Lecture # 09: Conservation Equations of Mass and Momentum in Differential Form-P1

Dr. Hui HU

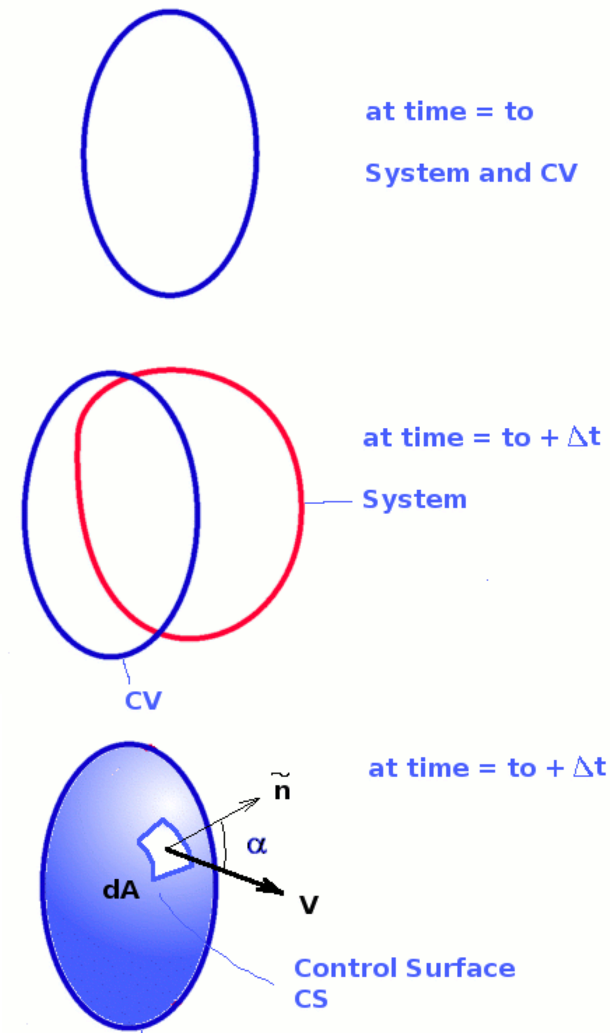
Department of Aerospace Engineering

Iowa State University, 2251 Howe Hall, Ames, IA 50011-2271

Tel: 515-294-0094 / Email: huhui@iastate.edu

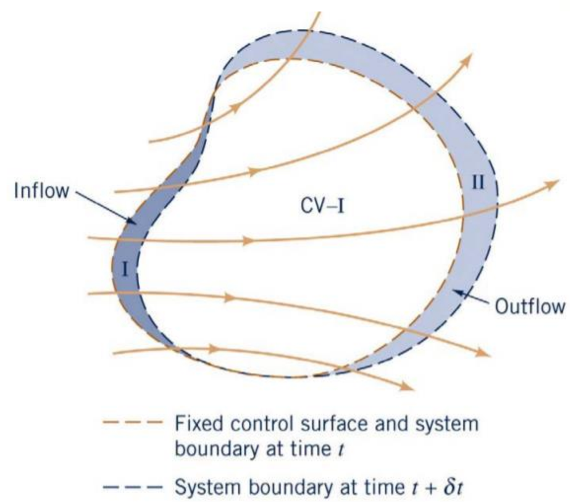
Reynolds Transport Theorem

$$\frac{DN_s}{Dt} = \frac{D\int_{\forall} \alpha \, \rho \, d\forall}{Dt} = \frac{\partial}{\partial t}\int_{C.V.} \alpha \, \rho \, d\forall + \int_{C.S.} (\alpha \, \rho \vec{V}) \bullet d\vec{A}$$



Where α is any intensive property corresponding to N. (i.e., $\alpha = N$ per unit mas and it can be used for different quantities as follows.

N_s	α
Mass	1
Linear momentum	$\vec{V}$
Angular momentum	$\vec{R} \times \vec{V}$
Energy	e
Entropy	s



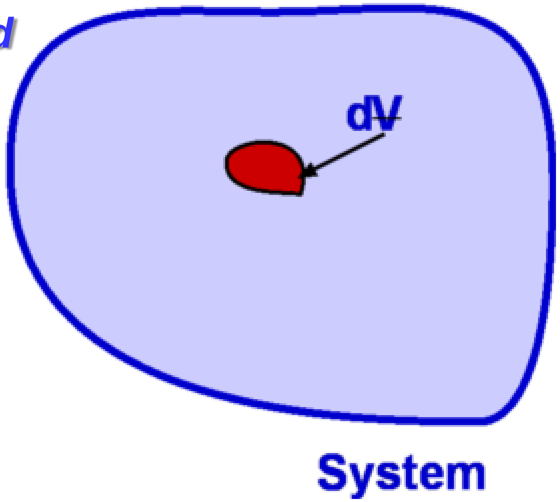
❑ Conservation of Mass

- *Physical principle: Mass can be neither created or destroyed*

$$m_{system} = constant$$

$$\left(\frac{dm}{dt}\right)_{system} = 0$$

with $m_{system} = \int_{system} dm = \int_{system} \rho d\forall$

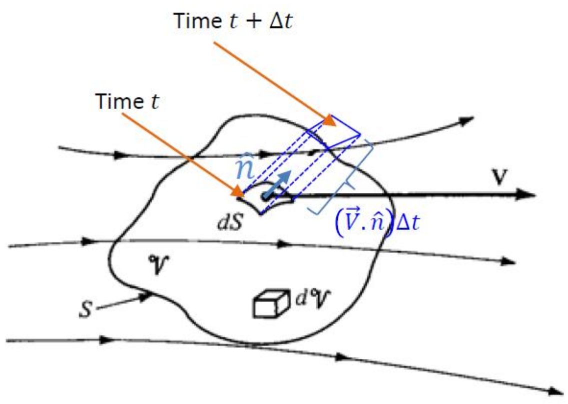


$$\frac{DN_s}{Dt} = \frac{D\int_{\forall} \alpha\,\rho\, d\forall}{Dt} = \frac{\partial}{\partial t}\int_{C.V.} \alpha\,\rho\, d\forall + \int_{C.S.} (\alpha\,\rho \vec{V}) \bullet d\vec{A}$$

$$Make: \quad \alpha = 1 \qquad \frac{DM_s}{Dt} = \frac{\partial}{\partial t}\int_{C.V.} \rho\, d\forall + \int_{C.S.} (\rho \vec{V}) \bullet d\vec{A} = 0$$

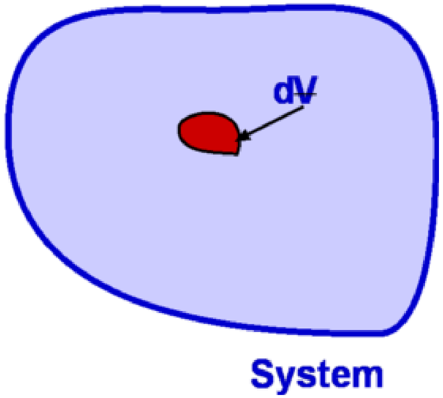
- *Integral form of the Mass Conservation Equation:*

$$\frac{\partial}{\partial t}\int_{C.V.} \rho\, d\forall + \int_{C.S.} (\rho \vec{V}) \bullet d\vec{A} = 0$$



❑ Conservation of Mass

- *Physical principle: Mass can be neither created or destroyed.*

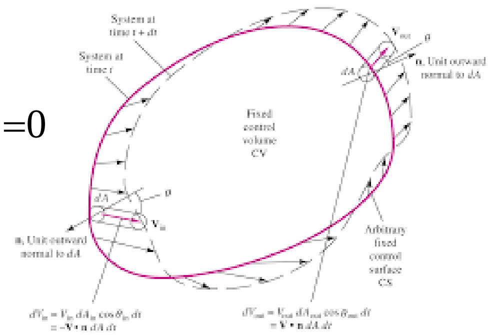


- *Integral form:*

$$\int_{C.\forall.} \frac{\partial \rho}{\partial t}\, d\forall + \int_{C.S.} (\rho \vec{V}) \bullet d\vec{A} = 0$$

- *Applying Guess divergence theorem, we convert the surface integral to volume integral to obtain:*

$$\int_{C.\forall.} \frac{\partial \rho}{\partial t}\, d\forall + \int_{C.S.} (\rho \vec{V}) \bullet d\vec{A} = \int_{C.V.} \frac{\partial \rho}{\partial t}\, d\forall + \int_{C.V.} \nabla \bullet (\rho \vec{V})\, d\forall = \int_{C.V.} [\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \vec{V})]\, d\forall = 0$$



- *Differential form of the mass conservation equation (or continuity equation):*

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \vec{V}) = \frac{\partial \rho}{\partial t} + \vec{V} \bullet \nabla \rho + \rho \nabla \bullet \vec{V}$$
$$= \frac{D\rho}{Dt} + \rho \nabla \bullet \vec{V} = 0$$

Simplifications:

Form incompressible flows:

ρ is constant, then $\frac{\partial \rho}{\partial t} = 0;$ $\nabla \rho = 0$

Therefore, $\nabla \bullet \vec{V} = 0$

Conservation of Mass

Example 01:

- ***The x-component velocity is given by $u(x,y)=Ay^2+C$ in an 2D incompressible flow.***
  - ***Please determine y-component velocity v(x,y) if v(x,0)=0 as would be the case in flow between parallel plates.***

❑ Conservation of Mass

Solution to Example 01:

$$\frac{D\rho}{Dt} + \rho \nabla \bullet \vec{V} = 0$$

$$\Rightarrow \nabla \bullet \vec{V} = 0$$

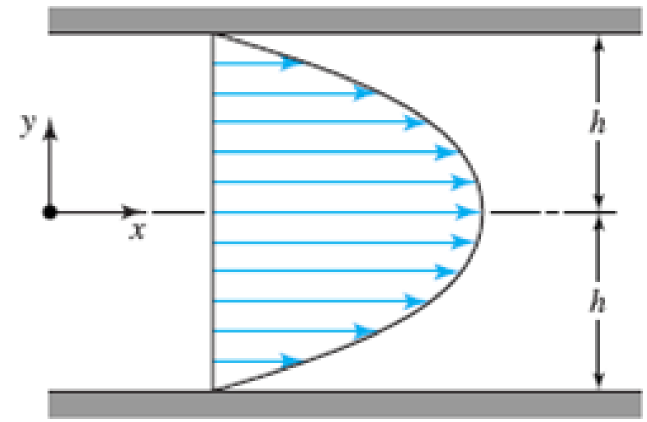
$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{\partial (Ay^2)}{\partial x} = 0$$

$$\frac{\partial y}{\partial y} = 0 \Rightarrow v(x,y) = f(x)$$

$$But \Rightarrow v(x,0) = 0 \Rightarrow f(x) = 0$$

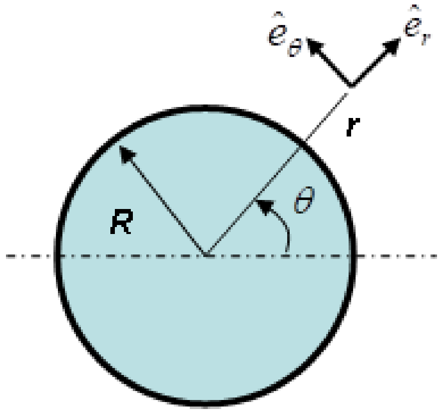
$$\Rightarrow v(x,y) = 0$$



(B) $u(y) = u_{max}\left[1 - \left(\frac{y}{h}\right)^2\right]$

❑ Conservation of Mass

Example 02:



For a two-dimensional steady, inviscid, incompressible flow around a cylinder of radius of R as shown in the figure, the velocity field is given as :

$$\vec{V}(r,\theta) = U_\infty(1-\frac{R^2}{r^2})\cos\theta\cdot\hat{e}_r - U_\infty(1+\frac{R^2}{r^2})\sin\theta\cdot\hat{e}_\theta;$$

where U_∞ is the velocity of the undisturbed stream (therefore, U_∞ is constant).

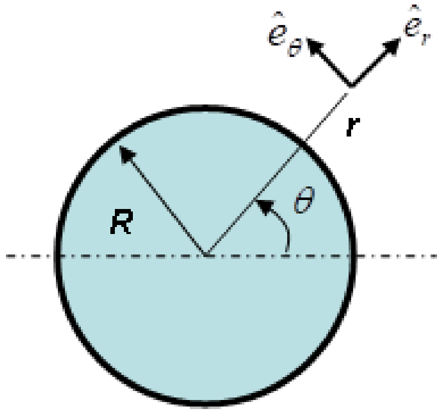
- ***Is the flow with the velocity field given above physically possible?***

$$\frac{\partial\rho}{\partial t} + \nabla\bullet(\rho\vec{V}) = \frac{\partial\rho}{\partial t} + \vec{V}\bullet\nabla\rho + \rho\nabla\bullet\vec{V}$$
$$= \frac{D\rho}{Dt} + \rho\nabla\bullet\vec{V} = 0$$

$$\nabla\bullet\vec{V} = \frac{1}{h_1 h_2 h_3}[\frac{\partial(h_2 h_3 V_1)}{\partial q_1} + \frac{\partial(h_1 h_3 V_2)}{\partial q_2} + \frac{\partial(h_1 h_2 V_3)}{\partial q_3}]$$

❑ Conservation of Mass

For a two-dimensional steady, inviscid, incompressible flow around a cylinder of radius of R as shown in the figure, the velocity field is given as :



$$\vec{V}(r,\theta) = U_\infty\left(1-\frac{R^2}{r^2}\right)\cos\theta\cdot\hat{e}_r - U_\infty\left(1+\frac{R^2}{r^2}\right)\sin\theta\cdot\hat{e}_\theta;$$

where U_∞ is the velocity of the undisturbed stream (therefore, U_∞ is constant).

- ***Is the flow with the velocity field given above physically possible?***

For a steady, inviscid, incompressible flow, the mass conservation equation will be: $\nabla\cdot\vec{V}=0$
Expend it in the cylindrical system, it will be:

$$\frac{1}{r}\left[\frac{\partial(rV_r)}{\partial r}+\frac{\partial V_\theta}{\partial\theta}+\frac{\partial(rV_z)}{\partial z}\right]=0 \quad\Rightarrow\quad \frac{\partial(rV_r)}{\partial r}+\frac{\partial V_\theta}{\partial\theta}+\frac{\partial(rV_z)}{\partial z}=0$$

$$\nabla\bullet\vec{V}=\frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3V_1)}{\partial q_1}+\frac{\partial(h_1h_3V_2)}{\partial q_2}+\frac{\partial(h_1h_2V_3)}{\partial q_3}\right]$$

For the velocity field given above:

$$V_r = U_\infty\left(1-\frac{R^2}{r^2}\right)\cos\theta;\quad V_\theta = -U_\infty\left(1+\frac{R^2}{r^2}\right)\sin\theta;\quad V_z=0$$

Since

$$\frac{\partial(rV_r)}{\partial r}=\frac{\partial}{\partial r}\left[U_\infty\left(r-\frac{R^2}{r}\right)\cos\theta\right]=U_\infty\left(1+\frac{R^2}{r^2}\right)\cos\theta$$

$$\frac{\partial V_\theta}{\partial\theta}=\frac{\partial}{\partial\theta}\left[-U_\infty\left(1+\frac{R^2}{r^2}\right)\sin\theta\right]=-U_\infty\left(1+\frac{R^2}{r^2}\right)\cos\theta$$

$$\frac{\partial(rV_z)}{\partial z}=0$$

Therefore:

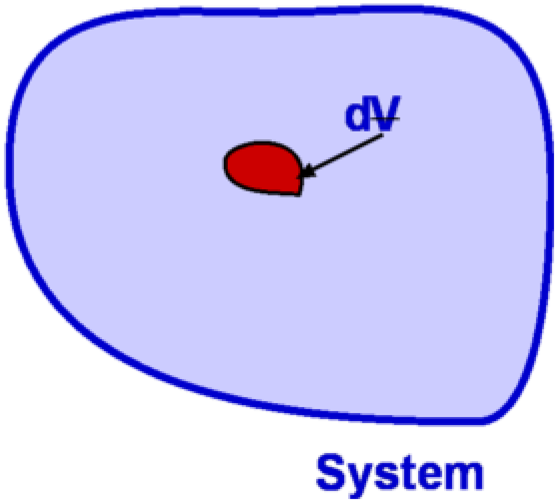
$$\frac{\partial(rV_r)}{\partial r}+\frac{\partial V_\theta}{\partial\theta}+\frac{\partial(rV_z)}{\partial z}=U_\infty\left(1+\frac{R^2}{r^2}\right)\cos\theta-U_\infty\left(1+\frac{R^2}{r^2}\right)\cos\theta+0=0$$

The velocity satisfies the mass conservation equation, i. e., with the velocity field given above is physically possible!

❑ Conservation of Momentum

- *Newton's second law states that:*
  [Time change rate of momentum of a system] = [Resultant external force acting on the system]

$$\frac{d\vec{M}_S}{dt} = \sum \vec{F}_S = \sum \vec{F}_{Surface} + \sum \vec{F}_{body}$$



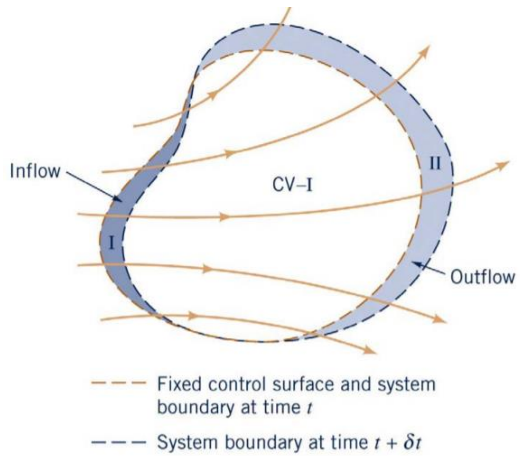
Reynolds Transport Theorem:

$$\left.\frac{dN_s}{dt}\right|_{system} = \frac{d\int_{V} \alpha\,\rho\,dV}{Dt} = \frac{\partial}{\partial t}\int_{C.V.} \alpha\,\rho\,dV + \int_{C.S.} (\alpha\,\rho\vec{V})\bullet d\vec{A}$$

$$Make: \quad \alpha = \vec{V} \qquad \left.\frac{DM_s}{Dt}\right|_{system} = \frac{\partial}{\partial t}\int_{C.V.} \rho\vec{V}\,dV + \int_{C.S.} (\vec{V}\,\rho\vec{V})\bullet d\vec{A} = 0$$

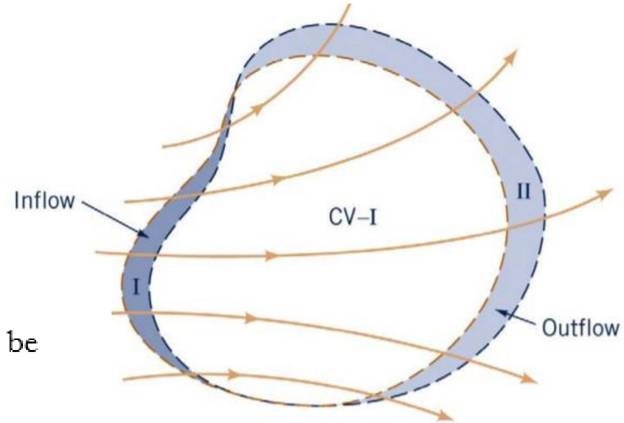
- ***Integral form of the Mass Conservation Equation:***

$$\frac{\partial}{\partial t}\int_{C.V.} \vec{V}\,\rho\,dV + \int_{C.S.} (\vec{V}\,\rho\vec{V})\bullet d\vec{A} = \vec{F}_{surface} + \vec{F}_{body}$$



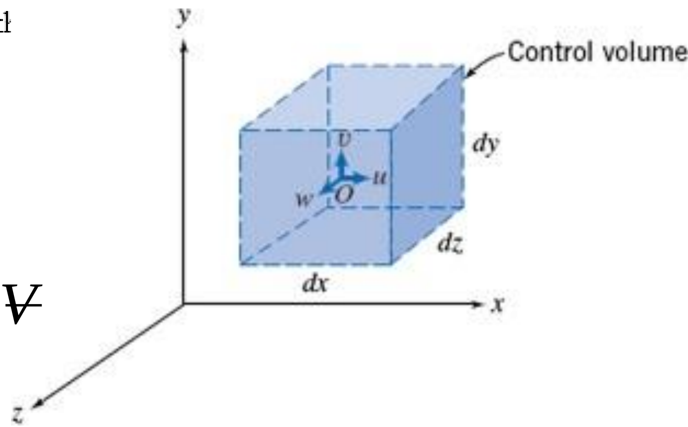
❑ Conservation of Momentum

$$\frac{\partial}{\partial t}\int_{C.V.}\vec{V}\rho\, d\forall + \int_{C.S.}(\vec{V}\rho\vec{V})\bullet d\vec{A} = \vec{F}_{surface} + \vec{F}_{body}$$



$\vec{F}_{surface}$ Surface forces such as pressure and shear stress. The surface forces usually can be expressed as $\vec{F}_{surface} = \int_{C.S.}\tilde{P}\bullet d\vec{A}$, where $\tilde{P}$ is the stress tensor exerted by the surroundings on the particle surface. $\tilde{P} = -P\tilde{I} + \tilde{\tau}$

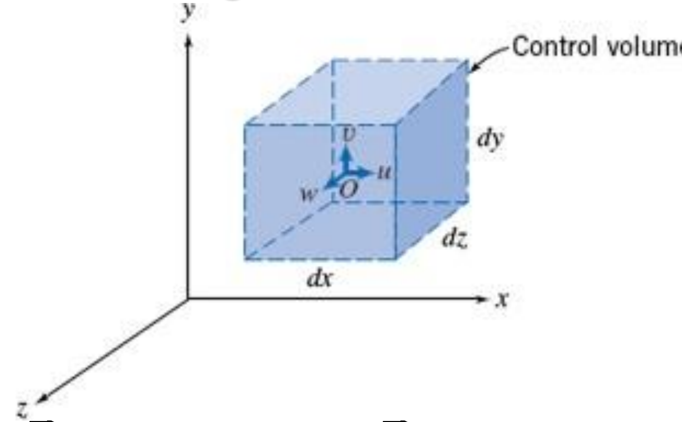
$\vec{F}_{body}$ Body forces such as electromagnetic, gravitational forces. Usually the body force can be expressed as $\vec{F}_{body} = \int_{C.V.}\rho\vec{f}\, d\forall$, where $\vec{f}$ is a vector which references tl resultant force per unit mass.



$$\frac{\partial}{\partial t}\int_{C.V.}\rho\vec{V}\, d\forall + \int_{C.S.}(\rho\vec{V}\vec{V})\bullet d\vec{A} = \int_{C.S.}\tilde{P}\bullet d\vec{A} + \int_{C.V.}\rho\vec{f}\, d\forall$$

Conservation of Momentum

- ***Using divergence theorem for the control surface integrals, we obtained following equation after noting that the limits do not change.***



$$\frac{\partial}{\partial t}\int_{C.V.} \rho\vec{V}\, d\forall + \int_{C.S.} \nabla\bullet(\rho\vec{V}\vec{V})\, d\forall = \int_{C.S.} \nabla\bullet\tilde{P}\, d\forall + \int_{C.V.} \rho\,\vec{f}\, d\forall$$

$$\Rightarrow \int_{C.V.} [\frac{\partial(\rho\vec{V})}{\partial t} + \nabla\bullet(\rho\vec{V}\vec{V}) - \nabla\bullet\tilde{P} - \rho\,\vec{f}\,]d\forall = 0$$

$$\Rightarrow \frac{\partial(\rho\vec{V})}{\partial t} + \nabla\bullet(\rho\vec{V}\vec{V}) - \nabla\bullet\tilde{P} - \rho\,\vec{f} = 0$$

- ***Expand the above equation using*** $\nabla\bullet(\phi\vec{A}) = (\vec{A}\bullet\nabla)\phi + \phi\nabla\bullet\vec{A}$

$$\frac{\partial(\rho\vec{V})}{\partial t} + \nabla\bullet(\rho\vec{V}\vec{V}) - \nabla\bullet\tilde{P} - \rho\,\vec{f} = 0$$

$$\Rightarrow \vec{V}\frac{\partial\rho}{\partial t} + \rho\frac{\partial\vec{V}}{\partial t} + \vec{V}\nabla\bullet(\rho\vec{V}) + (\rho\vec{V}\bullet\nabla)\vec{V} - \nabla\bullet\tilde{P} - \rho\,\vec{f} = 0$$

$$\Rightarrow \vec{V}[\frac{\partial\rho}{\partial t} + \nabla\bullet(\rho\vec{V})] + \rho\frac{\partial\vec{V}}{\partial t} + (\rho\vec{V}\bullet\nabla)\vec{V} - \nabla\bullet\tilde{P} - \rho\,\vec{f} = 0$$

$$\Rightarrow \rho\frac{\partial\vec{V}}{\partial t} + (\rho\vec{V}\bullet\nabla)\vec{V} - \nabla\bullet\tilde{P} - \rho\,\vec{f} = 0$$

$$\Rightarrow \rho\frac{D\vec{V}}{Dt} - \nabla\bullet\tilde{P} - \rho\,\vec{f} = 0$$

- ***The differential form of the momentum equation is:***

$$\rho\frac{D\vec{V}}{Dt} - \nabla\bullet\tilde{P} - \rho\,\vec{f} = 0$$

The Navier-Stokes Equations

$$\frac{\partial(\rho\vec{V})}{\partial t} + \nabla \bullet (\rho\vec{V}\vec{V}) - \nabla \bullet \widetilde{P} - \rho\vec{f} = 0$$

Where $\widetilde{P} = -P\widetilde{I} + \widetilde{\tau}$, and tensor $\widetilde{I} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$; and $\widetilde{\tau} = \begin{vmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{vmatrix}$

Re-writing the equation after substitution leads to:

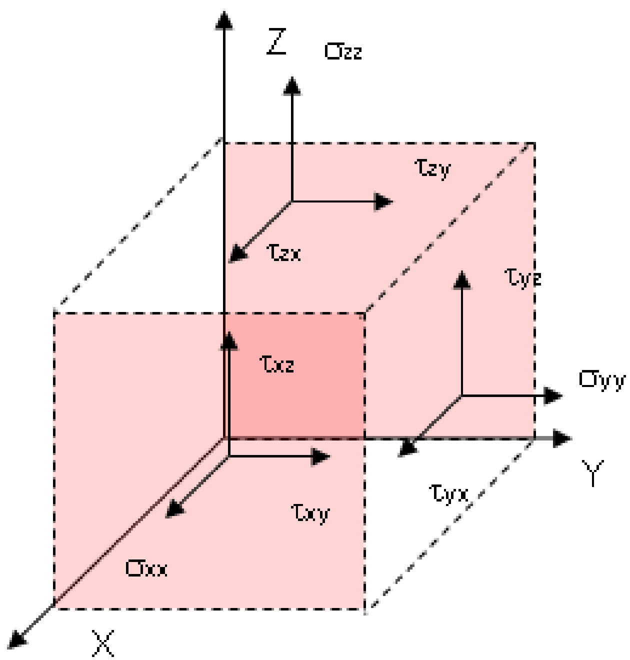
$$\frac{\partial(\rho\vec{V})}{\partial t} + \nabla \bullet (\rho\vec{V}\vec{V}) - \nabla \bullet (-P\widetilde{I} + \widetilde{\tau}) - \rho\vec{f} = 0$$

$$\frac{\partial(\rho\vec{V})}{\partial t} + \nabla \bullet (\rho\vec{V}\vec{V} + P\widetilde{I} - \widetilde{\tau}) - \rho\vec{f} = 0$$

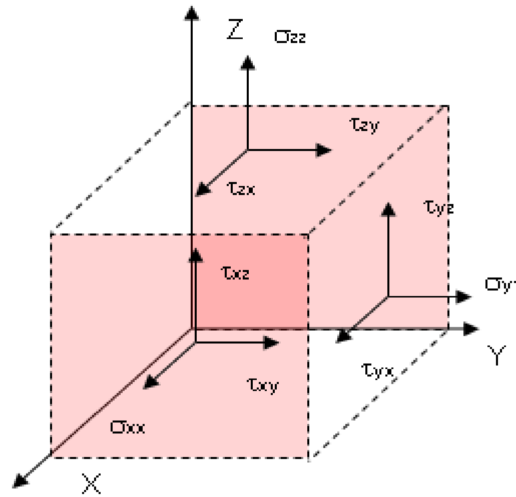
Since $\nabla \bullet (P\widetilde{I}) = P\nabla \bullet \widetilde{I} + (\widetilde{I} \bullet \nabla)P = \nabla P$

Therefore:

$$\frac{\partial(\rho\vec{V})}{\partial t} + \nabla \bullet (\rho\vec{V}\vec{V}) + \nabla P - \nabla \bullet \widetilde{\tau} - \rho\vec{f} = 0$$



❑ The Navier-Stokes Equations



Another form of Navier-Stokes equation:

$$\frac{\partial(\rho\vec{V})}{\partial t}+\nabla\bullet(\rho\vec{V}\vec{V})+\nabla P-\nabla\bullet\tilde{\tau}-\rho\vec{f}=0$$

$$\Rightarrow \rho\frac{\partial\vec{V}}{\partial t}+\vec{V}\frac{\partial\rho}{\partial t}+\vec{V}\nabla\bullet(\rho\vec{V})+(\rho\vec{V}\bullet\nabla)\vec{V})+\nabla P-\nabla\bullet\tilde{\tau}-\rho\vec{f}=0$$

$$\Rightarrow \vec{V}[\underbrace{\frac{\partial\rho}{\partial t}+\nabla\bullet(\rho\vec{V})}_{continuity\quad equation}]+\rho\frac{\partial\vec{V}}{\partial t}+(\rho\vec{V}\bullet\nabla)\vec{V})+\nabla P-\nabla\bullet\tilde{\tau}-\rho\vec{f}=0$$

$$\Rightarrow \rho(\underbrace{\frac{\partial\vec{V}}{\partial t}+\vec{V}\bullet\nabla\vec{V}}_{substtantial\quad derivative})+\nabla P-\nabla\bullet\tilde{\tau}-\rho\vec{f}=0\Rightarrow \rho\frac{D\vec{V}}{Dt}+\nabla P-\nabla\bullet\tilde{\tau}-\rho\vec{f}=0$$

i.e., $\rho\frac{D\vec{V}}{Dt}+\nabla P-\nabla\bullet\tilde{\tau}-\rho\vec{f}=0$

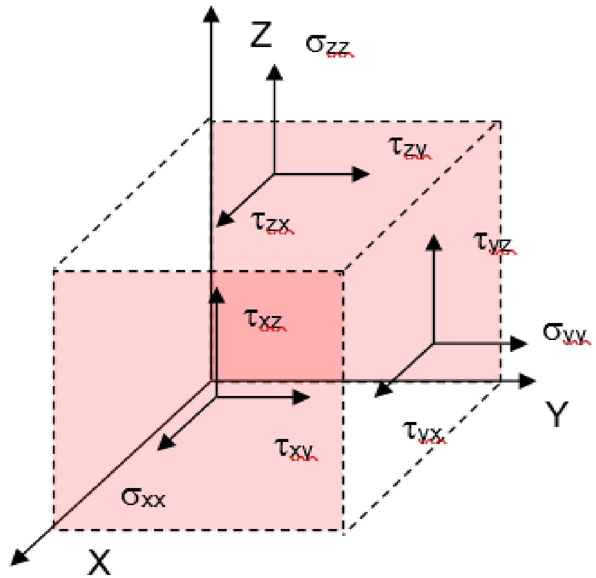
❑ The Navier-Stokes Equations

$$\rho \frac{D\vec{V}}{Dt} + \nabla P - \nabla \bullet \tilde{\tau} - \rho \vec{f} = 0$$

Stress Tensor

The stress tensor has nine components:

$$\tilde{\tau} = \begin{vmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{vmatrix}$$



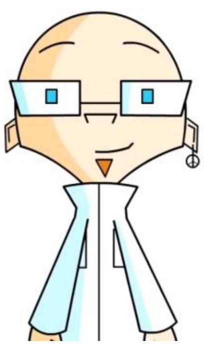
Newtonian fluid,

$$\tilde{\tau} = \mu[\nabla \vec{V} + (\nabla \vec{V})^T - \frac{2}{3}(\nabla \bullet \vec{V})\tilde{I}]$$

For incompressible flow, in Cartesian coordinate system

$$\tau_{ij} = \mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$$

$$\tau_{xy} = \tau_{yx}; \qquad \tau_{xz} = \tau_{zx} \qquad \tau_{zy} = \tau_{yz}$$



- *Newtonian and non-Newtonian fluids*