Lecture # 5: Conservation of Mass

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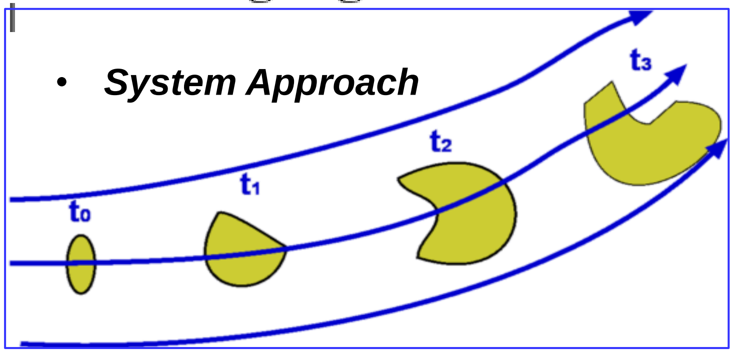
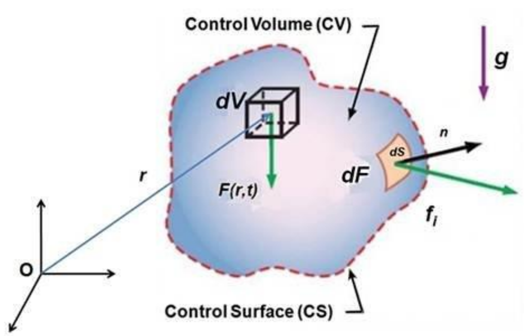
❑ System and Control Volume

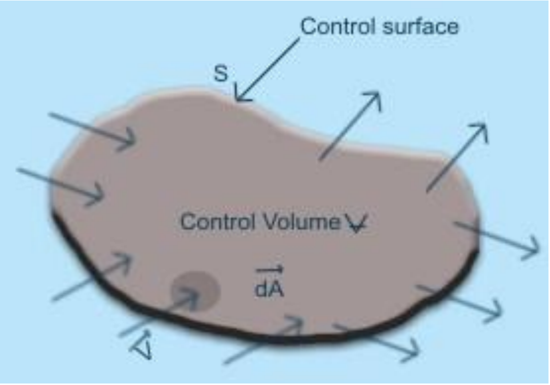
<u>System</u>: A system is a region enclosed by a rigid or flexible boundary with a quantity of matter of fixed mass and identity. Heat and work can cross the boundary of a system.

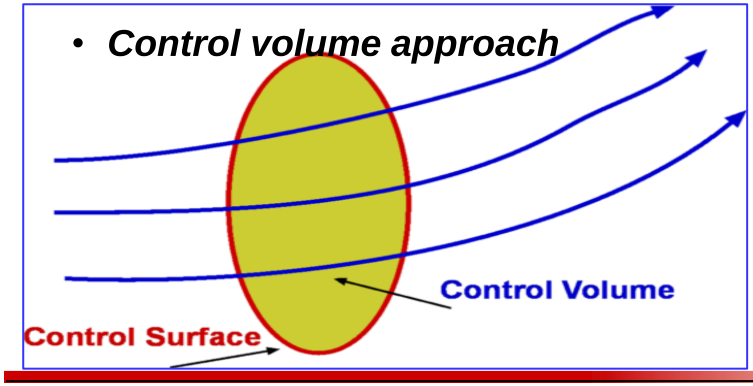
<u>Control volume</u>: A control volume is a finite region in space that may be fixed or moving in space. Mass, momentum, heat and work can cross the boundary of the region called the control surface.

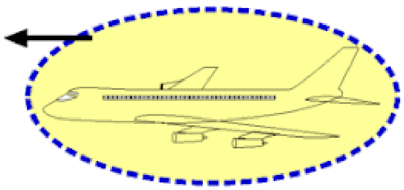
If the laws of physics are written for a fixed region of space, i.e., for different fluid particles occupy this region at different times, then, the frame of reference is said to be **Eulerian**. However, if the laws govern the same fluid particles in a particular region that moves with the fluid, the laws are written in **Lagrangian** reference frame.



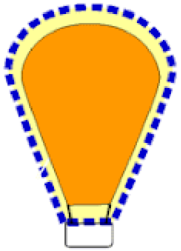






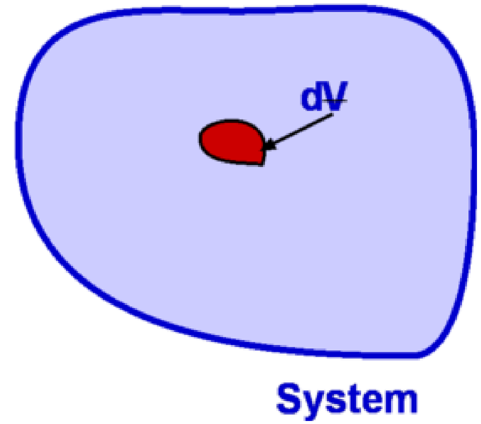


A moving Control Volume around a moving Aeroplane.



A collapsible Control Volume around a balloon.

Conservation Laws



Conservation of mass:

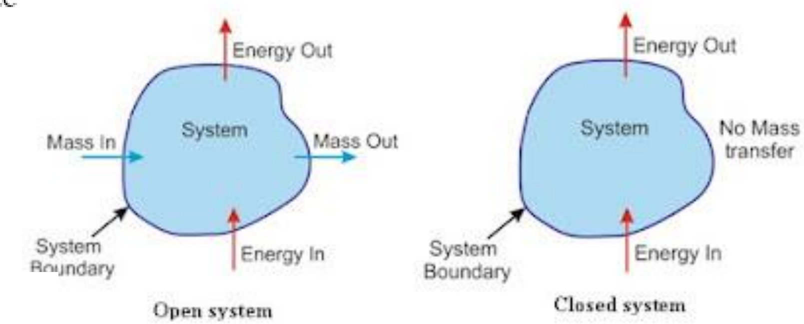
The conservation of mass simply states that the mass, M, of the system is constant. Writing as an equation, one obtained:

$$\frac{dM_S}{dt}=0 \qquad or \qquad \frac{d}{dt}\left(\int_S dm\right)=\frac{d}{dt}\left(\int_S \rho\, d\mathcal{V}\right)=0$$

Conservation of Linear Momentum (Newton's Second Law):

Sum of all external forces acting on the system is equal to the time change rate
momentum of the system.

$$\vec{F}\Big|_{acting\ on\ a\ system}=\frac{d}{dt}\left(\int_S \vec{V}\, dm\right)=\frac{d}{dt}\left(\int_S \vec{V}\rho\, d\mathcal{V}\right)$$

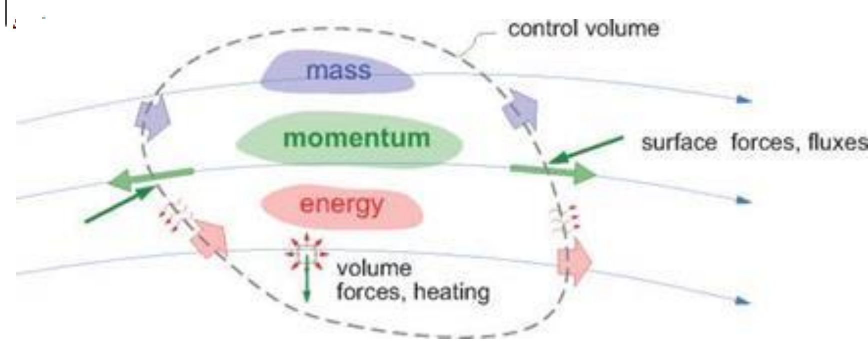


Conservation of Energy (First Law of Thermodynamics):

$$\delta W\Big|_{system}+\delta Q\Big|_{system}=\delta E\Big|_{system} \qquad or \qquad \dot{W}\Big|_{system}+\dot{Q}\Big|_{system}=\frac{dE}{dt}\Big|$$

$$E\Big|_{system}=\int_{system} e\, dm=\int_{\mathcal{V}} e\,\rho\, d\mathcal{V}$$

Where e is the specific inner energy.



❑ Conservation of Mass

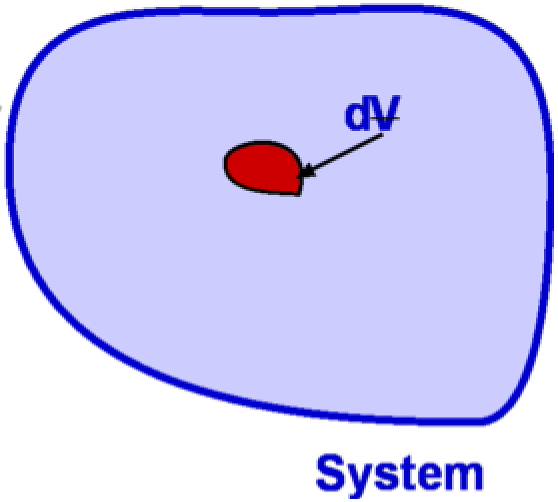
- *Physical principle: Mass can be neither created or destroyed*

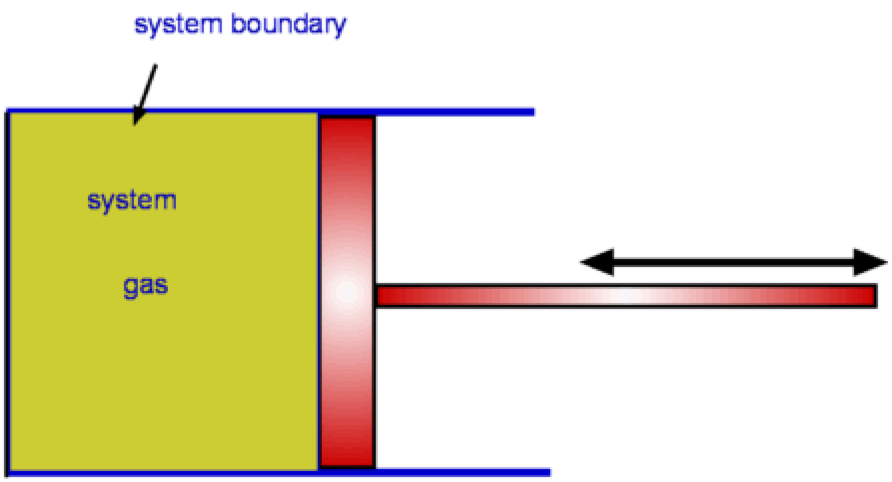
- *Consider a system of a fixed mass, M.*
- *We know that this mass does not change and is conserved.*
- *This leads to the law of conservation of mass , namely,*

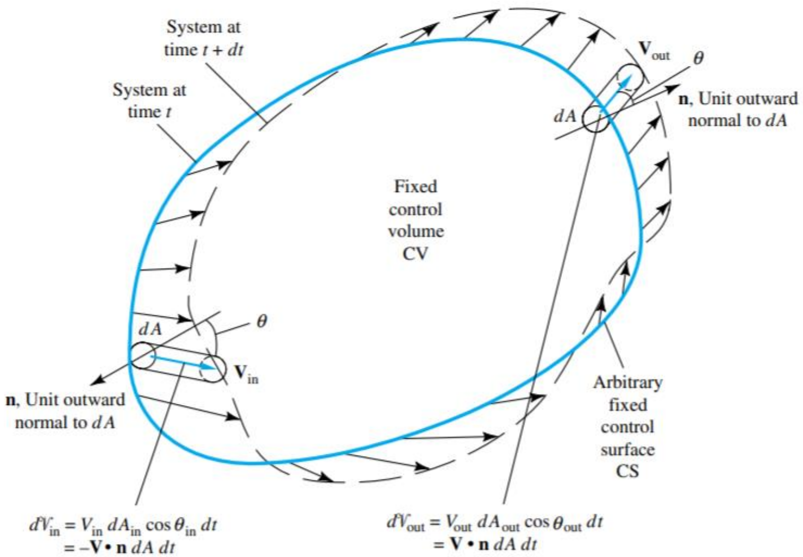
$$m_{system} = constant$$

$$\left(\frac{dm}{dt}\right)_{system} = 0$$

with $m_{system} = \int_{system} dm = \int_{system} \rho d\forall$







Conservation of Momentum

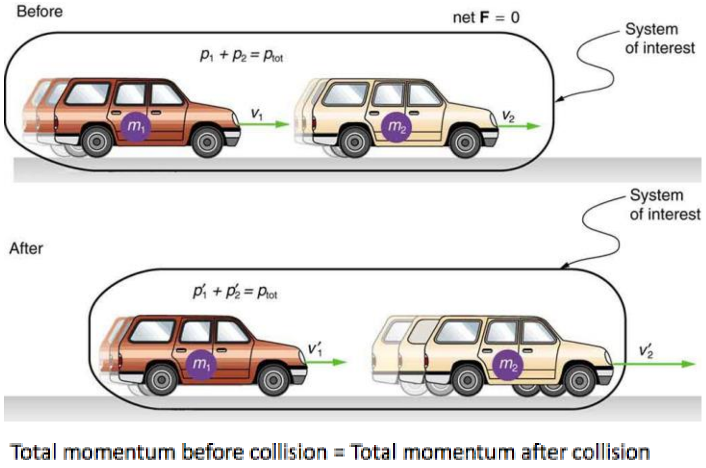
Newton's second law is the next one to be imposed upon fluid motion. It is known that the rate of change of momentum is proportional to the applied force. If F is the force upon a system,

$$F = \frac{dM}{dt}$$

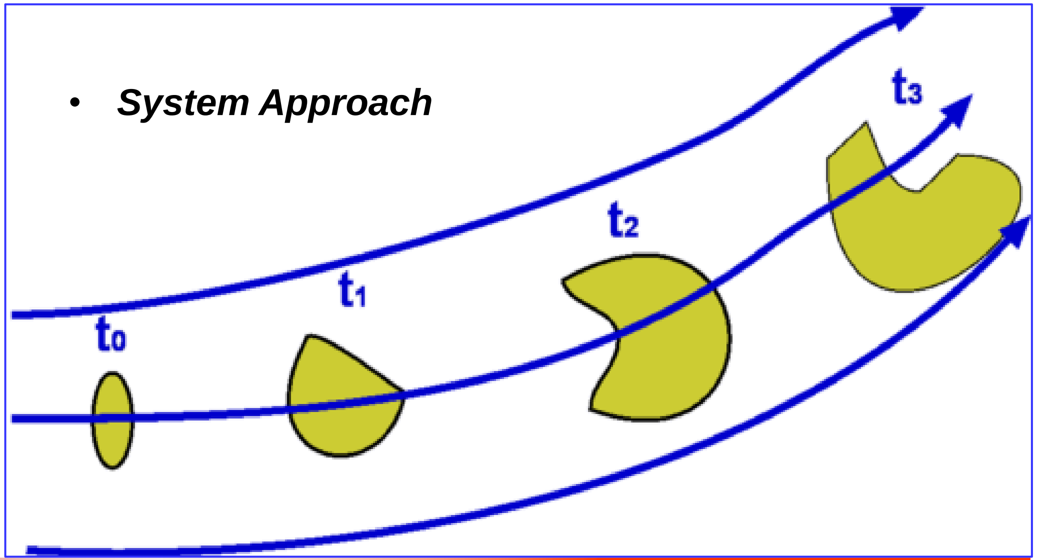
where **M** is the linear momentum. Further,

$$M = \int_{system} V\,dm = \int_{system} V\,\rho d\forall$$

It is to be realised that momentum M and velocity V are vectors and each of a component in each of the coordinate directions. Accordingly, Eq. 3.10 represents three equations.



$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_1 v_2$$



❑ Conservation of Energy

The first law of thermodynamics which is a statement of

$$dQ - dW = dE$$

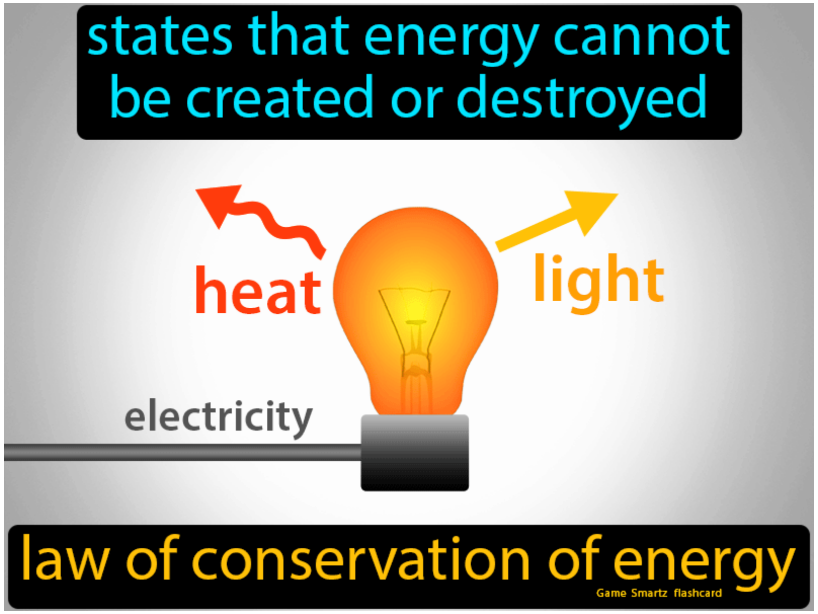
$$\text{i.e.,} \quad \frac{dQ}{dt} - \frac{dW}{dt} = \frac{dE}{dt}$$

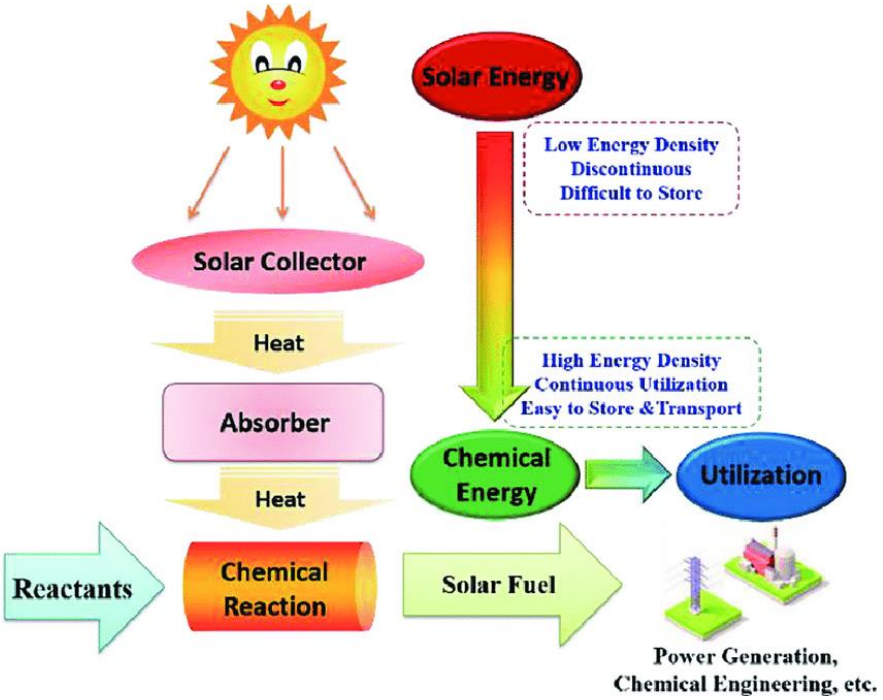
$$\dot{Q} + \dot{W} = \left.\frac{dE}{dt}\right|_{system}$$

$$E_{system} = \int_{M(sys)} e\,dm = \int_{V(sys)} e\rho\,dV$$

System Equation

where dQ is the heat added to the system, dW is the work done by the system and dE is the consequent change in energy of the system.

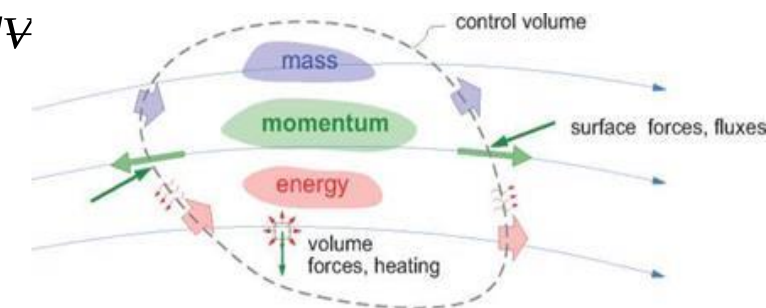




Reynolds Transport Theorem

- ***Let N be any extensive property of the identifiable fixed mass (system) such as total mass, momentum, or energy. The corresponding intensive property (extensive property per unit mass) will be designated as, α:***

$$N\Big|_{system} = \int_{system} \alpha\, dm = \int_{\forall} \alpha\, \rho\, d\forall$$



- ***The rate of change of N can be written:***

$$\frac{DN}{Dt}\Big|_{system} = \lim_{\delta t\to 0}\{\frac{1}{\delta t}[\int_{\forall(t+\delta t)} \alpha(t+\delta t)\,\rho(t+\delta t)\,d\forall - \int_{\forall(t)} \alpha(t)\,\rho(t)\,d\forall]\}$$

If we make: $\beta(t) = \alpha(t)\rho(t)$

$$\frac{DN}{Dt}\Big|_{system} = \lim_{\delta t\to 0}\{\frac{1}{\delta t}[\int_{\forall(t+\delta t)} \beta(t+\delta t)\,d\forall - \int_{\forall(t)} \beta(t)\, d\forall]\}$$

$$\frac{DN}{Dt}\Big|_{system} = \frac{D\int_{\forall} \beta\, d\forall}{Dt} = \lim_{\delta t\to 0}\{\frac{1}{\delta t}[\int_{\forall(t+\delta t)} \beta(t+\delta t)\,d\forall - \int_{\forall(t)} \beta(t+\delta t)\,d\forall + \int_{\forall(t)} \beta(t+\delta t)\,d\forall - \int_{\forall(t)} \beta(t)\, d\forall]\}$$

$$= \lim_{\delta t\to 0}\{\frac{1}{\delta t}[\overbrace{\int_{\forall(t+\delta t)} \beta(t+\delta t)\,d\forall - \int_{\forall(t)} \beta(t+\delta t)\,d\forall}^{first\,two} + \overbrace{\int_{\forall(t)} \beta(t+\delta t)\,d\forall - \int_{\forall(t)} \beta(t)\, d\forall}^{sencond\,two}]$$

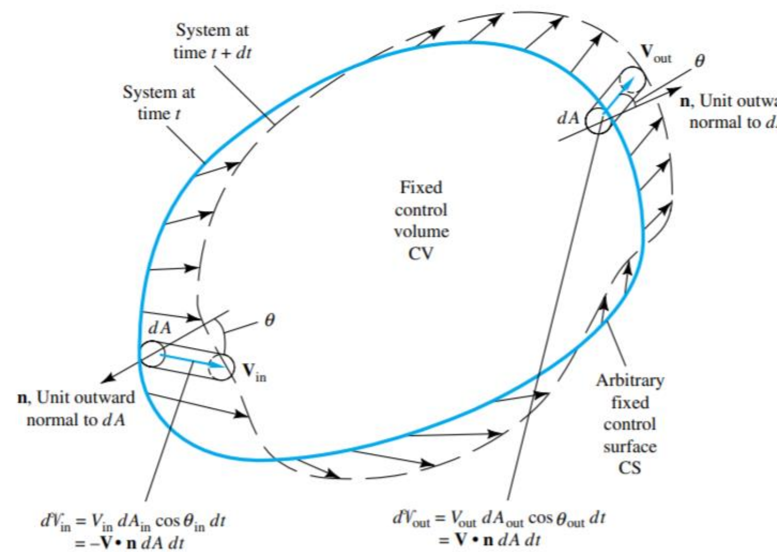
$$= \lim_{\delta t\to 0}\{\frac{1}{\delta t}(\overbrace{\int_{\forall(t+\delta t)} \beta(t+\delta t)\,d\forall - \int_{\forall(t)} \beta(t+\delta t)\,d\forall}^{first\,two})\} + \frac{\partial}{\partial t}\int_{\forall} \beta\, d\forall$$

$$= \lim_{\delta t\to 0}\{\frac{1}{\delta t}(\int_{S(t)} \beta(t+\delta t)(\vec{V}\bullet\hat{e}_n)\delta t\, dA)\} + \frac{\partial}{\partial t}\int_{\forall} \beta\, d\forall$$

$$= \lim_{\delta t\to 0}(\int_{S(t)} \beta(t+\delta t)(\vec{V}\bullet\hat{e}_n)\, dA) + \frac{\partial}{\partial t}\int_{\forall} \beta\, d\forall$$

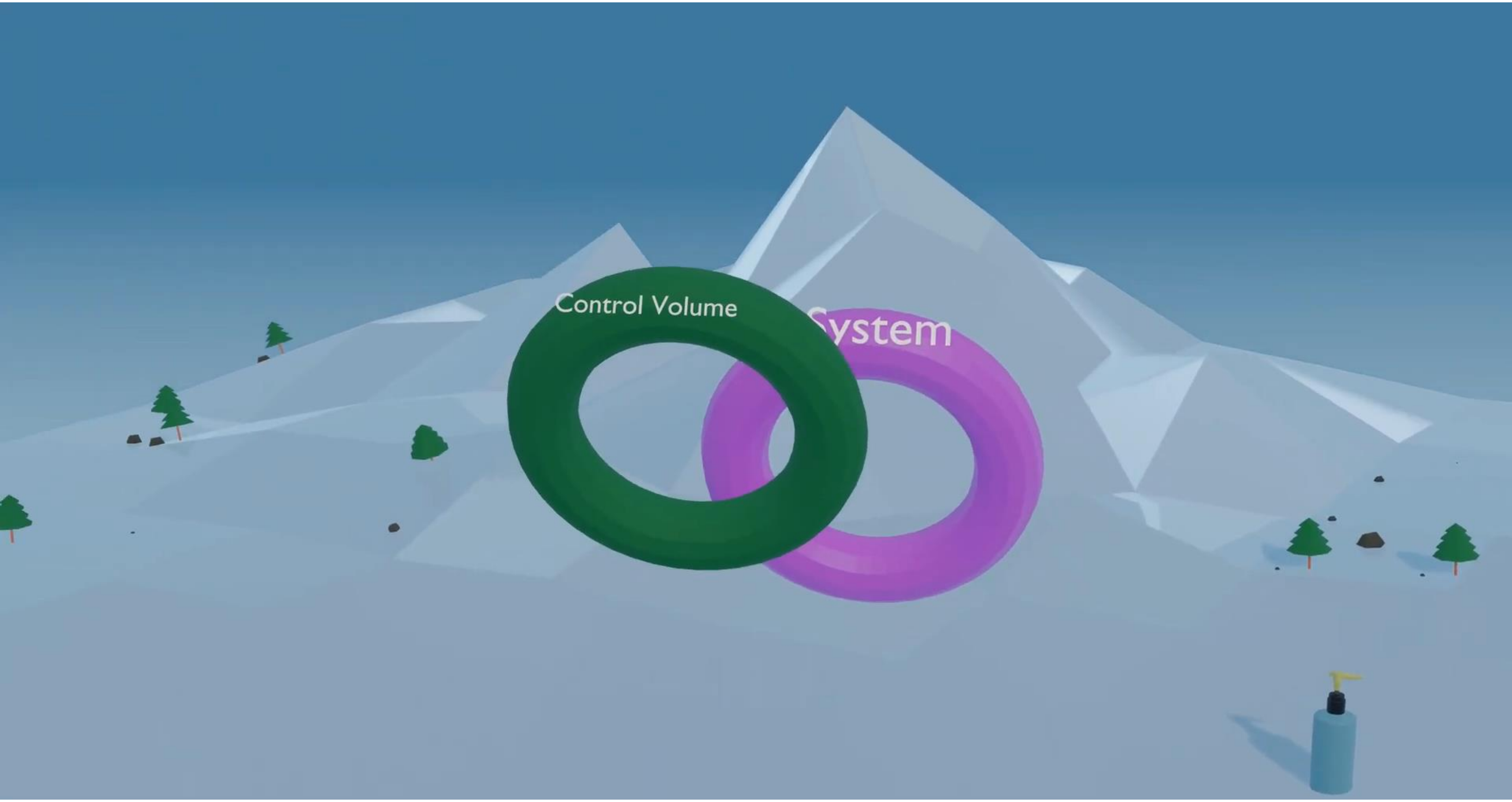
$$= \int_{C.S(t)} \beta(t)(\vec{V}\bullet\hat{e}_n)\, dA) + \frac{\partial}{\partial t}\int_{\forall} \beta\, d\forall$$

$$= \int_{C.S} \beta\vec{V}\bullet d\vec{A} + \frac{\partial}{\partial t}\int_{\forall} \beta\, d\forall = \int_{C.S} \alpha\,\rho\vec{V}\bullet d\vec{A} + \frac{\partial}{\partial t}\int_{\forall} \alpha\,\rho\, d\forall$$



❑ Reynolds Transport Theorem

- *https://www.youtube.com/watch?v=a0AXA1frIs4*

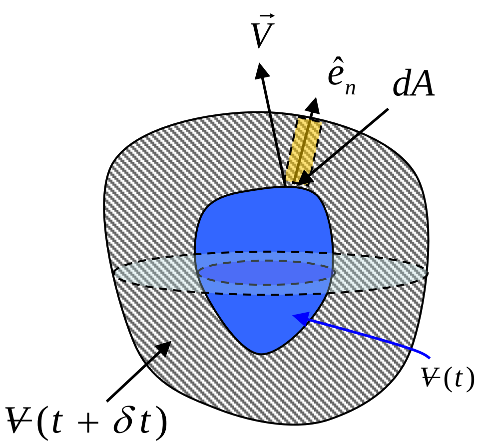


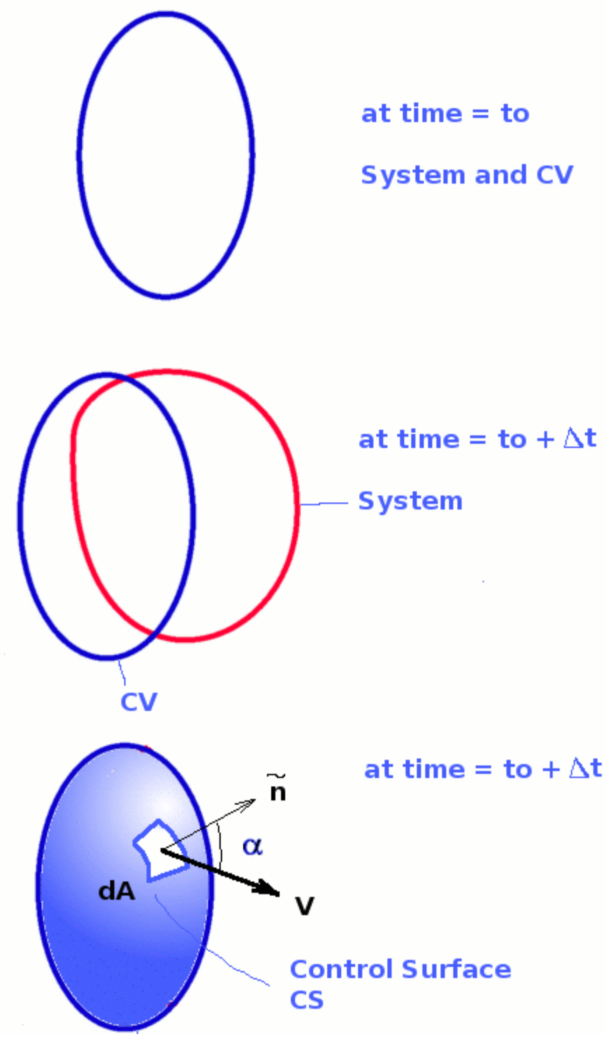
Reynolds Transport Theorem

$$\frac{DN_s}{Dt} = \frac{D\int_{\forall} \alpha\,\rho\,d\forall}{Dt} = \frac{\partial}{\partial t}\int_{C.V.} \alpha\,\rho\,d\forall + \int_{C.S.} (\alpha\,\rho\vec{V})\bullet d\vec{A}$$

Where α is any intensive property corresponding to N. (i.e., $\alpha = N$ per unit mass) and it can be used for different quantities as follows.

N_s	α
Mass	1
Linear momentum	$\vec{V}$
Angular momentum	$\vec{R}\times\vec{V}$
Energy	e
Entropy	s





❑ Conservation of Mass

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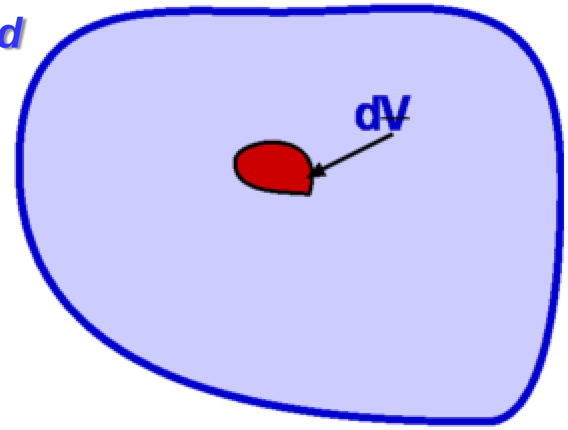
with $m_{system} = \int_{system} dm = \int_{system} \rho d\forall$



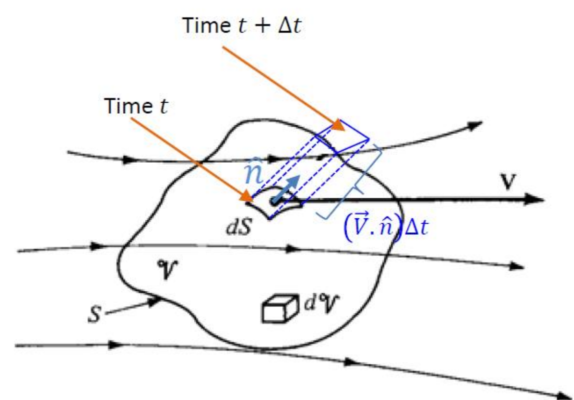
System

$$\frac{DN_s}{Dt} = \frac{D\int_{\forall} \alpha\,\rho\,d\forall}{Dt} = \frac{\partial}{\partial t}\int_{C.V.} \alpha\,\rho\,d\forall + \int_{C.S.} (\alpha\,\rho\vec{V})\bullet d\vec{A}$$

$$Make: \quad \alpha = 1 \qquad \frac{DM_s}{Dt} = \frac{\partial}{\partial t}\int_{C.V.} \rho\,d\forall + \int_{C.S.} (\rho\vec{V})\bullet d\vec{A} = 0$$

- ***Integral form of the Mass Conservation Equation:***

$$\frac{\partial}{\partial t}\int_{C.V.} \rho\,d\forall + \int_{C.S.} (\rho\vec{V})\bullet d\vec{A} = 0$$



❑ Integral Form of the Mas Conservation Equation

- ***Consider one-dimensional nozzle flow as shown in the figure. It is assumed that the flow is steady. Please prove that the following equation is correct:***

$$\frac{d\rho}{\rho}+\frac{dA}{A}+\frac{dV}{V}=0$$

$$\frac{\partial}{\partial t}\int_{C.V.}\rho\, d\forall+\int_{C.S.}(\rho\vec{V})\bullet d\vec{A}=0$$

$$Due\ to\ the\ steady\ flow \Rightarrow \frac{\partial}{\partial t}\int_{C.V.}\rho\, d\forall=0$$

$$Therefore: \int_{C.S.}(\rho\vec{V})\bullet d\vec{A}=0$$

$$\Rightarrow \rho_1 A_1 V_1=\rho_2 A_2 V_2 \Rightarrow \rho AV=const.$$

$$Take\ directive\ of\ \rho AV=const$$

$$\Rightarrow d(\rho AV)=AVd\rho+\rho VdA+\rho AdV=0$$

$$divided\ by\ \rho AV\ at\ two\ sides:$$

$$\frac{AVd\rho+\rho VdA+\rho AdV}{\rho AV}=\frac{0}{\rho AV}$$

$$\Rightarrow \frac{d\rho}{\rho}+\frac{dA}{A}+\frac{dV}{V}=0$$