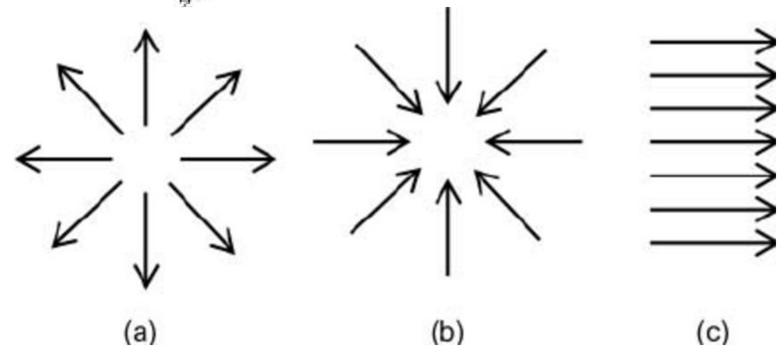
- ***The divergence of a vector at a point is the net outflow of the vector per unit volume enclosing the point.***

Consider vector $\vec{A}$ with component A_x, A_y, A_z at a point in the vector field surround by an element control volume ∇V with an element surface ∇S.



For simplicity, the element control volume with its center having a vector and components A_x, A_y, A_z are oriented with edges parallel to x, y and z axes, respectively.

Outflow of $\vec{A}$ thorough any side = component of $\vec{A}$ in the direction normal to side × Area of the side.



- *Divergence of a vector field: (a) positive divergence; (b) negative divergence; and (c) zero divergence.*

Net outflow of $\vec{A}$ in X-direction (Net outflow of $\vec{A}$ from the X-direction)

$$=[(A_x + \frac{\partial A_x}{\partial x}\frac{\Delta x}{2}) - (A_x - \frac{\partial A_x}{\partial x}\frac{\Delta x}{2})]\Delta y \Delta z = \frac{\partial A_x}{\partial x}\Delta x \Delta y \Delta z = \frac{\partial A_x}{\partial x}\Delta V$$

Similarly, net outflow of $\vec{A}$ in Y-direction (Net outflow of $\vec{A}$ from the Y-direction)

$$= \frac{\partial A_y}{\partial y}\Delta x \Delta y \Delta z = \frac{\partial A_y}{\partial y}\Delta V$$

net outflow of $\vec{A}$ in Z-direction (Net outflow of $\vec{A}$ from the Z-direction)

$$= \frac{\partial A_z}{\partial z}\Delta x \Delta y \Delta z = \frac{\partial A_z}{\partial z}\Delta V$$

Therefore, the total net outflow of $\vec{A}$ at the point

$$= (\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z})\Delta x \Delta y \Delta z = (\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z})\Delta V$$

$$\nabla \bullet \vec{A} = \lim_{\Delta V \to 0}[\frac{\text{total net outflow of } \vec{A} \text{ at the point in all direction}}{\Delta V}]$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Gauss Divergence Theorem

Recall that: $\nabla \bullet \vec{B} = Div\vec{B} = \lim_{\Delta V \to 0}[\frac{\oiint_{\Delta S} \vec{B} \bullet d\vec{A}}{\Delta V}]$

can be approximated as: $\nabla \bullet \vec{B} = \frac{1}{\Delta V}\oiint_{\Delta S} \vec{B} \bullet d\vec{A}$ or $(\nabla \bullet \vec{B})\Delta V = \oiint_{\Delta S} \vec{B} \bullet d\vec{A}$ for an element control volume.

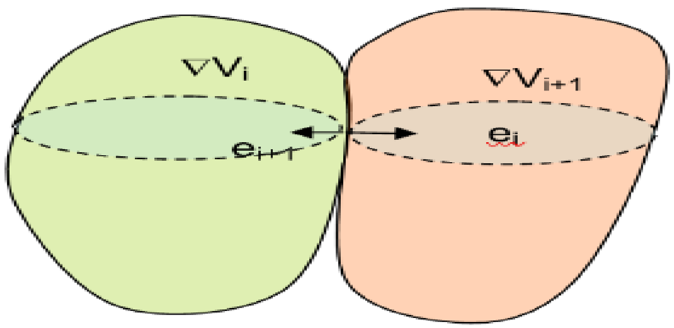
Now consider a finite control volume V in space subdivided into many smaller elemental sub-volumes.

Suppose $\nabla \bullet \vec{B}$ for all the sub volume are evaluated and summed:

$$\sum_{i=1}^{N}(\nabla \bullet \vec{B})_i \Delta V_i \approx \sum_{i=1}^{N}\oiint_{\Delta S} \vec{B} \bullet d\vec{A}$$

$$\underbrace{\lim_{\Delta V_i \to 0}\sum_{i=1}^{N}(\nabla \bullet \vec{B})_i \Delta V_i}_{volume\,integral\,by\,definition} \approx \lim_{\Delta V_i \to 0}\sum_{i=1}^{N}\oiint_{\Delta S} \vec{B} \bullet d\vec{A}$$

$$\iiint_{\Delta V}(\nabla \bullet \vec{B})\, dV = \lim_{\Delta V_i \to 0}\sum_{i=1}^{N}\oiint_{\Delta S} \vec{B} \bullet d\vec{A}$$



The flow of $\vec{B}$ through the common faces of adjacent volumes canceled because the inflow through one face equals the outflow through the other.

Thus, if we now sum the net outflow of $\vec{B}$ of all the sub-volumes, only faces on the surface enclosing the region will contribute to the summation.

Thus, Gauss divergence theorem states:

$$\iiint_{V}(\nabla \bullet \vec{B})\, dV = \oiint_{\Delta S} \vec{B} \bullet d\vec{A}$$

The Curl of a Vector Field

$$\nabla \times \vec{B} = Curl\ \vec{B}$$

$$\nabla q_1 = \frac{\hat{e}_1}{h_1}; \qquad \nabla q_2 = \frac{\hat{e}_2}{h_2}; \qquad \nabla q_3 = \frac{\hat{e}_3}{h_3}$$

$$\vec{B} = B_1\ \hat{e}_1 + B_2\ \hat{e}_2 + B_3\ \hat{e}_3$$

$$\nabla \times \vec{B} = \nabla \times (B_1\ \hat{e}_1 + B_2\ \hat{e}_2 + B_3\ \hat{e}_3)$$

Consider the first term

$$\nabla \times (B_1\ \hat{e}_1) = \nabla \times (B_1\ h_1 \nabla q_1) = \nabla \times (B_1\ h_1 \nabla q_1)$$

Since $\nabla \times (\phi\ \vec{A}) = \nabla \phi \times (\vec{A}) + \phi \nabla \times \vec{A}$

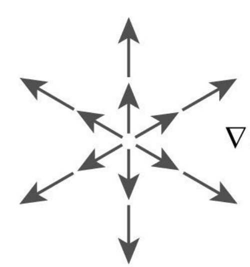
$$\begin{aligned}
\nabla \times (B_1\ \hat{e}_1) &= \nabla (B_1\ h_1) \times \nabla q_1 + (B_1\ h_1) \nabla \times \nabla q_1 \\
&= \nabla (B_1\ h_1) \times \nabla q_1 \\
&= \left[\frac{\hat{e}_1}{h_1}\frac{\partial (B_1\ h_1)}{\partial q_1} + \frac{\hat{e}_2}{h_2}\frac{\partial (B_1\ h_1)}{\partial q_2} + \frac{\hat{e}_3}{h_3}\frac{\partial (B_1\ h_1)}{\partial q_3}\right] \times \left(\frac{\hat{e}_1}{h_1}\right) \\
&= \frac{\hat{e}_2 \times \hat{e}_1}{h_2 h_1}\frac{\partial (B_1\ h_1)}{\partial q_2} + \frac{\hat{e}_3 \times \hat{e}_1}{h_3 h_1}\frac{\partial (B_1\ h_1)}{\partial q_3} \\
&= \frac{-\hat{e}_3}{h_2 h_1}\frac{\partial (B_1\ h_1)}{\partial q_2} + \frac{\hat{e}_2}{h_3 h_1}\frac{\partial (B_1\ h_1)}{\partial q_3} \\
&= \frac{1}{h_1}\left\{\frac{\hat{e}_2}{h_3}\frac{\partial (B_1\ h_1)}{\partial q_3} - \frac{\hat{e}_3}{h_2}\frac{\partial (B_1\ h_1)}{\partial q_2}\right\}
\end{aligned}$$

$$\begin{aligned}
\nabla \times \vec{B} &= \nabla \times (B_1\ \hat{e}_1 + B_2\ \hat{e}_2 + B_3\ \hat{e}_3) \\
&= \frac{1}{h_1}\left\{\frac{\hat{e}_2}{h_3}\frac{\partial (B_1\ h_1)}{\partial q_3} - \frac{\hat{e}_3}{h_2}\frac{\partial (B_1\ h_1)}{\partial q_2}\right\} \\
&+ \frac{1}{h_2}\left\{\frac{\hat{e}_3}{h_1}\frac{\partial (B_2\ h_2)}{\partial q_1} - \frac{\hat{e}_1}{h_3}\frac{\partial (B_2\ h_2)}{\partial q_3}\right\} \\
&+ \frac{1}{h_3}\left\{\frac{\hat{e}_1}{h_2}\frac{\partial (B_2\ h_2)}{\partial q_2} - \frac{\hat{e}_2}{h_1}\frac{\partial (B_2\ h_2)}{\partial q_1}\right\}
\end{aligned}$$

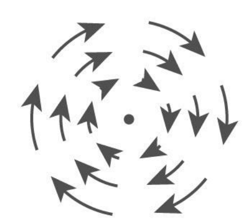
Or

$$\nabla \times \vec{B} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1\ B_1 & h_2\ B_2 & h_2\ B_2 \end{vmatrix}$$

VECTOR REVIEW: DIVERGENCE & CURL OF A VECTOR FIELD



$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$



$$\nabla \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

Divergence and curl of Vector fields

- *https://www.youtube.com/watch?v=rB83DpBJQsE&t=1s*