Lecture # 4: Description of Fluid Motion

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❑ DESCRIPTION OF FLOW MOTION

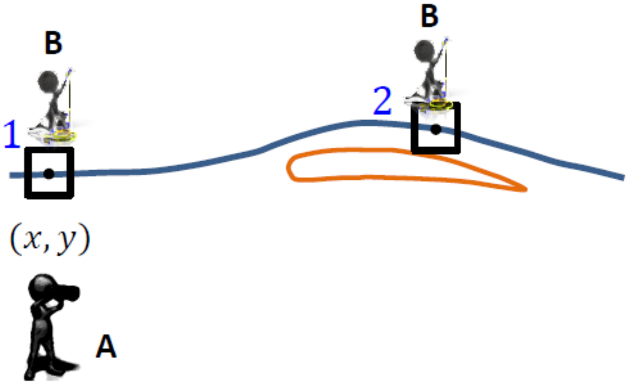
3.2.1 Lagrangian Method

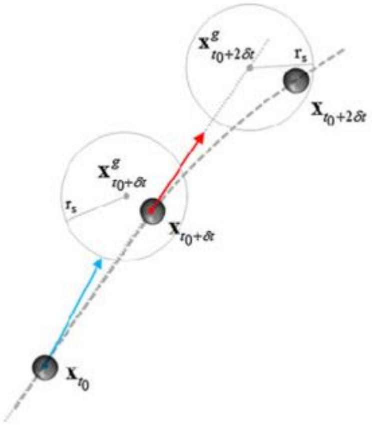
Lagrangian method, a natural extension of particle mechanics, considers the individual molecules and obtains conversation equations (mass, momentum and energy) based on individual molecular motion.

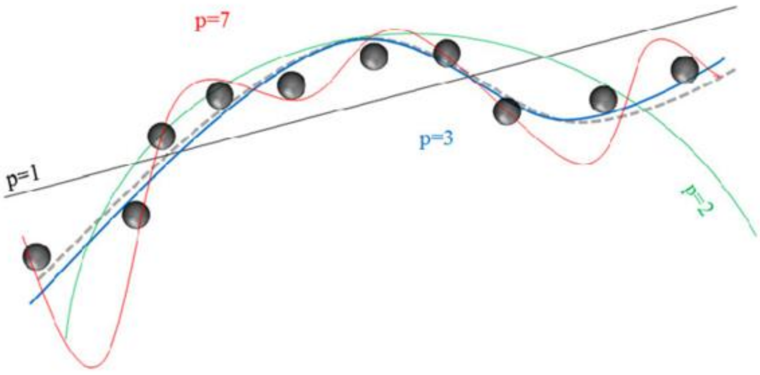
Attention is paid to what happens to the individual fluid particle (identified usually by its position at time *t=0*) in the course of time, what paths they described, what velocities or accelerations they possess, and so on.

The temperature in Lagrangian variables is given by $T = T(a,b,c,t)$, where (a,b,c) is the position of the particle at time *t=0*.

Also $\vec{R} = \vec{R}(a,b,c,t)$ is the position of the particle at time *t*, tagged by (a,b,c). t,a,b,c are the independent variables in Lagrangian frame. Since the fluid elements are continuously distributed, the values the parameters (a,b,c) will assume for the various elements that are continuous.







☐ DESCRIPTION OF FLOW MOTION

3.2.2 Eulerian Method

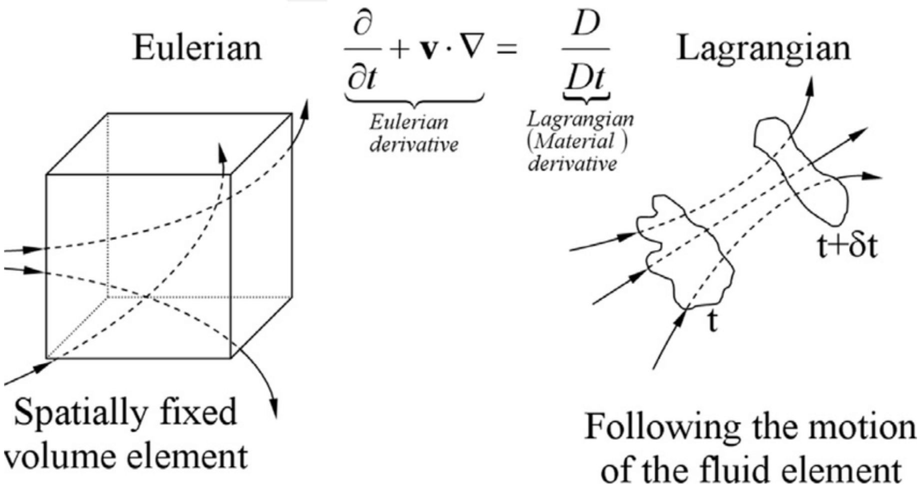
In Eulerian description, we describe the distribution of a macroscopic property as a function of space and time which we refer to as an Eulerian field of that property. Thus, we watch a fixed point (x, y, z) in space as time t proceeds.

The independent variables are the spatial coordinate (x, y, z) and time t.

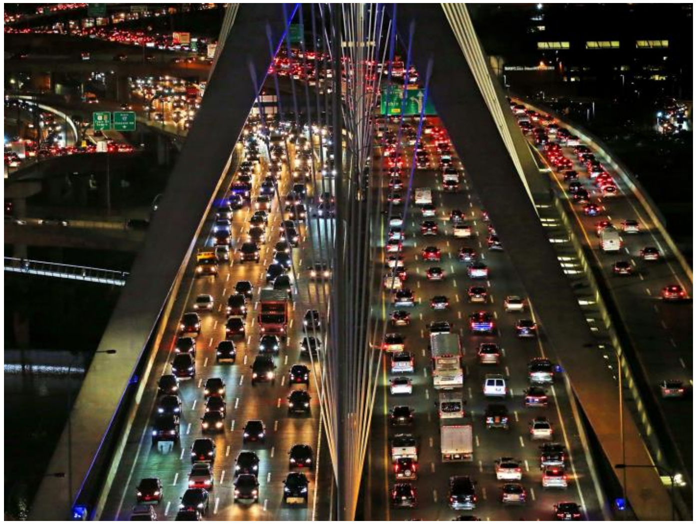
For example:

The temperature of the fluid is given by $T = T(x, y, z, t)$

- At a given (x, y, z), $T = T(x, y, z, t)$ gives the time history of T at that point.
- At a given time t, $T = T(x, y, z, t)$ gives the spatial variation of T.
- In order words, for any fluid quantity Q can be expressed as $Q = Q(\vec{R}, t)$ - a scalar or a vector field.



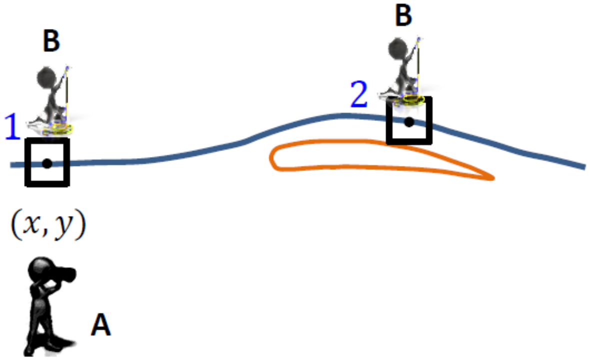
$$\underbrace{\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla}_{\text{Eulerian derivative}} = \underbrace{\frac{D}{Dt}}_{\text{Lagrangian (Material) derivative}}$$

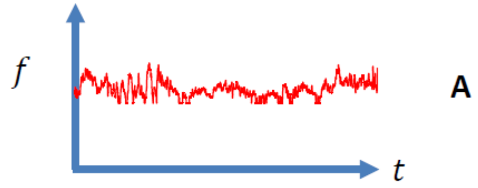


Eulerian/Lagrangian descriptions

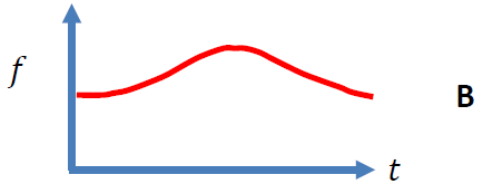
- Consider a function $f(x, y, t)$ in a flow field, could be $p, \rho, \vec{V}$, …
- Observer A is fixed looking at point 1, measuring f
  - $\partial f / \partial t$ shows the rate of change of f at a fixed point (x, y).
  - Eulerian description
- Observer B is following the fluid element going through point 1.
  - Now x, y, t are all changing (Lagrangian description)

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \underset{\substack{\downarrow \\ \boldsymbol{u}}}{\frac{dx}{dt}}\frac{\partial f}{\partial x} + \underset{\substack{\downarrow \\ \boldsymbol{v}}}{\frac{dy}{dt}}\frac{\partial f}{\partial y}$$





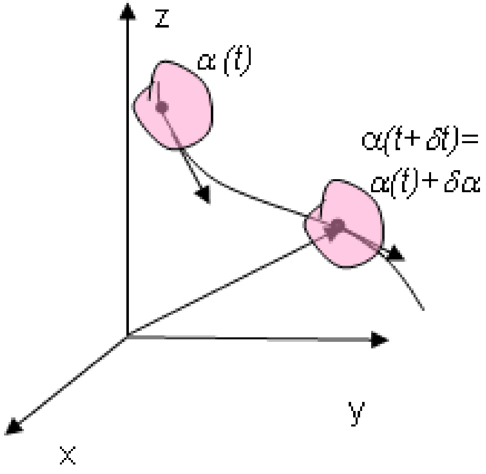
(x, y) fixed



(x, y) changing

❑ Eularian vs. Lagrangian Methods in Describing Flow Motion

3.3 Substantial/Material Derivative



Substantial/material derivative gives the relation between the derivatives in Lagrangian and Eulerian derivatives.

In Eulerian viewpoint, since one attention is focused upon specific points in space at various times, the history of the individual fluid particles is not explicit. The substantial derivative allows us to express the time rate of change of a particle property in terms of the spatial (Eulerian) derivatives of that property at a given point.

Let α be any fluid variable such as density, velocity or energy. From Eulerian reference frame,

$$\underbrace{\overbrace{\frac{D\alpha}{Dt}}^{\text{Time derivatives following a particle}}}_{\text{Lagrangian frame}} = \frac{\partial\alpha}{\partial t} + \frac{\partial\alpha}{\partial x}\cdot u + \frac{\partial\alpha}{\partial y}\cdot v + \frac{\partial\alpha}{\partial z}\cdot = \frac{\partial\alpha}{\partial t} + \vec{V}\bullet\nabla\alpha$$

$$= \underbrace{\left(\frac{\partial}{\partial t} + \vec{V}\bullet\nabla\right)\alpha}_{\text{Expression in Eulerian domain}}$$

$\frac{D}{Dt}$ is the change rate of a fluid property as the given fluid particle moves through space.

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{V}\bullet\nabla$$

$\frac{\partial}{\partial t}$ is the time change rate of fluid property of the given fixed point (i.e., local derivatives)

$\vec{V}\bullet\nabla$ is the time change rate of fluid property due to the movement of the fluid element from one location to another in the flow field where the flow property are spatially different (i.e., convective derivative).

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}$$

Rate of change following the fluid

Rate of change at a fixed point

In vector form

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \vec{V}.\vec{\nabla}f$$

Material Derivative

- Rate of change of f following the fluid along its path $x(t), y(t)$

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y}$$

In 3D

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}$$

Rate of change following the fluid

Rate of change at a fixed point

In vector form

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \vec{V}.\vec{\nabla} f$$

- D/Dt is called material derivative (also known as total or substantial derivative)
- D/Dt is an operator similar to d/dt, for example

$$\frac{D(AB)}{Dt} = A\frac{DB}{Dt} + B\frac{DA}{Dt}$$

$$\frac{D}{Dt}(A+B) = \frac{DA}{Dt} + \frac{DB}{Dt}$$

❑ Acceleration of a Fluid Particle

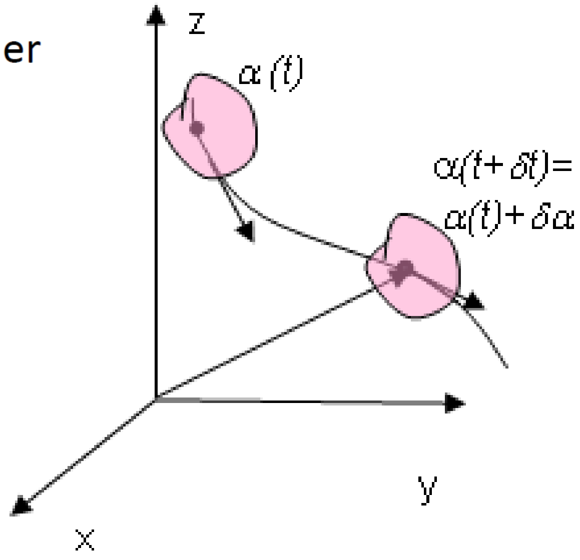
Acceleration Field

- The acceleration of a fluid element is the time rate of change of its velocity
  - Velocity can change as fluid flows from one point to another
  - Velocity can change in time at a given point

$$\vec{a} = \frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + \vec{V}.\vec{\nabla}\vec{V}$$

In cartesian system with $\vec{V} = (u, v, w)$

$$\vec{a} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$



- ***In general form:***

$$\boxed{\begin{aligned}\frac{D\vec{V}}{Dt} &= \frac{\partial \vec{V}}{\partial t} + \nabla \bullet \vec{V} = \left(\frac{\partial}{\partial t} + \frac{V_1}{h_1}\frac{\partial}{\partial q_1} + \frac{V_2}{h_2}\frac{\partial}{\partial q_2} + \frac{V_3}{h_3}\frac{\partial}{\partial q_3}\right)\vec{V} \\ &= \frac{\partial}{\partial t}(V_1 \hat{e}_1 + V_2 \hat{e}_2 + V_3 \hat{e}) + \left(\frac{V_1}{h_1}\frac{\partial}{\partial q_1} + \frac{V_2}{h_2}\frac{\partial}{\partial q_2} + \frac{V_3}{h_3}\frac{\partial}{\partial q_3}\right)\vec{V}\end{aligned}}$$

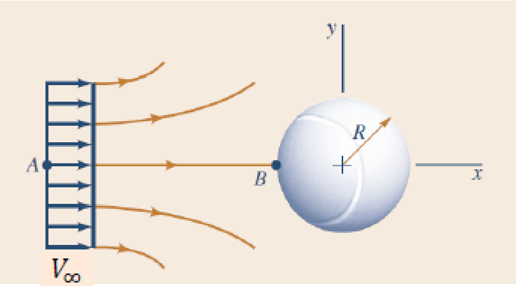
$$\boxed{\frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + \vec{V} \bullet \nabla \vec{V} = \frac{\partial \vec{V}}{\partial t} + \frac{1}{2}\nabla(\vec{V} \bullet \vec{V}) - \vec{V} \times (\nabla \times \vec{V})}$$

Acceleration of a Fluid Particle - Example

$$\boxed{\vec{a} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}$$

- The fluid velocity approaching a tennis ball with radius R may be approximated, along the straight streamline A-B, by

$$\vec{V} = u\hat{\imath} = V_\infty\left(1 + \frac{R^3}{x^3}\right)\hat{\imath}$$

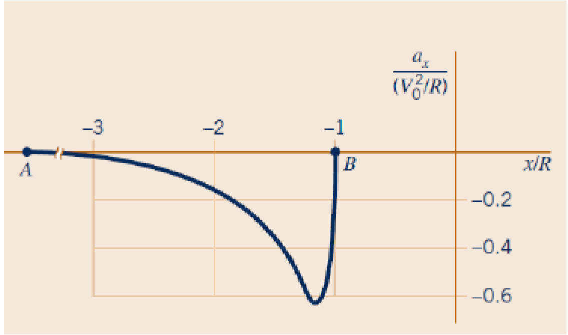


Find the fluid acceleration along this streamline

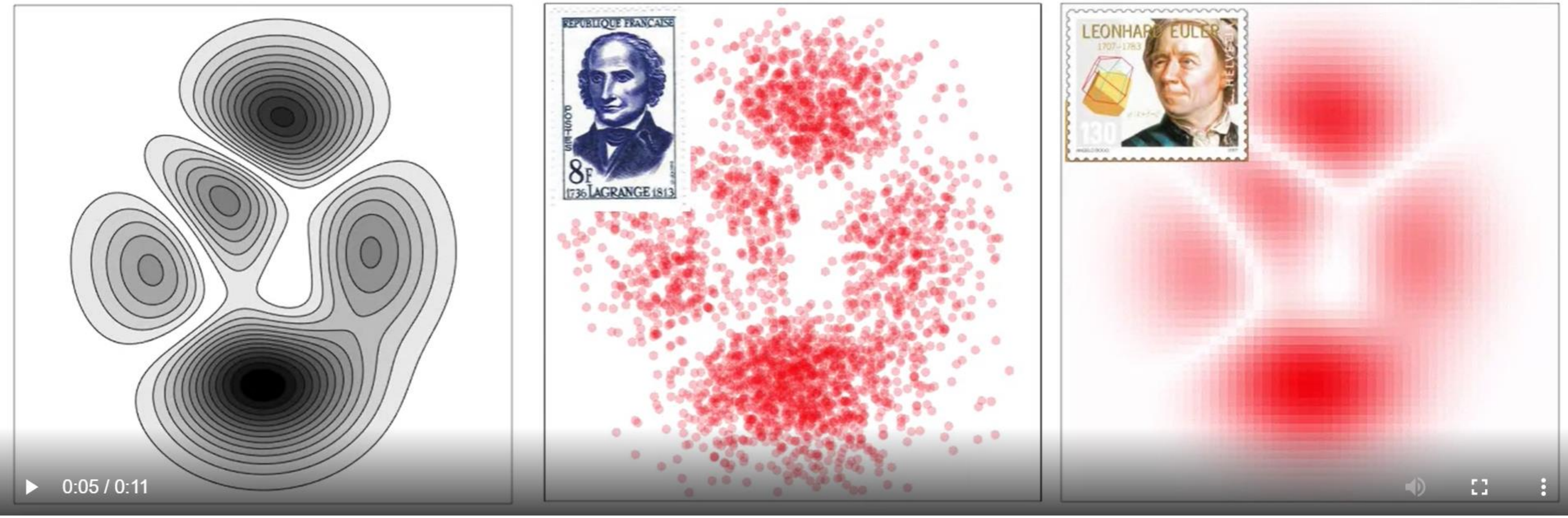
Since along A-B: $v = w = 0 \rightarrow a_y = a_z = 0$

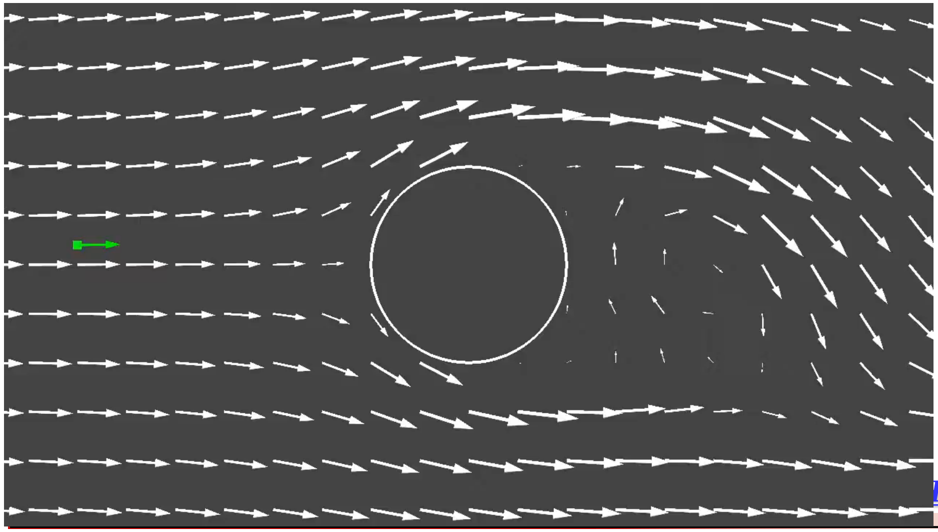
$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 + V_\infty\left(1 + \frac{R^3}{x^3}\right) \times V_\infty\left(-\frac{3R^3}{x^4}\right)$$

$$a_x = -\frac{3V_\infty^2}{R}\frac{1 + \left(\frac{R}{x}\right)^3}{\left(\frac{x}{R}\right)^4}$$



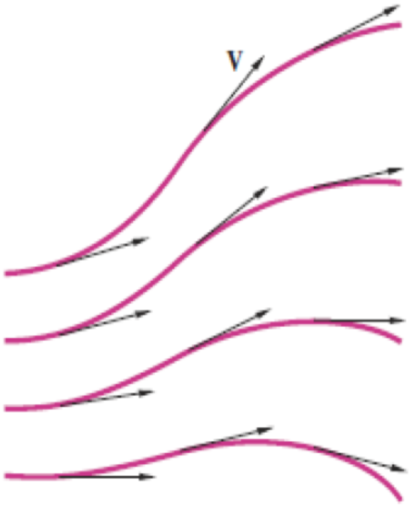
❑ Eularian vs. Lagrangian Methods in Describing Flow Motion





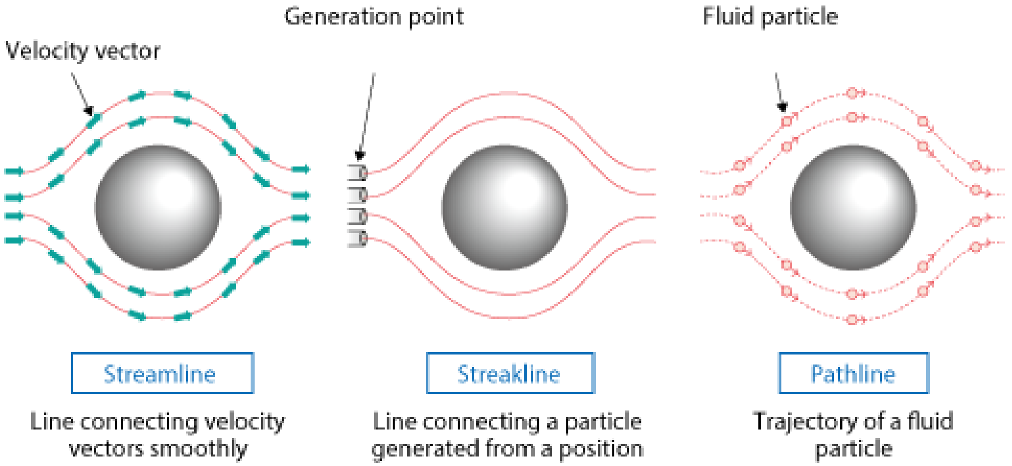
Flow patterns

- Streamline is a line everywhere tangent to the velocity vector at a given instant of time.
- Pathline is the actual path traversed by a given fluid element
- Streakline is the locus of particles that have earlier passed through a prescribed point.
- Timeline is a set of fluid elements that form a line at a given instant of time.
- Streamline is convenient to calculate/draw mathematically but the other three are easier to generate/observe experimentally.
- **Streamlines, pathlines and streaklines are all identical in steady flow.**



• Streamlines; Streaklines, and Pathline

- ***Streamlines*** *are a family of curves that are instantaneously tangent to the velocity vector of the flow. These show the direction in which a massless fluid element will travel at any point in time (Eularian approach).*
- ***Streaklines*** *are the loci of points of all the fluid particles that have passed continuously through a particular spatial point in the past. Dye steadily injected into the fluid at a fixed point extends along a streakline (Langragian approach).*
- ***Pathlines*** *are the trajectories that individual fluid particles follow. These can be thought of as "recording" the path of a fluid element in the flow over a certain period. The direction the path takes will be determined by the streamlines of the fluid at each moment in time (Langragian approach).*



Streamlines, streaklines, and pathlines

☐ IN-CLASS PRACTICE

An idealized velocity field is given by

$$\vec{V} = 4tx\hat{\imath} - 2t^2\hat{\jmath} + 4xz\hat{k}$$

where (x, y, z) are the Cartesian coordinates and t represents time. At point $(x, y, z) = (-1,1,0)$ calculate the acceleration vector and an in-plane unit vector normal to the acceleration

❑ IN-CLASS PRACTICE

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$$\vec{V} = 4tx\hat{\imath} - 2t^2\hat{\jmath} + 4xz\hat{k}$$

$$\vec{a} = \frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + \vec{V}.\vec{\nabla}\vec{V} = \frac{\partial \vec{V}}{\partial t} + \vec{V}.\vec{\nabla}u\hat{\imath} + \vec{V}.\vec{\nabla}v\hat{\jmath} + \vec{V}.\vec{\nabla}w\hat{k}$$

$$\frac{\partial \vec{V}}{\partial t} = 4x\hat{\imath} - 4t\hat{\jmath} + 0\hat{k} == 4x\hat{\imath} - 4t\hat{\jmath}$$

$$\vec{V}.\vec{\nabla}u = \left(4tx\hat{\imath} - 2t^2\hat{\jmath} + 4xz\hat{k}\right).\left[\frac{\partial}{\partial x}(4tx)\hat{\imath} + \frac{\partial}{\partial y}(4tx)\hat{\jmath} + \frac{\partial}{\partial z}(4tx)\hat{k}\right]$$

$$= \left(4tx\hat{\imath} - 2t^2\hat{\jmath} + 4xz\hat{k}\right).(4t\hat{\imath}) = 16t^2x$$

$$\vec{V}.\vec{\nabla}v = \left(4tx\hat{\imath} - 2t^2\hat{\jmath} + 4xz\hat{k}\right).\left[\frac{\partial}{\partial x}(-2t^2)\hat{\imath} + \frac{\partial}{\partial y}(-2t^2)\hat{\jmath} + \frac{\partial}{\partial z}(-2t^2)\hat{k}\right]$$

$$= \left(4tx\hat{\imath} - 2t^2\hat{\jmath} + 4xz\hat{k}\right).(0) = 0$$

$$\vec{V}.\vec{\nabla}w = \left(4tx\hat{\imath} - 2t^2\hat{\jmath} + 4xz\hat{k}\right).\left[\frac{\partial}{\partial x}(4xz)\hat{\imath} + \frac{\partial}{\partial y}(4xz)\hat{\jmath} + \frac{\partial}{\partial z}(4xz)\hat{k}\right]$$

$$= \left(4tx\hat{\imath} - 2t^2\hat{\jmath} + 4xz\hat{k}\right).\left(4z\hat{\imath} + 4x\hat{k}\right) = 16txz + 16x^2z$$

$$\vec{a} = 4x\hat{\imath} - 4t\hat{\jmath} + 16t^2x\hat{\imath} + (16txz + 16x^2z)\hat{k}$$

$$\vec{a} = 4x(1 + 4t^2)\hat{\imath} - 4t\hat{\jmath} + 16t^2x\hat{\imath} + 16xz(t + x)\hat{k}$$

At (-1,1,0):

$$\vec{a}(-1,1,0) = -4(1 + 4t^2)\hat{\imath} - 4t\hat{\jmath}$$

For an in-plane normal vector in the form of $\hat{n} = \alpha\hat{\imath} + \beta\hat{\jmath}$

$$\vec{a}.\hat{n} = 0 \rightarrow -4(1 + 4t^2)\alpha - 4t\beta = 0 \rightarrow \beta = -\frac{1 + 4t^2}{4t}\alpha$$

Unit vector has a magnitude of 1, hence $\alpha^2 + \beta^2 = 1$

$$\rightarrow \alpha^2 + \frac{(1 + 4t^2)^2}{16t^2}\alpha^2 = 1 \rightarrow \alpha = \frac{4t}{\sqrt{16t^2 + (1 + 4t^2)^2}}, \beta = -\frac{1 + 4t^2}{\sqrt{16t^2 + (1 + 4t^2)^2}}$$

$$\hat{n} = \left(\frac{4t}{\sqrt{16t^2 + (1 + 4t^2)^2}}, -\frac{1 + 4t^2}{\sqrt{16t^2 + (1 + 4t^2)^2}}\right)$$