# Lecture # 06: Conservation of Mass

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# ❑ Feed back of the In-class Quiz #1 (43 response)

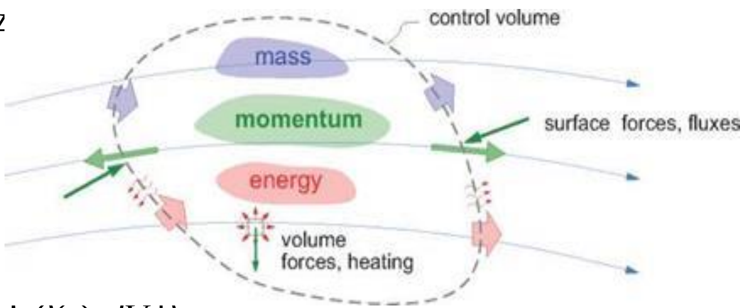
- ***Class teaching speed:***
  - *Speed is okay: ~50%*
  - *Speed is too fast: ~45%*
  - *Speed is too slow: ~ 5%*

- ***Other comments:***
  - *Fast-paced with many derivations.*
  - *Some mathematical terms can be expanded upon and explained more thoroughly*
  - *Why use complicated formula to describe simple problems.*
  - *More examples to link the equations to real word applications.*
  - *More Practice problems on the slides, which can help prepare for HW and Tests.*

# Reynolds Transport Theorem

- ***Let N be any extensive property of the identifiable fixed mass (system) such as total mass, momentum, or energy. The corresponding intensive property (extensive property per unit mass) will be designated as, $\alpha$:***

$$N\Big|_{system} = \int_{system} \alpha \, dm = \int_{\forall} \alpha \, \rho \, d\forall$$




- ***The rate of change of N can be written:***

$$\frac{DN}{Dt}\Big|_{system} = \lim_{\delta t \to 0}\{\frac{1}{\delta t}[\int_{\forall(t+\delta t)} \alpha(t+\delta t)\,\rho(t+\delta t)\,d\forall - \int_{\forall(t)} \alpha(t)\,\rho(t)\,d\forall]\}$$

If we make: $\beta(t) = \alpha(t)\rho(t)$

$$\frac{DN}{Dt}\Big|_{system} = \lim_{\delta t \to 0}\{\frac{1}{\delta t}[\int_{\forall(t+\delta t)} \beta(t+\delta t)\,d\forall - \int_{\forall(t)} \beta(t)\,d\forall]\}$$

$$\frac{DN}{Dt}\Big|_{system} = \frac{D\int_{\forall} \beta \, d\forall}{Dt} = \lim_{\delta t \to 0}\{\frac{1}{\delta t}[\int_{\forall(t+\delta t)} \beta(t+\delta t)\,d\forall - \int_{\forall(t)} \beta(t+\delta t)\,d\forall + \int_{\forall(t)} \beta(t+\delta t)\,d\forall - \int_{\forall(t)} \beta(t)\,d\forall]\}$$

$$= \lim_{\delta t \to 0}\{\frac{1}{\delta t}[\overbrace{\int_{\forall(t+\delta t)} \beta(t+\delta t)\,d\forall - \int_{\forall(t)} \beta(t+\delta t)\,d\forall}^{first\,two} + \overbrace{\int_{\forall(t)} \beta(t+\delta t)\,d\forall - \int_{\forall(t)} \beta(t)\,d\forall}^{sencond\,two}]$$

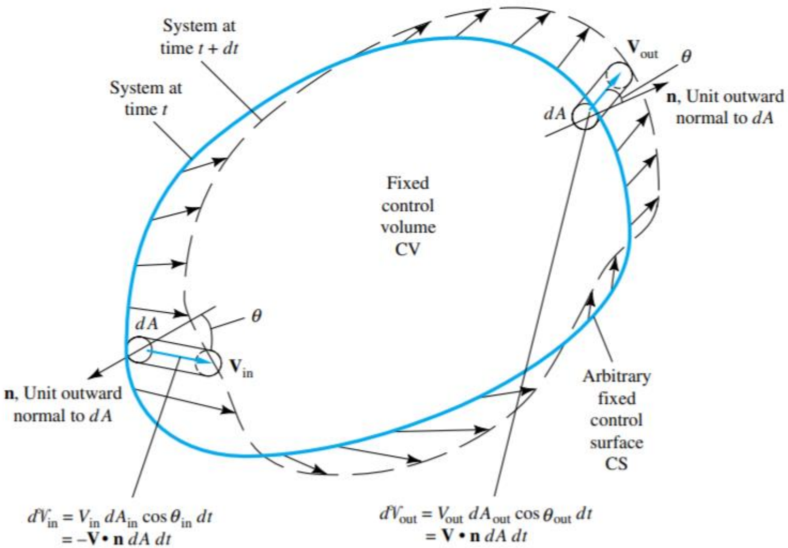
$$= \lim_{\delta t \to 0}\{\frac{1}{\delta t}(\overbrace{\int_{\forall(t+\delta t)} \beta(t+\delta t)\,d\forall - \int_{\forall(t)} \beta(t+\delta t)\,d\forall}^{first\,two})\} + \frac{\partial}{\partial t}\int_{\forall} \beta \, d\forall$$

$$= \lim_{\delta t \to 0}\{\frac{1}{\delta t}(\int_{S(t)} \beta(t+\delta t)(\vec{V} \bullet \hat{e}_n)\delta t \, dA)\} + \frac{\partial}{\partial t}\int_{\forall} \beta \, d\forall$$

$$= \lim_{\delta t \to 0}(\int_{S(t)} \beta(t+\delta t)(\vec{V} \bullet \hat{e}_n)\,dA) + \frac{\partial}{\partial t}\int_{\forall} \beta \, d\forall$$

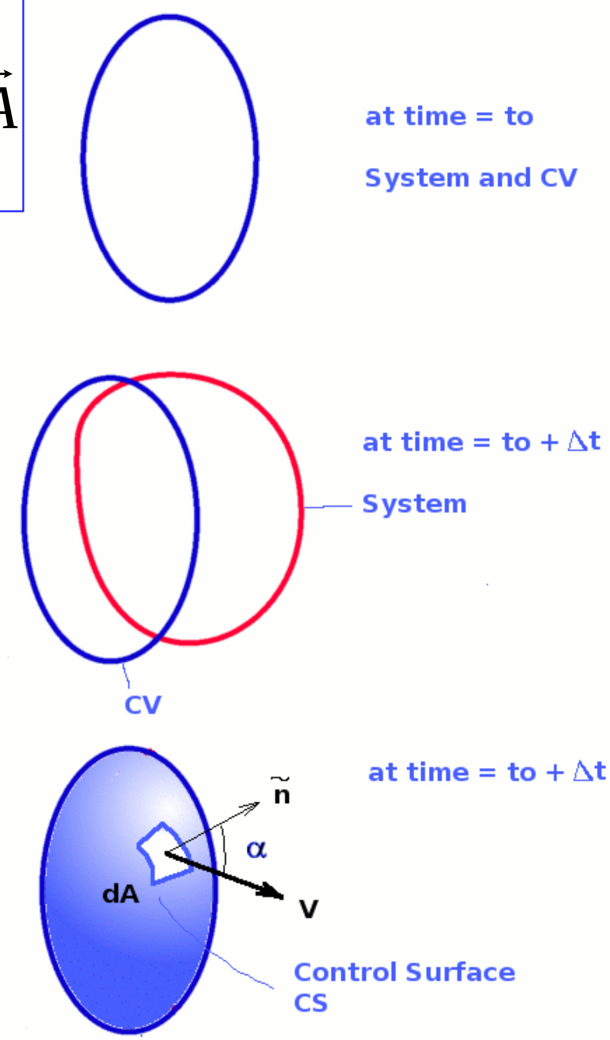
$$= \int_{C.S(t)} \beta(t)(\vec{V} \bullet \hat{e}_n)\,dA) + \frac{\partial}{\partial t}\int_{\forall} \beta \, d\forall$$

$$= \int_{C.S} \beta \vec{V} \bullet d\vec{A} + \frac{\partial}{\partial t}\int_{\forall} \beta \, d\forall = \int_{C.S} \alpha \, \rho \vec{V} \bullet d\vec{A} + \frac{\partial}{\partial t}\int_{\forall} \alpha \, \rho \, d\forall$$



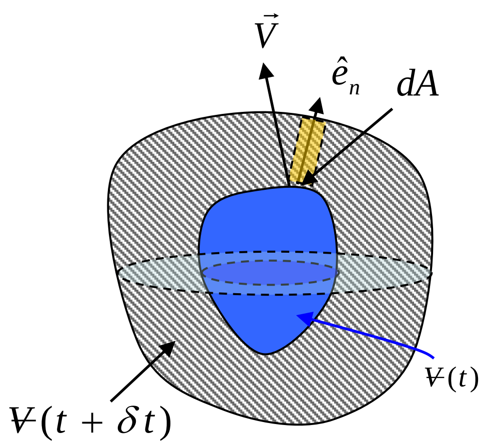
# ❑ Reynolds Transport Theorem

$$\frac{DN_s}{Dt} = \frac{D\int_{\forall} \alpha\,\rho\, d\forall}{Dt} = \frac{\partial}{\partial t}\int_{C.V.} \alpha\,\rho\, d\forall + \int_{C.S.} (\alpha\,\rho \vec{V}) \bullet d\vec{A}$$



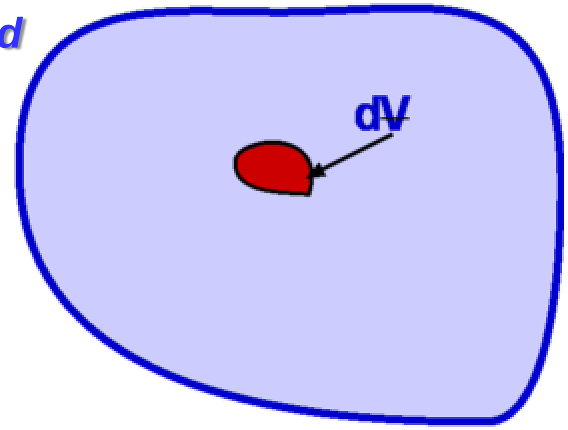

Where $\alpha$ is any intensive property corresponding to $N$. (i.e., $\alpha = N$ per unit mass), and it can be used for different quantities as follows.

| $N_s$ | $\alpha$ |
|---|---|
| Mass | 1 |
| Linear momentum | $\vec{V}$ |
| Angular momentum | $\vec{R} \times \vec{V}$ |
| Energy | $e$ |
| Entropy | $s$ |



# ❑ Conservation of Mass

- *Physical principle: Mass can be neither created or destroyed*




System

$$m_{system} = constant$$

$$\left(\frac{dm}{dt}\right)_{system} = 0$$

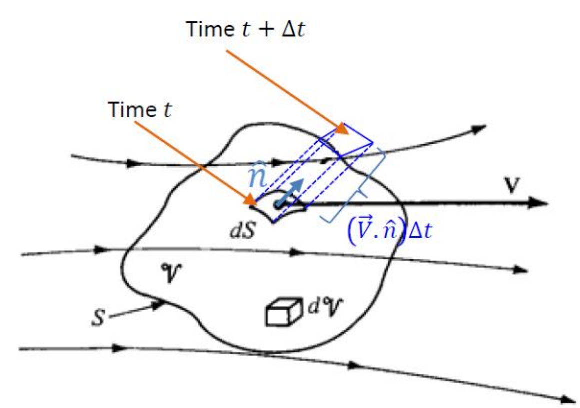
with $m_{system} = \int_{system} dm = \int_{system} \rho d\forall$

$$\frac{DN_s}{Dt} = \frac{D\int_{\forall} \alpha\, \rho\, d\forall}{Dt} = \frac{\partial}{\partial t}\int_{C.V.} \alpha\, \rho\, d\forall + \int_{C.S.} (\alpha\, \rho \vec{V}) \bullet d\vec{A}$$

$$Make: \quad \alpha = 1 \qquad \frac{DM_s}{Dt} = \frac{\partial}{\partial t}\int_{C.V.} \rho\, d\forall + \int_{C.S.} (\rho \vec{V}) \bullet d\vec{A} = 0$$

- *Integral form of the Mass Conservation Equation:*

$$\frac{\partial}{\partial t}\int_{C.V.} \rho\, d\forall + \int_{C.S.} (\rho \vec{V}) \bullet d\vec{A} = 0$$

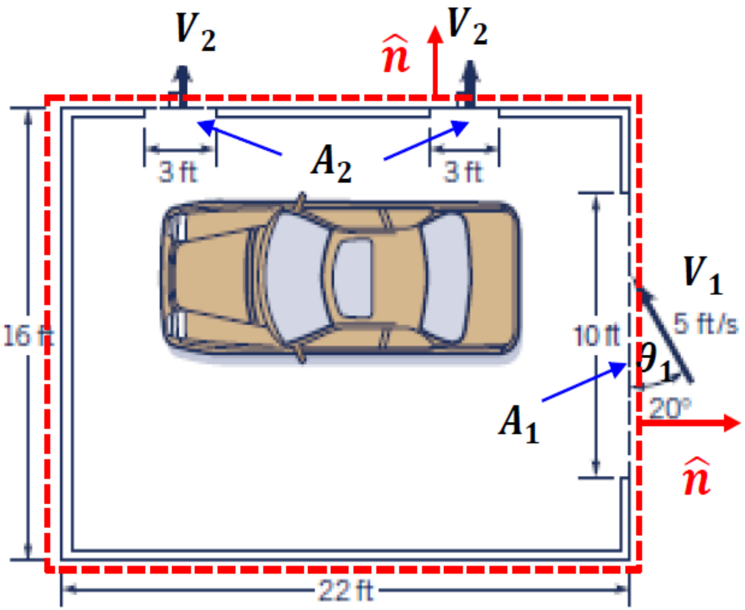


# ❑ Integral Form of the Mas Conservation Equation

- Wind blows through a 7ft × 10ft garage door opening with speed of $V_1 = 5\ ft/s$ as shown in figure below. Determine the average speed, $V_2$, of the air through the two 3 ft× 4ft window openings.

Solution procedure

- Choose and draw an appropriate control volume (the simplest CV along the surfaces we have information on or need to find)
- Write conservation law(s) for the chosen CV
- Evaluate volume integrals over the entire CV volume and surface integrals over control surfaces
- Simplify to find the unknown quantity




Fixed CV

# ❑ Integral Form of the Mas Conservation Equation

Conservation of mass

$$\frac{\partial}{\partial t}\iiint_V \rho dV + \iint_S \rho\vec{V}.\hat{n}dS = 0$$

Here we can assume steady state condition (wind steadily blows in and leaves through the windows), hence

$$\frac{\partial}{\partial t}\iiint_V \rho dV = 0$$

$$\iint_S \rho\vec{V}.\hat{n}dS = \iint_{A_1} \rho\vec{V}.\hat{n}dS + 2\iint_{A_2} \rho\vec{V}.\hat{n}dS$$

Note: there are two identical windows, so it is reasonable to assume average velocity is the same through both.

$$\iint_{A_1} \rho\vec{V}.\hat{n}dS = \iint_{A_1} \rho V_1(-\sin(\theta)\,\hat{\imath} + \cos(\theta)\hat{\jmath}).\hat{\imath}dS$$

$$= -\rho V_1 \sin(\theta)\iint_{A_1} dS = -\rho V_1 \sin(\theta)\, A_1$$

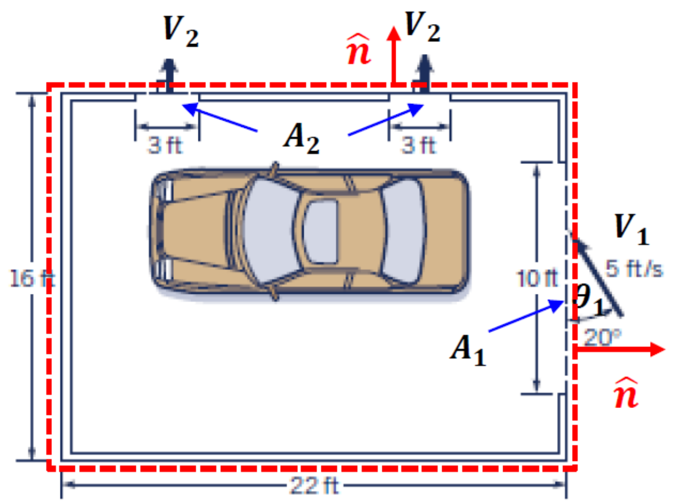
$$\iint_{A_2} \rho\vec{V}.\hat{n}dS = \iint_{A_2} \rho V_2\hat{\jmath}.\hat{\jmath}dS = \rho V_2 A_2$$

- ***Sin(20) ≈0.342***

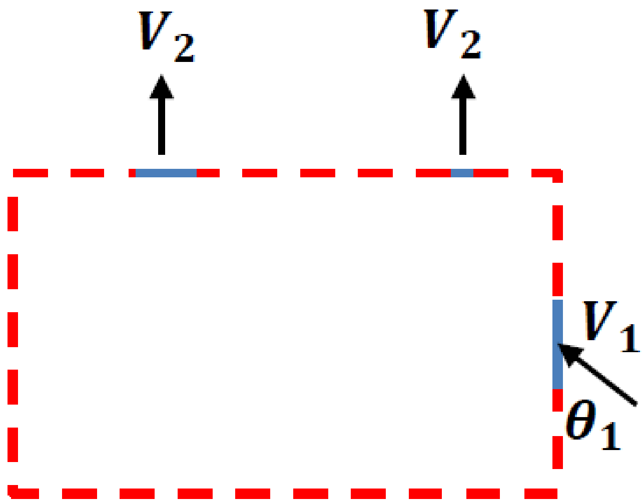
Therefore, conservation of mass becomes

$$-\rho V_1 \sin(\theta)\, A_1 + 2\rho V_2 A_2 = 0 \rightarrow V_2 = V_1 \sin(\theta)\frac{A_1}{2A_2}$$

$$V_2 = 5 \times \sin(20) \times \frac{10 \times 7}{2 \times 3 \times 4} \rightarrow V_2 = 4.99\ ft/s$$

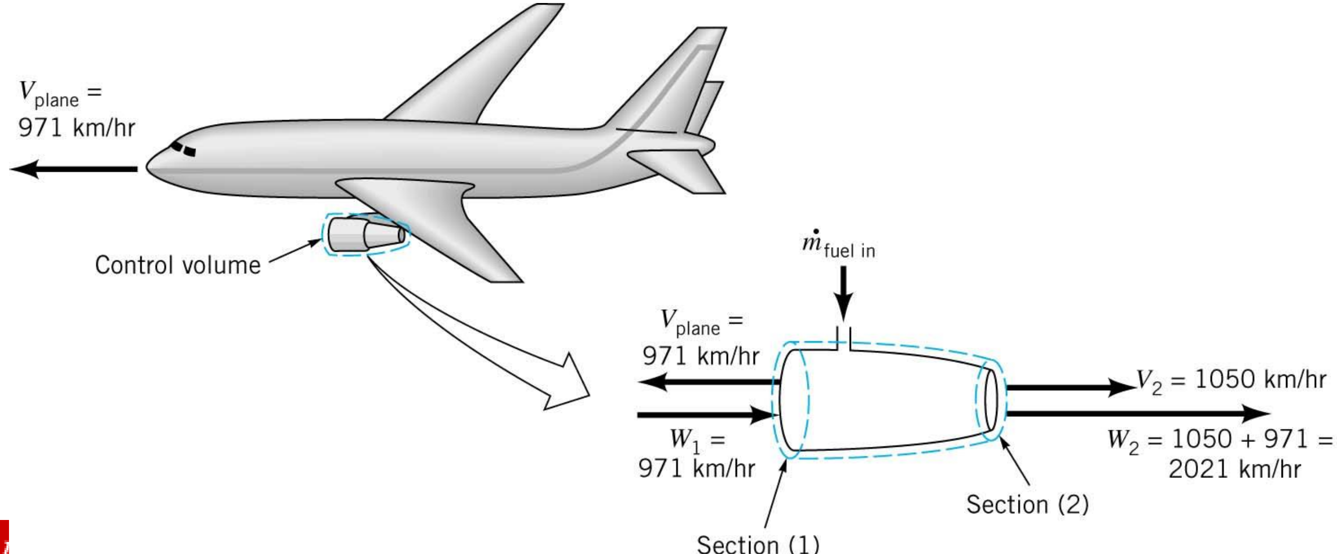



Fixed CV



# ❑ Integral Form of the Mas Conservation Equation

- An airplane moves forward at a speed of 971 km/hr. The frontal area of the intake to one of the jet engines is $0.80 m^2$ , and the entering air density is $0.736 kg/m^3$ . A stationary observer estimates that relative to earth, the jet engine exhaust gases move away from the engine with a speed of 1,050 km/hr. The engine exhaust area is $0.558 m^2$ and the exhaust gas density is $0.515 kg/m^3$ .
  - ***Question: Please estimate the mass flow rate of fuel into the engine in kg/hr.***



# ❑ Integral Form of the Mas Conservation Equation

$$\frac{\partial}{\partial t}\int_{cv} \rho \, d\forall + \int_{cs} \rho \mathbf{W} \cdot \hat{\mathbf{n}} \, dA = 0$$

0 (flow relative to moving control volume is considered steady on a time-average basis)

Assuming one-dimensional flow, we evaluate the surface integral in Eq. 1 and get

$$-\dot{m}_{\substack{\text{fuel}\\\text{in}}} - \rho_1 A_1 W_1 + \rho_2 A_2 W_2 = 0$$

or

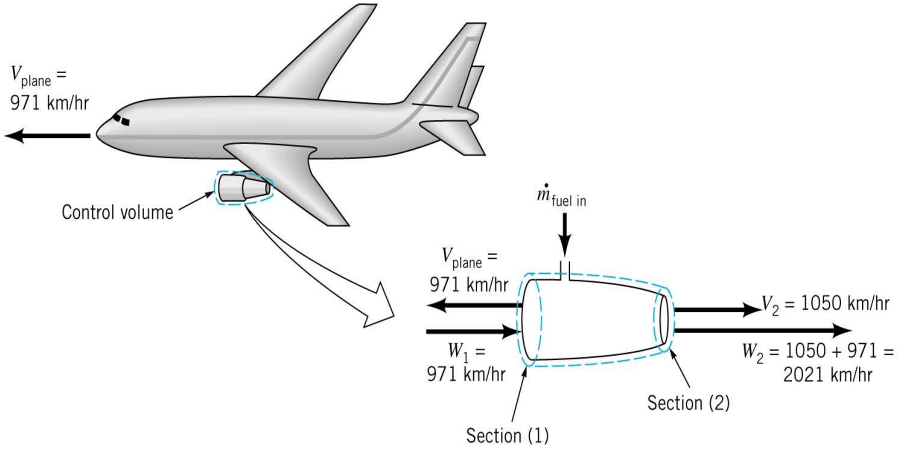
$$\dot{m}_{\substack{\text{fuel}\\\text{in}}} = \rho_2 A_2 W_2 - \rho_1 A_1 W_1$$

We consider the intake velocity, $W_1$, relative to the moving control volume, as being equal in magnitude to the speed of the airplane, 971 km/hr. The exhaust velocity, $W_2$, also needs to be measured relative to the moving control volume. Since a fixed observer noted that the exhaust gases were moving away from the engine at a speed of 1050 km/hr, the speed of the exhaust gases relative to the moving control volume, $W_2$, is determined as follows by using Eq. 5.14

$$V_2 = W_2 + V_{\text{plane}}$$

or

$$W_2 = V_2 - V_{\text{plane}} = 1050 \text{ km/hr} + 971 \text{ km/hr} = 2021 \text{ km/hr}$$



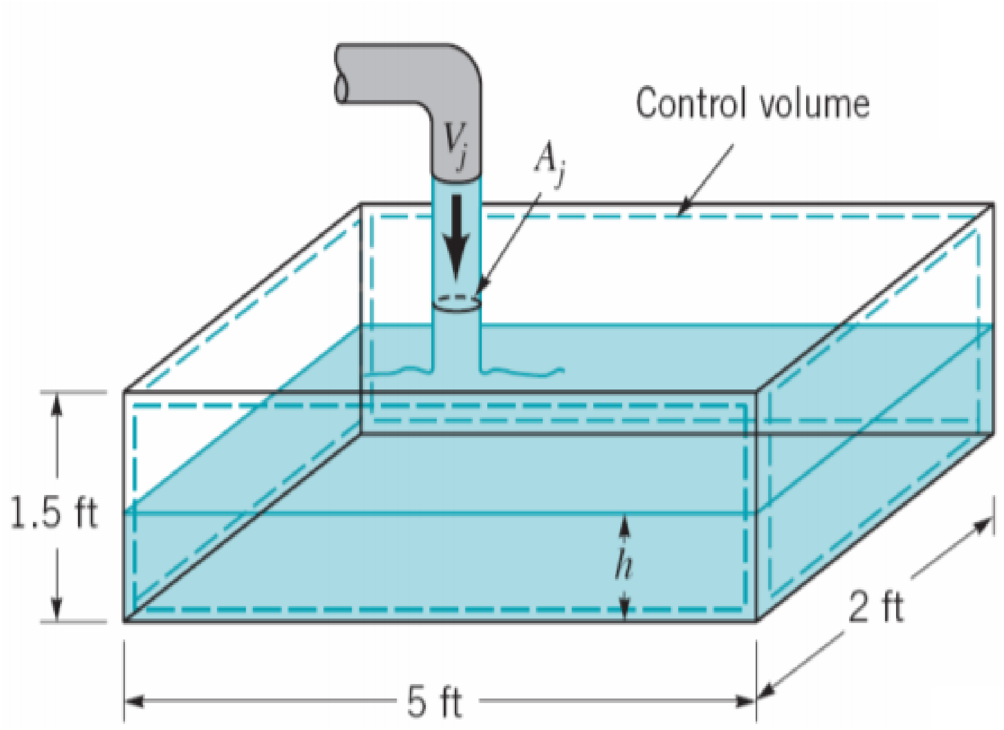

From Eq. 2,

$$\dot{m}_{\substack{\text{fuel}\\\text{in}}} = (0.515 \text{ kg/m}^3)(0.558 \text{ m}^2)(2021 \text{ km/hr})(1000 \text{ m/km}) - (0.736 \text{ kg/m}^3)(0.80 \text{ m}^2)(971 \text{ km/hr})(1000 \text{ m/km})$$
$$= (580{,}800 - 571{,}700) \text{ kg/hr}$$
$$\dot{m}_{\substack{\text{fuel}\\\text{in}}} = 9100 \text{ kg/hr} \qquad \textbf{(Ans)}$$

Note that the fuel flowrate was obtained as the difference of two large, nearly equal numbers. Precise values of $W_2$ and $W_1$ are needed to obtain a modestly accurate value of $\dot{m}_{\substack{\text{fuel}\\\text{in}}}$.

# ❑ Integral Form of the Mas Conservation Equation

- ***A bathtub is being filled with water from a faucet. The rate of flow from the faucet is steady at 9 gal/min. The tub volume is approximated by a rectangular space as indicate the figure. Please estimate the time rate of change of the depth of water in at any instant.***



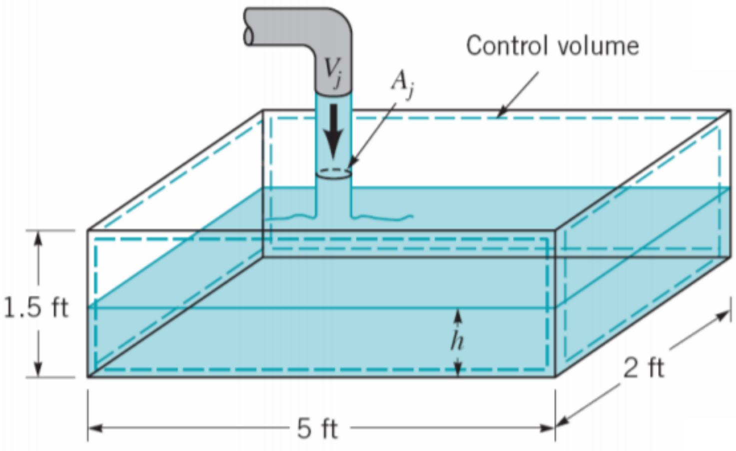
# ❑ Integral Form of the Mas Conservation Equation

- *Conservation of mass equation*

The continuity equation

$$\frac{\partial}{\partial t}\int_{CV}\rho d\forall + \int_{CS}\rho\vec{V}\cdot\vec{n}dA = 0$$

$$= \frac{\partial}{\partial t}\int_{\text{air volume}}\rho_{\text{air}} d\forall_{\text{air}} + \frac{\partial}{\partial t}\int_{\text{water volume}}\rho_{\text{water}} d\forall_{\text{water}} - \dot{m}_{\text{water}} + \dot{m}_{\text{air}}$$




For air $\frac{\partial}{\partial t}\int_{\text{air volume}}\rho_{\text{air}} d\forall_{\text{air}} + \dot{m}_{\text{air}} = 0$

For water $\frac{\partial}{\partial t}\int_{\text{water volume}}\rho_{\text{water}} d\forall_{\text{water}} = \dot{m}_{\text{water}}$

$$\int_{\text{water volume}}\rho_{\text{water}} d\forall_{\text{water}} = \rho_{\text{water}}[h(2ft)(5ft) + (1.5ft - h)A_j]$$

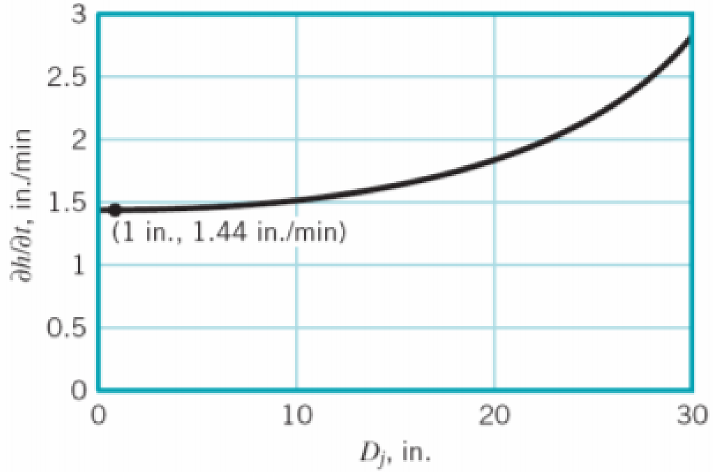
$$\Rightarrow \rho_{\text{water}}(10ft^2 - A_j)\frac{\partial h}{\partial t} = \dot{m}_{\text{water}}$$

and, thus

$$\frac{\partial h}{\partial t} = \frac{Q_{\text{water}}}{(10\text{ ft}^2 - A_j)}$$

For $A_j \ll 10\text{ ft}^2$ we can conclude that

$$\frac{\partial h}{\partial t} = \frac{Q_{\text{water}}}{(10\text{ ft}^2)} = \frac{(9\text{ gal/min})(12\text{ in./ft})}{(7.48\text{ gal/ft}^3)(10\text{ ft}^2)} = 1.44\text{ in./min}$$



# ❑ Integral Form of the Mas Conservation Equation

A balloon is being inflated with an air supply by a small tube. The velocity within the 1-cm-diameter tube is 10 m/s.

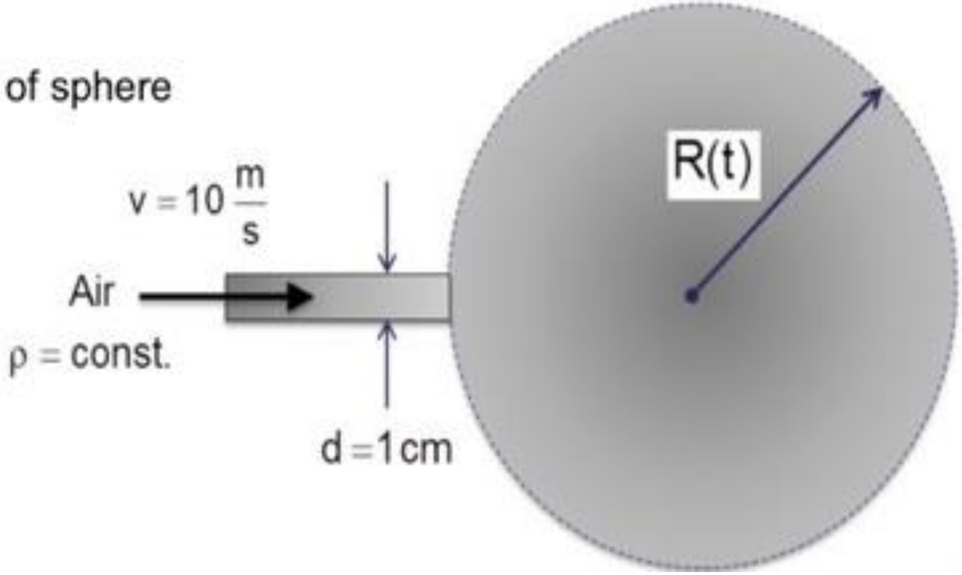
Find the rate of growth of the radius R at the instant when R = 0.5 m.

Given:

Surface of a sphere $A_{Sphere} = 4\pi R^2$ $\quad Ballon \quad volume = \frac{4}{3}\pi R^3$

Assumption:

- Density of air remain constant
- Cross-sectional area of tube << Surface of sphere

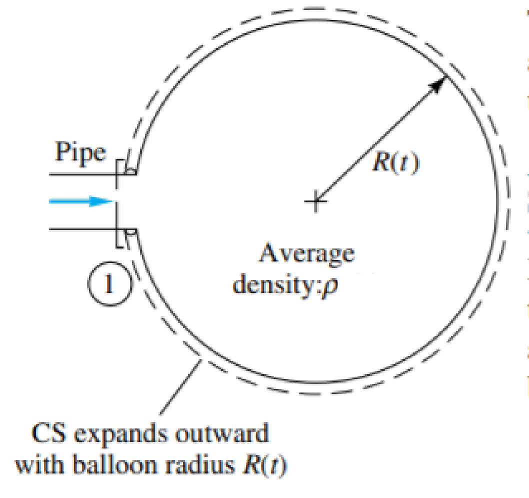


# Integral Form of the Mas Conservation Equation

**Solution**

It is convenient to define a deformable control surface just outside the balloon, expanding at the same rate $R(t)$. Equation (3.16) applies with $V_r = 0$ on the balloon surface and $V_r = V_1$ at the pipe entrance. For mass change, we take $B = m$ and $\beta = dm/dm = 1$. Equation (3.16) becomes

$$\left.\frac{Dm}{Dt}\right|_{system} = \frac{\partial}{\partial t}\int_{C.V.} \rho \, d\mathcal{V} + \int_{C.S.} (\rho \vec{V}) \bullet d\vec{A} = 0$$




Mass flux occurs only at the inlet, so that the control-surface integral reduces to the single negative term $-\rho_1 A_1 V_1$. The fluid mass within the control volume is approximately the average density times the volume of a sphere. The equation thus becomes

$$\left.\frac{Dm}{Dt}\right|_{system} = \frac{\partial}{\partial t}\left(\frac{4}{3}\pi R^3 \rho\right) + \int_{C.S.} (\rho \vec{V}) \bullet d\vec{A} = 0$$

$$\Rightarrow \frac{4}{3}\pi\rho\, 3R^2 \frac{\partial R}{\partial t} - \rho V_1 A_1 = 0$$

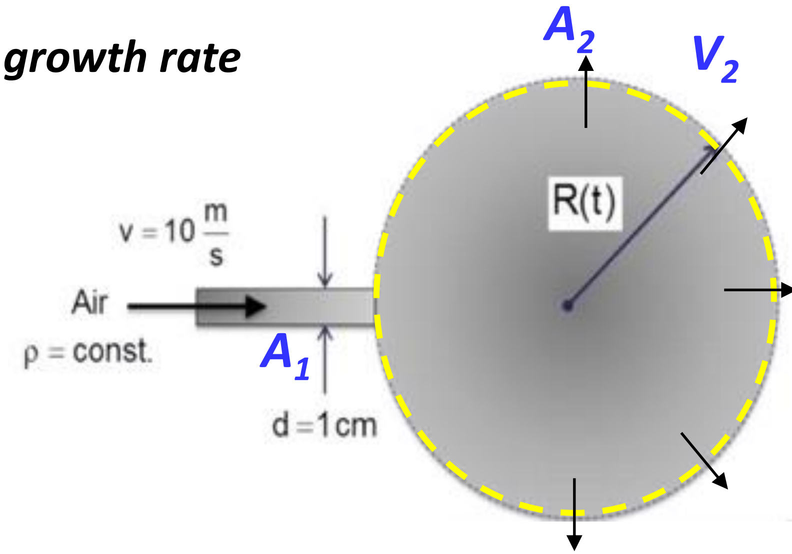
$$\Rightarrow \frac{\partial R}{\partial t} = \frac{V_1 A_1}{4\pi R^2} = \frac{V_1 * \pi r_1^2}{4\pi R^2} = \frac{V_1 * r_1^2}{4R^2} = \frac{10 * 0.005^2}{4 * 0.5^2} = 2.5\times 10^{-4} \quad m/s$$

# ❑ Integral Form of the Mas Conservation Equation

$$\left.\frac{Dm}{Dt}\right|_{system} = \frac{\partial}{\partial t}\int_{C.V.} \rho \, d\forall + \int_{C.S.} (\rho \vec{V}) \bullet d\vec{A} = 0$$

- ***Chose the control volume inside the balloon***
- ***Flow velocity will be the same as the balloon radius growth rate***

$$\frac{dR}{dt} = V_2$$




- ***Flow inside the control volume is steady, therefore:***

$$\left.\frac{Dm}{Dt}\right|_{system} = \frac{\partial}{\partial t}\int_{C.V.} \rho \, d\forall + \int_{C.S.} (\rho \vec{V}) \bullet d\vec{A} = 0$$

$$\left.\frac{Dm}{Dt}\right|_{system} = 0 + \int_{C.S.} (\rho \vec{V}) \bullet d\vec{A} = 0$$

$$\Rightarrow \rho A_2 V_2 - \rho V_1 A_1 = 0$$

$$\Rightarrow \rho 4\pi R^2 \frac{\partial R}{\partial t}_2 - \rho V_1 \pi r_1^2 = 0$$

$$\Rightarrow \frac{\partial R}{\partial t} = V_2 = \frac{V_1 * \pi r_1^2}{4\pi R^2} = \frac{V_1 * r_1^2}{4R^2} = \frac{10 * 0.005^2}{4 * 0.5^2} == 2.5\times 10^{-4} \quad m/s$$