Lecture # 08: Conservation of Momentum

Dr. Hui HU

Department of Aerospace Engineering

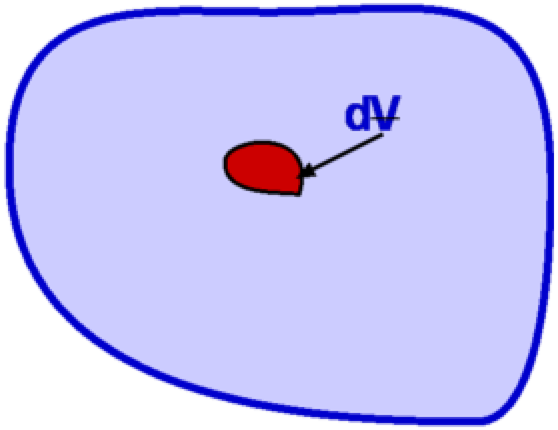
Iowa State University, 2251 Howe Hall, Ames, IA 50011-2271

Tel: 515-294-0094 / Email: huhui@iastate.edu

❑ Conservation of Momentum

- *Newton's second law states that:*
  [Time change rate of momentum of a system] = [Resultant external force acting on the system]

$$\boxed{\frac{d\vec{M}_S}{dt} = \sum \vec{F}_S = \sum \vec{F}_{Surface} + \sum \vec{F}_{body}}$$



System

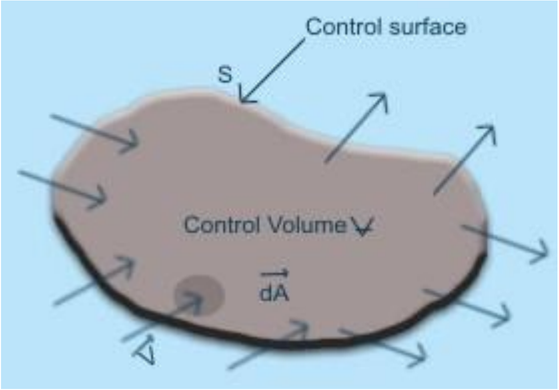
Reynolds Transport Theorem:

$$\left.\frac{dN_s}{dt}\right|_{system} = \frac{d\int_{\forall} \alpha\, \rho\, d\forall}{Dt} = \frac{\partial}{\partial t}\int_{C.V.} \alpha\, \rho\, d\forall + \int_{C.S.} (\alpha\, \rho\vec{V})\bullet d\vec{A}$$

$$Make: \quad \alpha = \vec{V} \qquad \left.\frac{DM_s}{Dt}\right|_{system} = \frac{\partial}{\partial t}\int_{C.V.} \rho\vec{V}\, d\forall + \int_{C.S.} (\vec{V}\, \rho\vec{V})\bullet d\vec{A} = 0$$

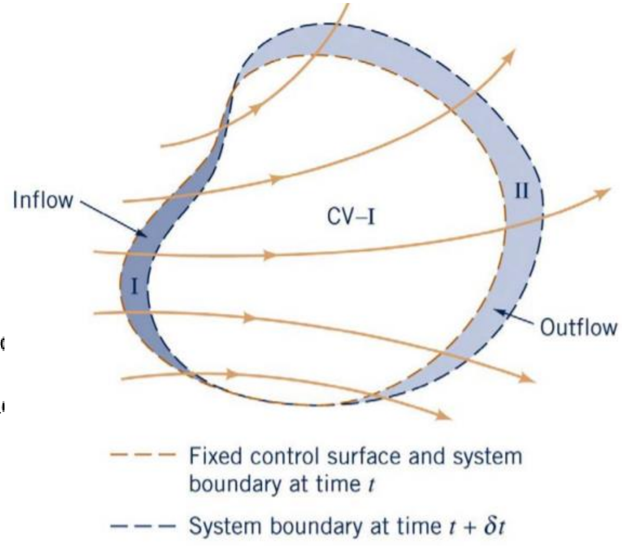
- ***Integral form of the Mass Conservation Equation:***

$$\frac{\partial}{\partial t}\int_{C.V.} \vec{V}\, \rho\, d\forall + \int_{C.S.} (\vec{V}\, \rho\vec{V})\bullet d\vec{A} = \vec{F}_{surface} + \vec{F}_{body}$$



❑ Conservation of Momentum – Integral form

$$\frac{\partial}{\partial t}\int_{C.V.}\vec{V}\,\rho\, d\forall + \int_{C.S.}(\vec{V}\,\rho\vec{V})\bullet d\vec{A} = \vec{F}_{surface}+\vec{F}_{body}$$



$\vec{F}_{surface}$

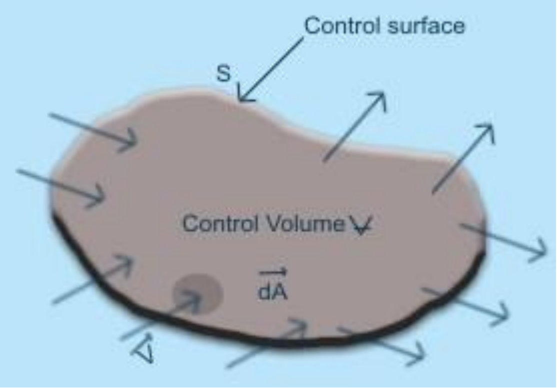
Surface forces such as pressure and shear stress. The surface forces usually

expressed as $\vec{F}_{surface} = \int_{C.S.}\tilde{P}\bullet d\vec{A}$, where $\tilde{P}$ is the stress tensor exerted by th

surroundings on the particle surface. $\tilde{P} = -P\tilde{I} + \tilde{\tau}$

$\vec{F}_{body}$

Body forces such as electromagnetic, gravitational forces. Usually the body force can be expressed as $\vec{F}_{body} = \int_{C.V.}\rho\,\vec{f}\,d\forall$, where $\vec{f}$ is a vector which references the resultant force per unit mass.



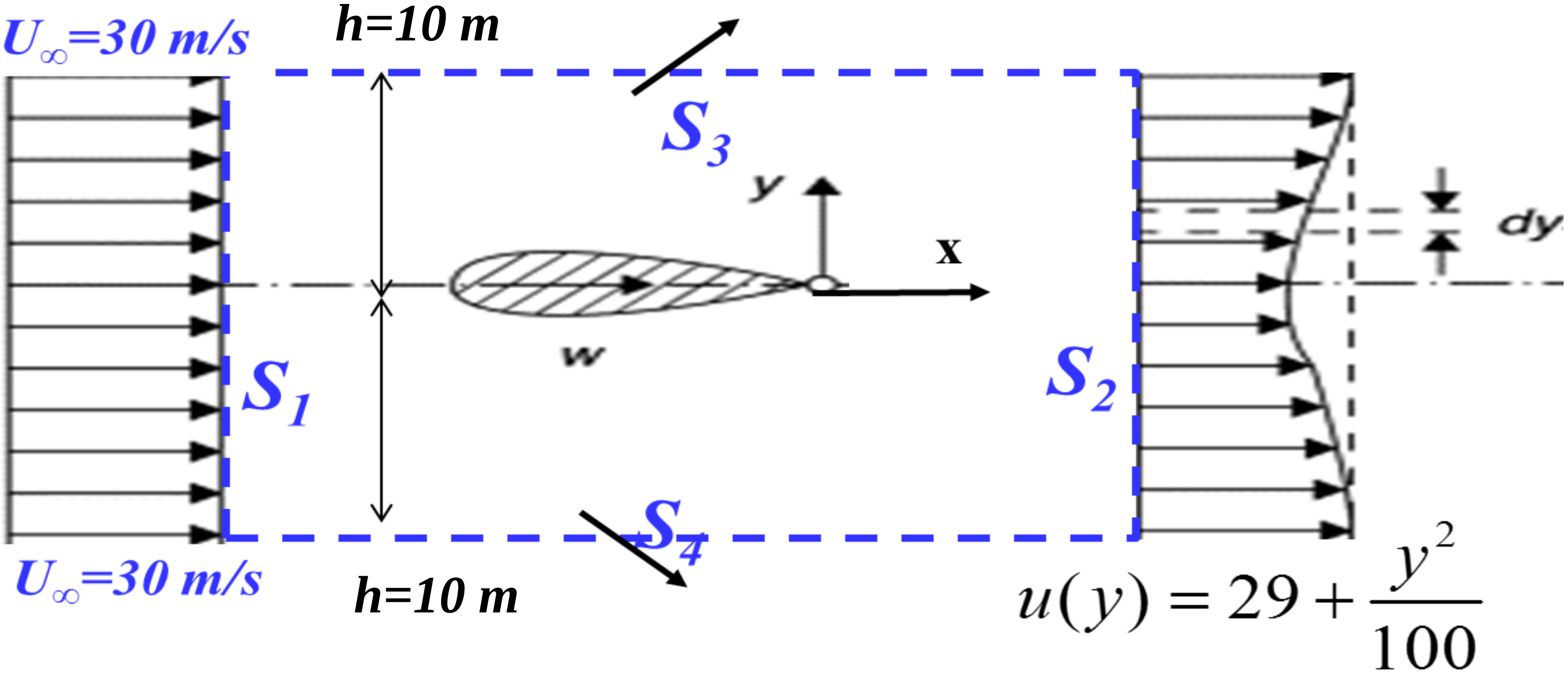
$$\frac{\partial}{\partial t}\int_{C.V.}\rho\vec{V}\, d\forall + \int_{C.S.}(\rho\vec{V}\vec{V})\bullet d\vec{A} = \int_{C.S.}\tilde{P}\bullet d\vec{A}+\int_{C.V.}\rho\,\vec{f}\, d\forall$$

❑ Conservation of Momentum – Integral form

Example-Drag on an airfoil

- Problem : Two-dimensional steady flow around the airfoil where velocity profile is given (measured) at upstream and a downstream location. Find an expression for the drag force on the airfoil.

- ***Pressure around the control volume is a constant p_∞***
- ***The air density is $\rho = 1.23\ kg/m^3$***

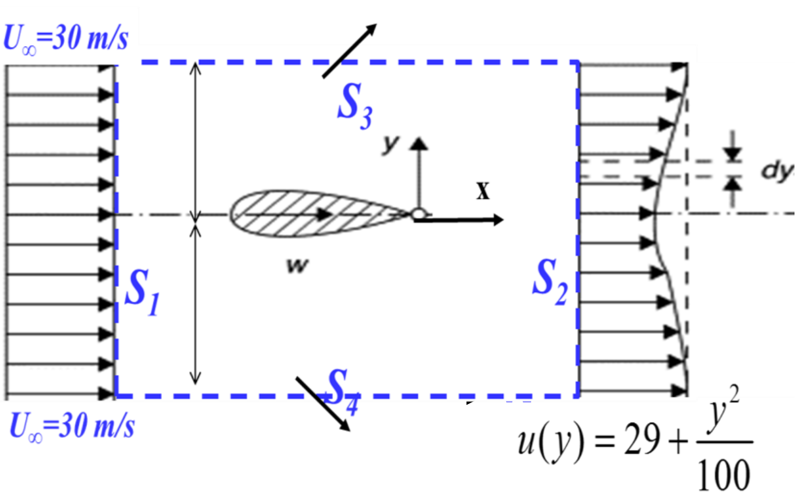


$$u(y) = 29 + \frac{y^2}{100}$$

❑ Conservation of Momentum – Integral form

$$\frac{\partial}{\partial t}\int_{C.V.}\rho\, d\forall + \int_{C.S.}(\rho\vec{V})\bullet d\vec{A} = 0$$

- **Steady flow:** $$\frac{\partial}{\partial t}\int_{C.V.}\rho\, d\forall = 0$$



$$\int_{C.S.}(\rho\vec{V})\bullet d\vec{A} = \int_{S_1}(\rho\vec{V})\bullet d\vec{A} + \int_{S_2}(\rho\vec{V})\bullet d\vec{A} + \int_{S_3}(\rho\vec{V})\bullet d\vec{A} + \int_{S_4}(\rho\vec{V})\bullet d\vec{A}$$

$$\Rightarrow \dot{m}_3 + \dot{m}_4 = 2\dot{m}_3 = -[\int_{S_1}(\rho\vec{V})\bullet d\vec{A} + \int_{S_2}(\rho\vec{V})\bullet d\vec{A}]$$

$$= -\rho[-\int_{-h}^{h} U_\infty dy + \int_{-h}^{h} u(y)dy]$$

$$= \rho[\int_{-h}^{h}[U_\infty - u(y)]dy = 1.23[\int_{-10}^{10}[30-(29+\frac{y^2}{100})]dy = 16.4\ Kg/s\ per\ unit\ span$$

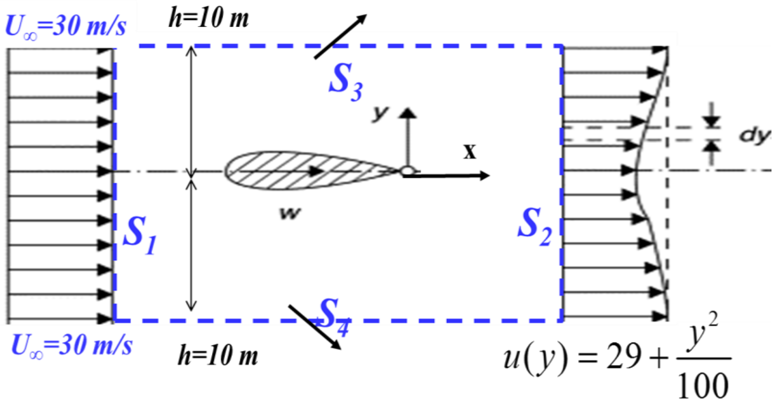
$$\Rightarrow \dot{m}_3 = 8.2Kg/s\ per\ unit\ span$$

❑ Conservation of Momentum – Integral form

Problem Solution:

1. ***Chose a control volume:***
2. ***Applying momentum conservation equation.***

$$\frac{\partial}{\partial t}\int_{C.V.}\vec{V}\rho\, d\forall+\int_{C.S.}(\vec{V}\rho\vec{V})\bullet d\vec{A}=\sum\vec{F}_{surface}+\sum\vec{F}_{body}$$



$$u(y)=29+\frac{y^2}{100}$$

- Since the flow is steady, the momentum equation is simplified as:

$$\int_{C.S.}(\vec{V}\rho\vec{V})\bullet d\vec{A}=\sum\vec{F}_{surface}+\sum\vec{F}_{body}$$

- Forces along X- direction:

$$\sum\vec{F}=-D$$

- Momentum along X- direction:

$$\int_{C.S.}(\vec{V}\rho\vec{V})\bullet d\vec{A}=\int_{S_1}(u\,\rho\vec{V})\bullet d\vec{A}+\int_{S_2}(u\,\rho\vec{V})\bullet d\vec{A}+\int_{S_3}(u\,\rho\vec{V})\bullet d\vec{A}+\int_{S_4}(u\,\rho\vec{V})\bullet d$$

$$=-\int_{S_1}\rho U_\infty{}^2\bullet dy+\int_{S_2}\rho u(y)^2\,dy+\int_{S_3}U_\infty(\rho\vec{V})\bullet d\vec{A}+\int_{S_4}U_\infty(\rho\vec{V})\bullet d\vec{A}$$

$$=-20*1.23*30^2+1.23*\int_{S_2}[29+\frac{y^2}{100}]^2\,dy+U_\infty\dot{m}_3+U_\infty\dot{m}_4$$

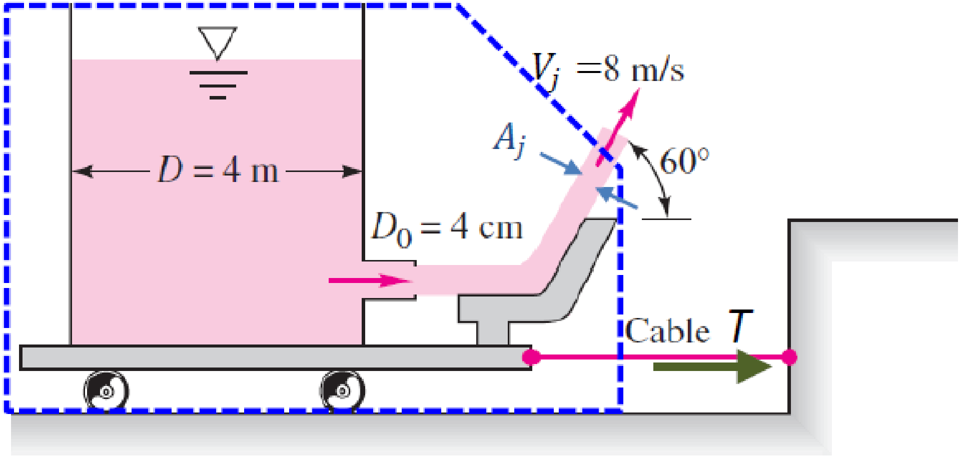
$$=-20*1.23*30^2+1.23*\int_{S_2}[29+\frac{y^2}{100}]^2\,dy+20*16.4$$

$$\Rightarrow D=22140-21170-492=478N/m$$

❑ Conservation of Momentum – Integral form

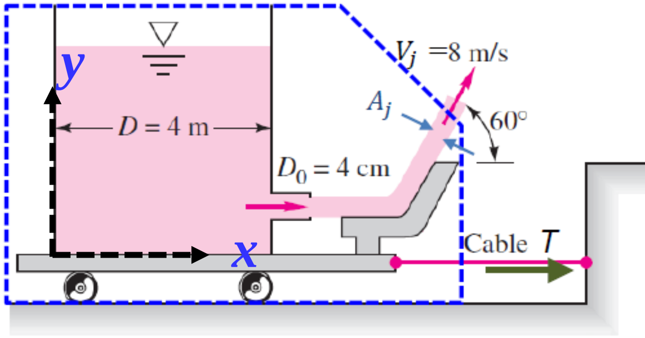
Example 2

- The water tank in the figure stands on a frictionless cart and feeds a jet of diameter D_j =4 cm and velocity V_j =8 m/s, which is deflected θ =60° by a vane. Compute the tension in the supporting cable.



Conservation of Momentum – Integral form

$$\frac{\partial}{\partial t}\int_{C.V.}\rho\vec{V}\,d\forall+\int_{C.S.}(\rho\vec{V}\vec{V})\bullet d\vec{A}=\int_{C.S.}\tilde{\vec{P}}\bullet d\vec{A}+\int_{C.V.}\rho\,\vec{f}\,d\forall$$



Conservation of x-momentum

$$\frac{\partial}{\partial t}\iiint\limits_{V}\rho u dV+\iint\limits_{S}\rho u(\vec{V}.\hat{n})dS=\iint -(p\hat{n})_x dS+\iiint\limits_{V}\rho f_x dV+F_{x_{vis}}+T$$

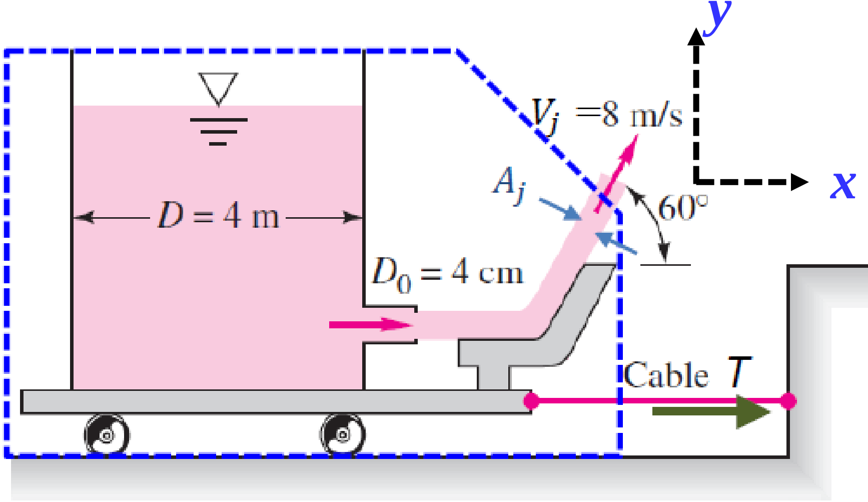
Since the volume of the tank is much larger than the jet diameter, we can assume the flow inside the tank is mostly vertical, i.e. $u = 0$ inside the tank, therefore

$$\frac{\partial}{\partial t}\iiint\limits_{V}\rho u dV=0$$

There is no body force on CV and no viscous force on control surfaces.

Also $\iint \;-(p\hat{n})_x dS=0$ since pressure is atmospheric everywhere.

Conservation of Momentum – Integral form



Conservation of x-momentum

$$\underbrace{\cancel{\frac{\partial}{\partial t}\iiint_V \rho u dV}}_{= 0} + \iint_S \rho u(\vec{V}.\hat{n})dS = \underbrace{\cancel{\iint -(p\hat{n})_x dS}}_{= 0} + \underbrace{\cancel{\iiint_V \rho f_x dV}}_{= 0} + \underbrace{\cancel{F_{x_{vis}}}}_{= 0} + T$$

- The x-momentum then simplifies to

$$\iint_S \rho u(\vec{V}.\hat{n})dS = T$$

$$\iint_{A_j} \rho V_j \cos\theta \,(V_j) dS = T$$

$$T = \rho A_j V_j^2 \cos\theta$$

$$T = 1000\frac{kg}{m^3}\left(\frac{\pi}{4} \times 0.04^2 m^2\right) \times 8^2 \frac{m^2}{s^2} \times \cos 60 \approx 40\ N$$

❑ Conservation of Momentum – Integral form

- *Conservation of mass for a non-inertial frame of reference*

$$\frac{\partial}{\partial t}\iiint_{V} \rho dV + \iint_{S} \rho(\vec{V} - \vec{V}_b).\hat{n}dS = 0$$

- *Conservation of Momentum for a non-inertial frame of reference*

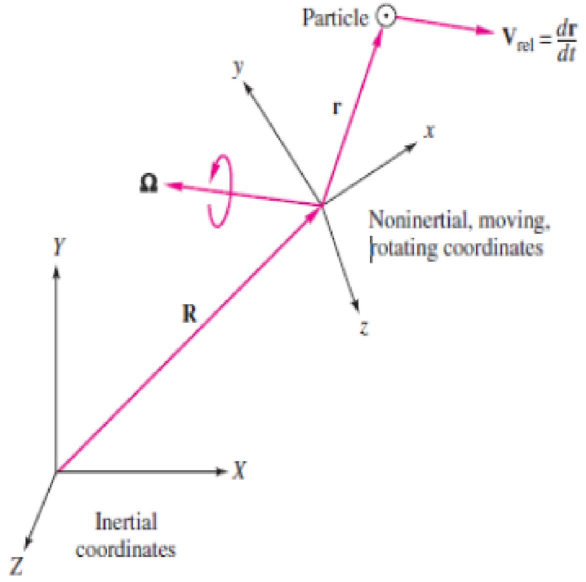
$$\frac{\partial}{\partial t}\iiint_{V} \rho\vec{V} dV + \iint_{S} \rho\vec{V}(\vec{V} - \vec{V}_b).\hat{n}dS = \iint -p\hat{n}dS + \iiint_{V} \rho\vec{f} dV + \vec{F}_{vis}$$

$\vec{V}$s are in an inertial reference frame

For a non-inertial frame of reference

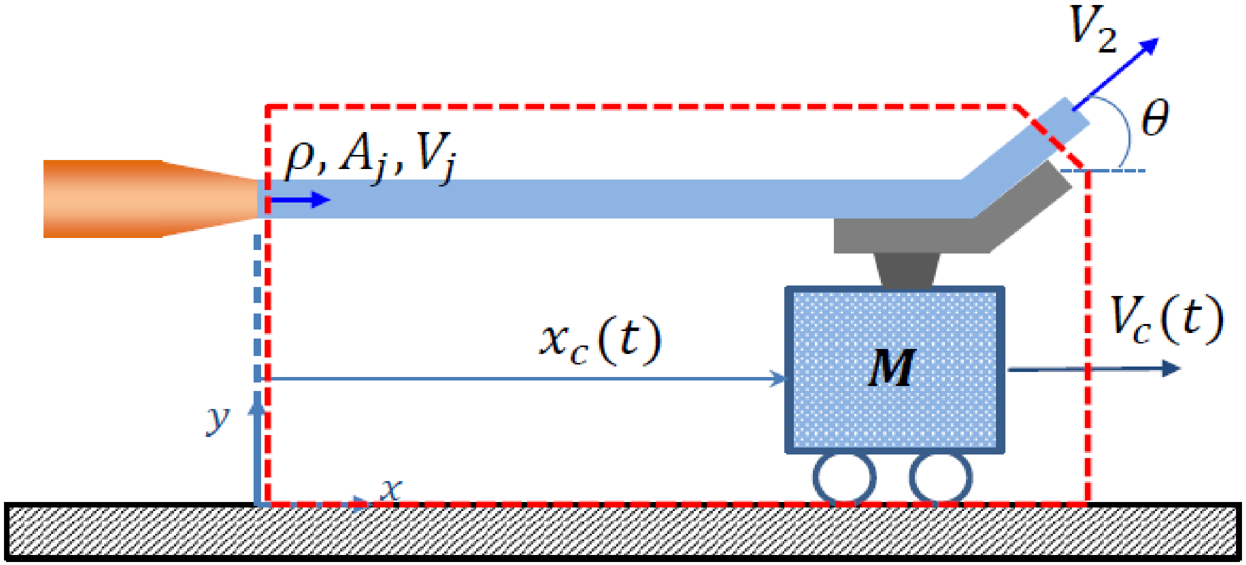
$$\frac{\partial}{\partial t}\iiint_{V} \rho\vec{V} dV + \iint_{S} \rho\vec{V}(\vec{V} - \vec{V}_b).\hat{n}dS$$

$$= \iint -p\hat{n}dS + \iiint_{V} \rho\vec{f} dV + \vec{F}_{vis} - \iiint_{V} \vec{a}_{rel}\, dm$$



☐ Conservation of Momentum – Integral form

- Problem : In figure below, the jet strikes a vane that moves to the right at velocity V_c on a frictionless cart with mass M. The cart is at rest initially, find an expression for cart velocity $V_c(t)$. What is the terminal velocity?



- *Control Volume*

❑ Conservation of Momentum – Integral form

Example-solution

Conservation of mass

$$\frac{\partial}{\partial t}\iiint_{\mathrm{V}} \rho d\mathrm{V} + \iint_{S} \rho(\vec{V}-\vec{V}_b).\hat{n}dS = 0$$

Consider the mass inside the control volume. (we can ignore the mass of water in contact with the vane, $m_{jv} \approx 0$):

$$\frac{\partial}{\partial t}\iiint_{\mathrm{V}} \rho d\mathrm{V} = \frac{d}{dt}(M + \rho A_j x_c + m_{jv}) = \rho A_j \frac{dx_c}{dt} = \rho A_j V_c$$

$\iint_S \rho(\vec{V}-\vec{V}_b).\hat{n}dS$ on the left surface is simply $-\rho A_j V_j$. On the right surface, where the jet leaves the control volume:

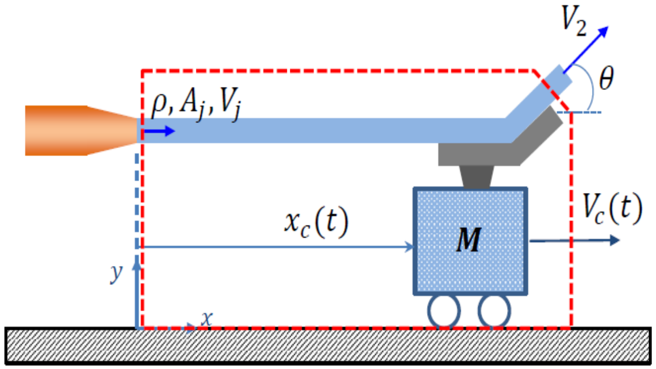
$$\vec{V} = \vec{V_2} + V_c\hat{\imath}\,, \qquad \vec{V_b} = V_c\hat{\imath} \rightarrow \vec{V}-\vec{V}_b = \vec{V}_2$$

And the normal unit vector is in the direction of $\vec{V}_2$. Hence:

$$\iint_{S} \rho(\vec{V}-\vec{V}_b).\hat{n}dS = -\rho A_j V_j + \rho A_j V_2$$

Therefore

$$\rho A_j V_c - \rho A_j V_j + \rho A_j V_2 = 0 \rightarrow V_2 = V_j - V_c$$



❑ Conservation of Momentum – Integral form

Example-continued

Conservation of x-momentum

$$\frac{\partial}{\partial t}\iiint_{\mathrm{V}} \rho u d\mathrm{V} + \iint_{S} \rho u\left(\vec{V} - \vec{V}_b\right).\hat{n}dS$$

$$= \iint -(p\hat{n})_x dS + \iiint_{\mathrm{V}} \rho f_x d\mathrm{V} + F_{x_{vis}}$$

Ignoring the viscous effects and noting $f_x = 0$ as well as $\iint \ -(p\hat{n})_x dS = 0$ (atmospheric everywhere):

$$\frac{\partial}{\partial t}\iiint_{\mathrm{V}} \rho u d\mathrm{V} + \iint_{S} \rho u(\vec{V} - \vec{V}_b).\hat{n}dS = 0$$

$$\frac{d}{dt}(MV_c + \rho A_j V_j x_c) - \rho A_j V_j^2 + \rho(V_2 \cos\theta + V_c)V_2 A_j = 0$$

❑ Conservation of Momentum – Integral form

Example-continued

$$\begin{aligned} &M\frac{dV_c}{dt} + \rho A_j V_j V_c - \rho A_j V_j^2 + \rho(V_j \\ &- V_c)\big((V_j - V_c)\cos\theta + V_c\big)A_j = 0 \end{aligned}$$

$$\begin{aligned} &M\frac{dV_c}{dt} + \rho A_j V_j V_c - \rho A_j V_j^2 + \rho A_j (V_j - V_c)^2 \cos\theta \\ &+ \rho A_j V_c (V_j - V_c) = 0 \end{aligned}$$

$$M\frac{dV_c}{dt} + \rho A_j (V_j - V_c)^2 \cos\theta - \rho A_j (V_j^2 - 2V_c V_j + V_c^2) = 0$$

$$M\frac{dV_c}{dt} + \rho A_j (V_j - V_c)^2 (\cos\theta - 1) = 0$$

❑ Conservation of Momentum – Integral form

Example-continued

$$M\frac{dV_c}{dt} = \rho A_j\left(V_j - V_c\right)^2(1 - \cos\theta)$$

$$\frac{dV_c}{\left(V_j - V_c\right)^2} = \frac{\rho A_j}{M}(1 - \cos\theta)$$

Define $K = \frac{\rho A_j}{M}(1 - \cos\theta)$ and Integrate from $t = 0$ to time t

$$V_c(t) = \frac{V_j^2 Kt}{1 + V_j Kt}$$

Terminal velocity when $t \to \infty$

$$V_c(t \to \infty) = V_j$$