Lecture # 10: Conservation Equations of Mass and Momentum in Differential Form-P2

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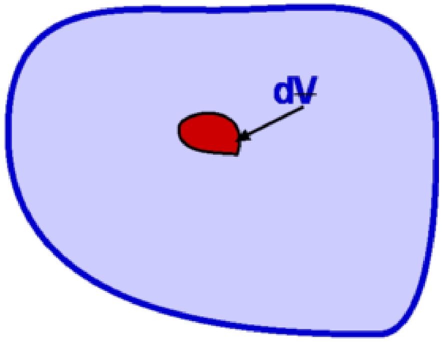
❑ Conservation of Mass

- ***Physical principle: Mass can be neither created or destroyed.***



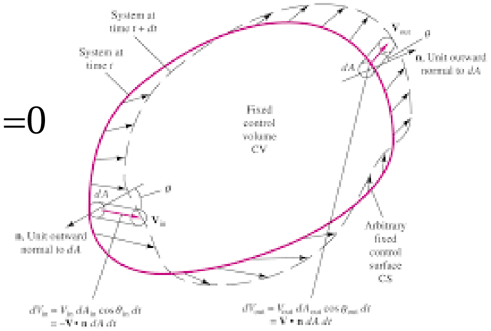
System

- ***Integral form:***

$$\int_{C.\forall.} \frac{\partial \rho}{\partial t}\, d\forall + \int_{C.S.} (\rho \vec{V}) \bullet d\vec{A} = 0$$

- ***Applying Guess divergence theorem, we convert the surface integral to volume integral to obtain:***

$$\int_{C.\forall.} \frac{\partial \rho}{\partial t}\, d\forall + \int_{C.S.} (\rho \vec{V}) \bullet d\vec{A} = \int_{C.V.} \frac{\partial \rho}{\partial t}\, d\forall + \int_{C.V.} \nabla \bullet (\rho \vec{V})\, d\forall = \int_{C.V.} [\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \vec{V})]\, d\forall = 0$$



- ***Differential form of the mass conservation equation (or continuity equation):***

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \vec{V}) = \frac{\partial \rho}{\partial t} + \vec{V} \bullet \nabla \rho + \rho \nabla \bullet \vec{V}$$

$$= \frac{D\rho}{Dt} + \rho \nabla \bullet \vec{V} = 0$$

Simplifications:

Form incompressible flows:

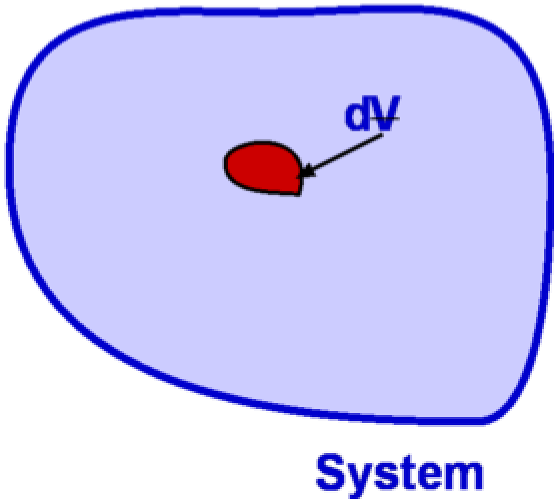
ρ is constant, then $\quad \frac{\partial \rho}{\partial t} = 0; \qquad \nabla \rho = 0$

Therefore, $\quad \nabla \bullet \vec{V} = 0$

❑ Conservation of Momentum

- ***Newton's second law states that:***
  [Time change rate of momentum of a system] = [Resultant external force acting on the system]

$$\frac{d\vec{M}_S}{dt} = \sum \vec{F}_S = \sum \vec{F}_{Surface} + \sum \vec{F}_{body}$$



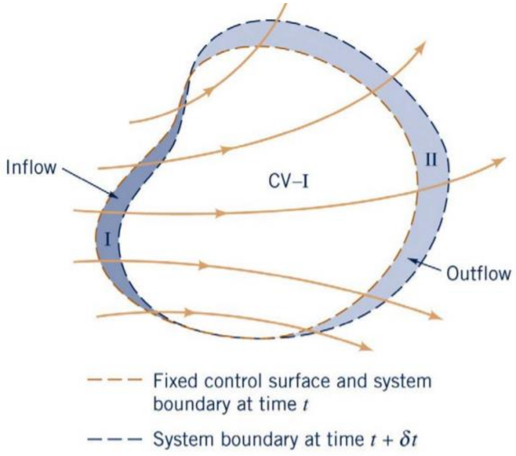
Reynolds Transport Theorem:

$$\left.\frac{dN_s}{dt}\right|_{system} = \frac{d\int_{V} \alpha\,\rho\, dV}{Dt} = \frac{\partial}{\partial t}\int_{C.V.} \alpha\,\rho\, dV + \int_{C.S.} (\alpha\,\rho\vec{V})\bullet d\vec{A}$$

$$Make: \quad \alpha = \vec{V} \qquad \left.\frac{DM_s}{Dt}\right|_{system} = \frac{\partial}{\partial t}\int_{C.V.} \rho\vec{V}\, dV + \int_{C.S.} (\vec{V}\,\rho\vec{V})\bullet d\vec{A} = 0$$

- ***Integral form of the Mass Conservation Equation:***

$$\frac{\partial}{\partial t}\int_{C.V.} \vec{V}\,\rho\, dV + \int_{C.S.} (\vec{V}\,\rho\vec{V})\bullet d\vec{A} = \vec{F}_{surface} + \vec{F}_{body}$$



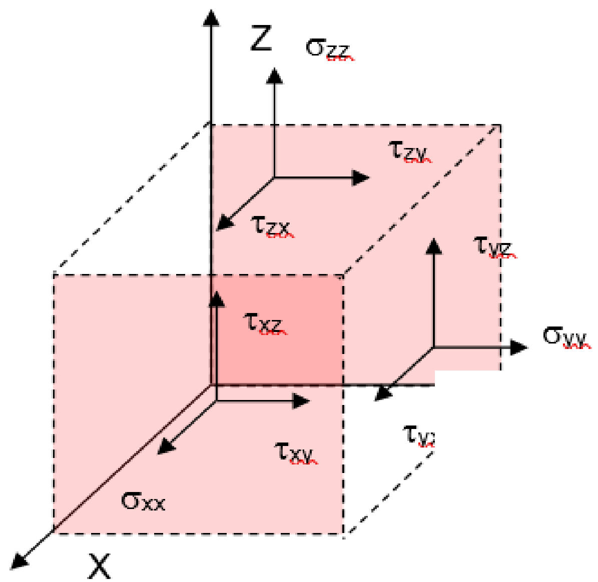
The Navier-Stokes Equations

$$\rho\frac{D\vec{V}}{Dt}+\nabla P-\nabla\bullet\tilde{\tau}-\rho\vec{f}=0$$

Stress Tensor

The stress tensor has nine components:

$$\tilde{\tau}=\begin{vmatrix}\sigma_{xx} & \tau_{yx} & \tau_{zx}\\ \tau_{xy} & \sigma_{yy} & \tau_{zy}\\ \tau_{xz} & \tau_{yz} & \sigma_{zz}\end{vmatrix}$$



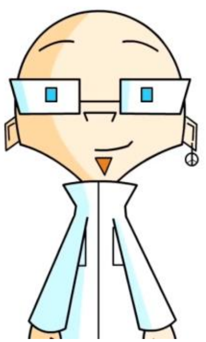
Newtonian fluid,

$$\tilde{\tau}=\mu[\nabla\vec{V}+(\nabla\vec{V})^T-\frac{2}{3}(\nabla\bullet\vec{V})\tilde{I}]$$

For incompressible flow, in Cartesian coordinate system

$$\tau_{ij}=\mu(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i})$$

$$\tau_{xy}=\tau_{yx};\qquad \tau_{xz}=\tau_{zx}\qquad \tau_{zy}=\tau_{yz}$$



- *Newtonian and non-Newtonian fluids*

❑ THE NAVIER-STOKES EQUATION IN CARTESIAN COORDINATE

$$\rho \frac{D\vec{V}}{Dt} + \nabla P - \nabla \bullet \tilde{\tau} - \rho \vec{f} = 0$$

For incompressible flow, Navier-Stokes equation in Cartesian coordinate system will be:

x-direction :

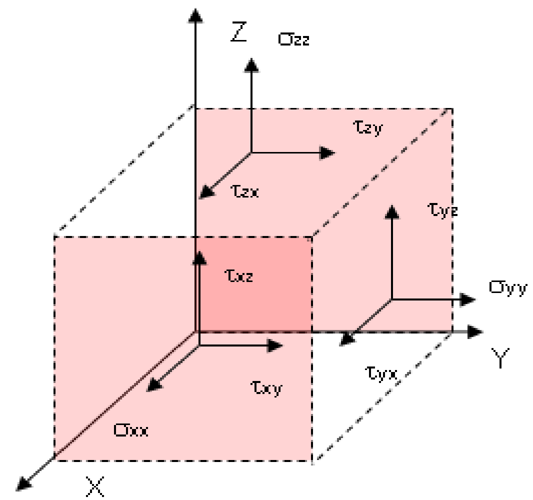
$$\begin{cases} \rho \dfrac{Du}{Dt} = -\dfrac{\partial p}{\partial x} + \rho f_x + \mu[\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}] \\ \quad or \\ \rho[\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z}] = -\dfrac{\partial p}{\partial x} + \rho f_x + \mu[\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}] \end{cases}$$

y-direction:

$$\begin{cases} \rho \dfrac{Dv}{Dt} = -\dfrac{\partial p}{\partial y} + \rho f_y + \mu[\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} + \dfrac{\partial^2 v}{\partial z^2}] \\ \quad or \\ \rho[\dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z}] = -\dfrac{\partial p}{\partial x} + \rho f_y + \mu[\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} + \dfrac{\partial^2 v}{\partial z^2}] \end{cases}$$

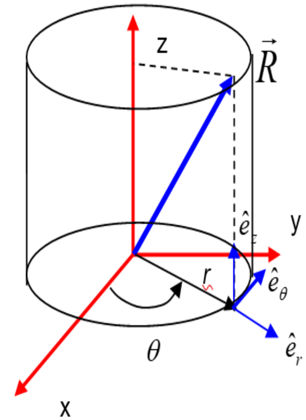
z-direction:

$$\begin{cases} \rho \dfrac{Dw}{Dt} = -\dfrac{\partial p}{\partial z} + \rho f_z + \mu[\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2}] \\ \quad or \\ \rho[\dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + v\dfrac{\partial w}{\partial y} + w\dfrac{\partial w}{\partial z}] = -\dfrac{\partial p}{\partial z} + \rho f_z + \mu[\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2}] \end{cases}$$



❑ Expansion of the Navier-Stokes Equation in Cylindric Coordinate

$$\rho \frac{D\vec{V}}{Dt} + \nabla P - \nabla \bullet \tilde{\tau} - \rho \vec{f} = 0$$



Cylindrical coordinate system (r, θ, z)

- ***Expansion of the Navier-Stokes Equation in Cylindrical Coordinate***

- **r-direction:**

$$\frac{\partial(\rho V_r)}{\partial t} + \frac{\partial(\rho V_r V_r)}{\partial r} + \frac{\partial(\rho V_\theta V_r)}{r\partial\theta} + \frac{\partial(\rho V_Z V_r)}{\partial Z} + \rho\frac{V_r V_r}{r} - \rho\frac{V_\theta V_\theta}{r}$$

$$= -\frac{\partial p}{\partial r} + [\frac{\partial(r\sigma_{rr})}{r\partial r} + \frac{\partial\tau_{r\theta}}{r\partial\theta} + \frac{\partial\tau_{zr}}{\partial Z} - \frac{\sigma_{\theta\theta}}{r}] + \rho f_r$$

or

$$\frac{\partial(\rho V_r)}{\partial t} + \frac{\partial(\rho r V_r V_r)}{r\,\partial r} + \frac{\partial(\rho V_\theta V_r)}{r\partial\theta} + \frac{\partial(\rho V_Z V_r)}{\partial Z} - \rho\frac{V_\theta V_\theta}{r}$$

$$= -\frac{\partial p}{\partial r} + [\frac{\partial(r\sigma_{rr})}{r\partial r} + \frac{\partial\tau_{r\theta}}{r\partial\theta} + \frac{\partial\tau_{zr}}{\partial Z} - \frac{\sigma_{\theta\theta}}{r}] + \rho f_r$$

- **θ-direction:**

$$\frac{\partial(\rho V_\theta)}{\partial t} + \frac{\partial(\rho V_r V_\theta)}{\partial r} + \frac{\partial(\rho V_\theta V_\theta)}{r\partial\theta} + \frac{\partial(\rho V_Z V_\theta)}{\partial Z} + \rho\frac{V_r V_\theta}{r}$$

$$= -\frac{1}{r}\frac{\partial p}{\partial\theta} + [\frac{\partial(r\tau_{r\theta})}{r\partial r} + \frac{\partial\sigma_{\theta\theta}}{r\partial\theta} + \frac{\partial\tau_{z\theta}}{\partial Z} + \frac{\tau_{\theta r}}{r}] + \rho f_\theta$$

or

$$\frac{\partial(\rho V_\theta)}{\partial t} + \frac{\partial(\rho r V_r V_\theta)}{r\,\partial r} + \frac{\partial(\rho V_\theta V_\theta)}{r\partial\theta} + \frac{\partial(\rho V_Z V_\theta)}{\partial Z} + \rho\frac{V_r V_\theta}{r}$$

$$= -\frac{1}{r}\frac{\partial p}{\partial\theta} + [\frac{\partial(r\tau_{r\theta})}{r\partial r} + \frac{\partial\sigma_{\theta\theta}}{r\partial\theta} + \frac{\partial\tau_{z\theta}}{\partial Z} + \frac{\tau_{\theta r}}{r}] + \rho f_\theta$$

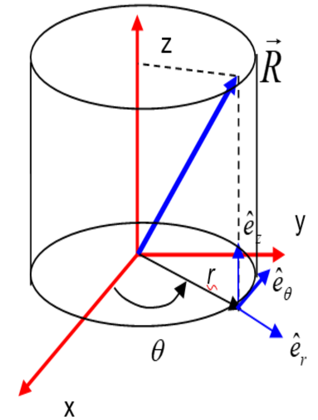
- **z-direction:**

$$\frac{\partial(\rho V_Z)}{\partial t} + \frac{\partial(\rho V_r V_Z)}{\partial r} + \frac{\partial(\rho V_\theta V_Z)}{r\partial\theta} + \frac{\partial(\rho V_Z V_Z)}{\partial Z} = -\frac{\partial p}{\partial Z} + [\frac{\partial(r\tau_{rZ})}{r\partial r} + \frac{\partial\tau_{\theta Z}}{r\partial\theta} + \frac{\partial\sigma_{zz}}{\partial Z}] + \rho f_Z$$

❑ Expansion of the Navier-Stokes Equation in Cylindric Coordinate

$$\rho\frac{D\vec{V}}{Dt}+\nabla P-\nabla\bullet\tilde{\tau}-\rho\vec{f}=0$$

- *Expansion of the Navier-Stokes Equation in Cylindrical Coordinate*



Cylindrical coordinate system (r,θ,z)

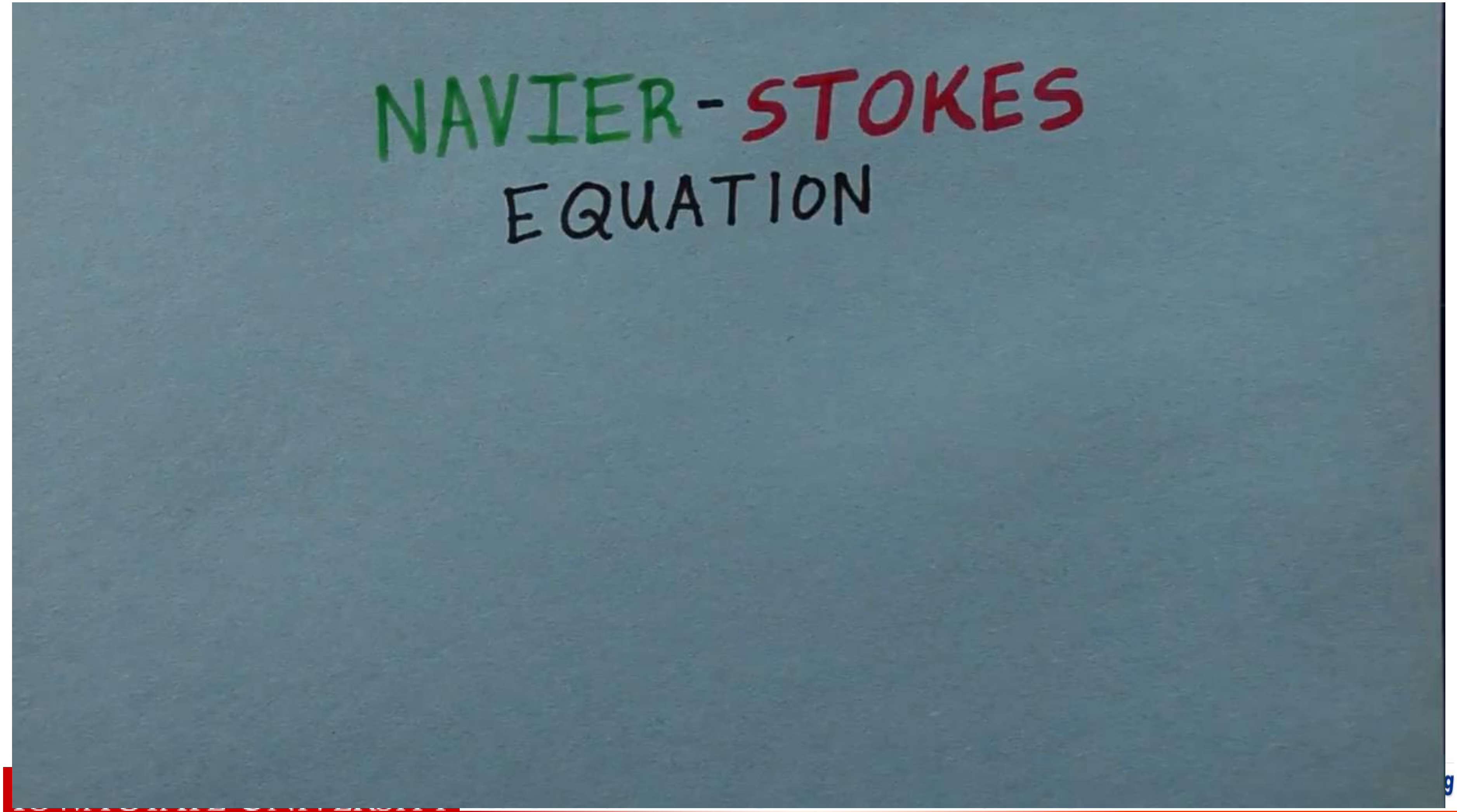
$$\tilde{\tau}=\mu[\nabla\vec{V}+(\nabla\vec{V})^T-\frac{2}{3}\mu(\nabla\bullet\vec{V})\tilde{I}]$$

$$=\mu\begin{pmatrix} 2\frac{\partial V_r}{\partial r}-\frac{2}{3}(\nabla\bullet\vec{V}) & \frac{1}{r}\frac{\partial V_r}{\partial\theta}-\frac{V_\theta}{r}+\frac{\partial V_\theta}{\partial r} & \frac{\partial V_r}{\partial Z}+\frac{\partial V_Z}{\partial r} \\ \frac{1}{r}\frac{\partial V_r}{\partial\theta}-\frac{V_\theta}{r}+\frac{\partial V_\theta}{\partial r} & 2(\frac{V_r}{r}+\frac{1}{r}\frac{\partial V_\theta}{\partial\theta})-\frac{2}{3}(\nabla\bullet\vec{V}) & \frac{\partial V_\theta}{\partial Z}+\frac{1}{r}\frac{\partial V_Z}{\partial\theta} \\ \frac{\partial V_Z}{\partial r}+\frac{\partial V_r}{\partial Z} & \frac{1}{r}\frac{\partial V_Z}{\partial\theta}+\frac{\partial V_\theta}{\partial Z} & \frac{\partial V_Z}{\partial Z}-\frac{2}{3}(\nabla\bullet\vec{V}) \end{pmatrix}$$

$$=\begin{pmatrix} \sigma_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \sigma_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{Z\theta} & \tau_{ZZ} \end{pmatrix}$$

❑ Navier-Stokes Equations

- ***Navier Stokes Equation | A Million-Dollar Question in Fluid Mechanics***
  - https://www.youtube.com/watch?v=XoefjJdFq6k

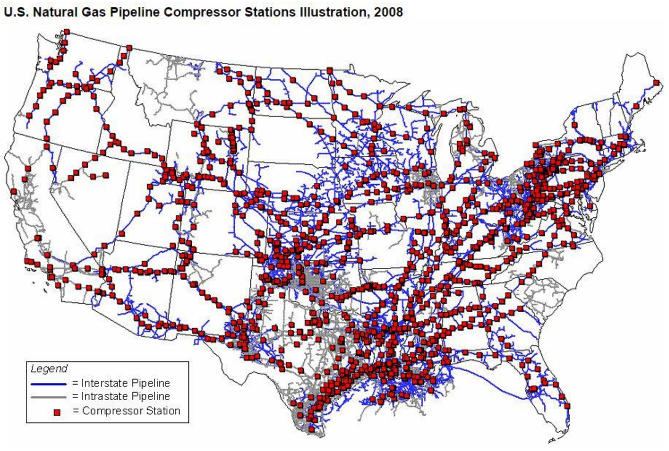


❑ Hagan-Poiseuille Flow

- ***Hagan-Poisuille flow results when the flow through a circular pipe has attained what is called a fully developed profile.***

Assumptions:

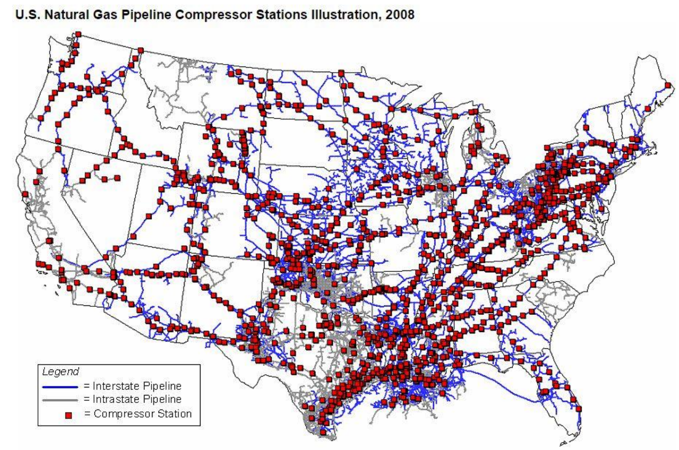
1. Steady flow: $\Rightarrow$ $\dfrac{\partial}{\partial t} = 0$
2. Incompressible flows $\Rightarrow$ ρ, μ are constant.
3. Axial symmetry $\Rightarrow$ $\dfrac{\partial}{\partial \theta} = 0$
4. In addition, we impose on velocity $V_r = 0$
5. No body forces.



Questions:

- ***a). Determine the velocity profile in the pipeline.***
- ***b). Determine the pressure drop along the pipeline.***

❑ Hagan-Poiseuille Flow



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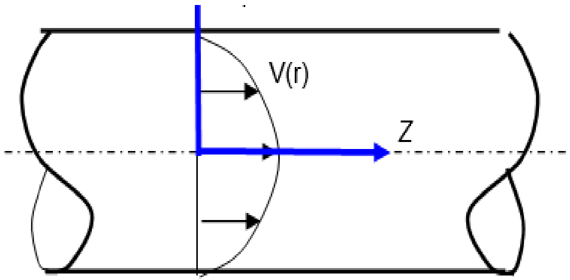


Continuity equation:

$$\frac{\partial \rho}{\partial t}+\nabla \bullet (\rho \vec{V})=0 \quad \Rightarrow \quad \nabla \bullet \vec{V}=0$$

$$\frac{1}{r}\left[\frac{\partial (rV_r)}{\partial r}+\frac{\partial V_\theta}{\partial \theta}+\frac{\partial (rV_z)}{\partial z}\right]=0$$

$$\Rightarrow \frac{\partial (rV_r)}{\partial r}+\frac{\partial V_\theta}{\partial \theta}+\frac{\partial (rV_z)}{\partial z}=0$$

$$\Rightarrow \frac{\partial (rV_z)}{\partial z}=0 \Rightarrow \frac{\partial V_z}{\partial z}=0$$

$$\Rightarrow V_z=V_z(r)$$

V_z is only the function of r, not functions of Z and θ [due to the axial symmetry of $\dfrac{\partial}{\partial \theta}=0$].

❑ Hagan-Poiseuille Flow

Momentum equation:

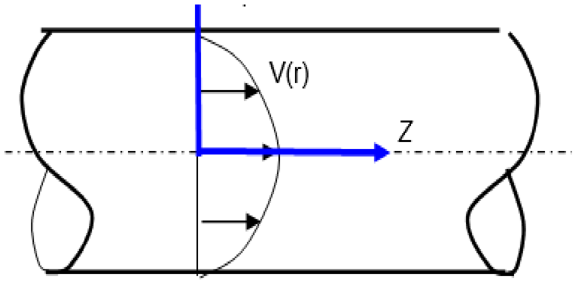
r-direction:

$$\frac{\partial(\rho V_r)}{\partial t}+\frac{\partial(\rho\, r\, V_r V_r)}{r\,\partial r}+\frac{\partial(\rho V_\theta V_r)}{r\partial\theta}+\frac{\partial(\rho V_Z V_r)}{\partial Z}-\rho\frac{V_\theta V_\theta}{r}$$

$$=-\frac{\partial p}{\partial r}+[\frac{\partial(r\sigma_{rr})}{r\partial r}+\frac{\partial\tau_{r\theta}}{r\partial\theta}+\frac{\partial\tau_{zr}}{\partial Z}-\frac{\sigma_{\theta\theta}}{r}]+\rho f_r$$

$$\begin{pmatrix}\sigma_{rr} & \tau_{r\theta} & \tau_{rz}\\ \tau_{\theta r} & \sigma_{\theta\theta} & \tau_{\theta z}\\ \tau_{zr} & \tau_{Z\theta} & \tau_{ZZ}\end{pmatrix}=\mu\begin{pmatrix}2\frac{\partial V_r}{\partial r}-\frac{2}{3}(\nabla\bullet\vec{V}) & \frac{1}{r}\frac{\partial V_r}{\partial\theta}-\frac{V_\theta}{r}+\frac{\partial V_\theta}{\partial r} & \frac{\partial V_r}{\partial Z}+\frac{\partial V_Z}{\partial r}\\ \frac{1}{r}\frac{\partial V_r}{\partial\theta}-\frac{V_\theta}{r}+\frac{\partial V_\theta}{\partial r} & 2(\frac{V_r}{r}+\frac{1}{r}\frac{\partial V_\theta}{\partial\theta})-\frac{2}{3}(\nabla\bullet\vec{V}) & \frac{\partial V_\theta}{\partial Z}+\frac{1}{r}\frac{\partial V_Z}{\partial\theta}\\ \frac{\partial V_Z}{\partial r}+\frac{\partial V_r}{\partial Z} & \frac{1}{r}\frac{\partial V_Z}{\partial\theta}+\frac{\partial V_\theta}{\partial Z} & \frac{\partial V_Z}{\partial Z}-\frac{2}{3}(\nabla\bullet\vec{V})\end{pmatrix}$$

$$=\begin{pmatrix}0 & 0 & \mu\frac{\partial V_Z}{\partial r}\\ 0 & 0 & 0\\ \mu\frac{\partial V_Z}{\partial r} & 0 & 0\end{pmatrix}$$



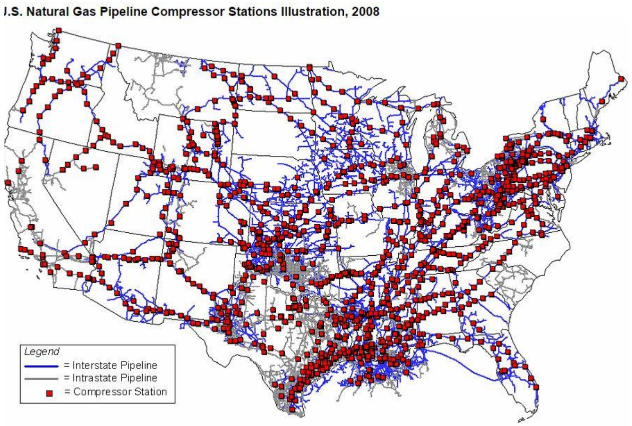
therefore

$$0+0+0+0-0=-\frac{\partial p}{\partial r}+[0+0+\frac{\partial(\mu\frac{\partial V_Z}{\partial r})}{\partial Z}-0]+0$$

$$\Rightarrow\frac{\partial p}{\partial r}=\frac{\partial(\mu\frac{\partial V_Z}{\partial r})}{\partial Z}=\mu\frac{\partial^2 V_Z}{\partial Z\partial r}=\mu\frac{\partial}{\partial r}(\frac{\partial V_Z}{\partial Z})=0$$

$$\Rightarrow\frac{\partial p}{\partial r}=0$$

Leads to the conclusion of $P=P(Z)$



❑ Hagan-Poiseuille Flow

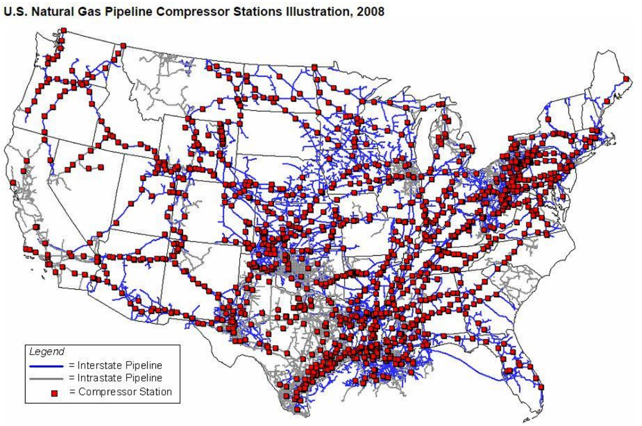
θ-direction:

$$\frac{\partial(\rho V_\theta)}{\partial t}+\frac{\partial(\rho\, r\, V_r V_\theta)}{r\,\partial r}+\frac{\partial(\rho V_\theta V_\theta)}{r\partial\theta}+\frac{\partial(\rho V_Z V_\theta)}{\partial Z}+\rho\frac{V_r V_\theta}{r}$$

$$=-\frac{1}{r}\frac{\partial p}{\partial\theta}+[\frac{\partial(r\tau_{r\theta})}{r\partial r}+\frac{\partial\sigma_{\theta\theta}}{r\partial\theta}+\frac{\partial\tau_{z\theta}}{\partial Z}+\frac{\tau_{\theta r}}{r}]+\rho f_\theta$$

$$\Rightarrow 0+0+0+0+0=-0+[0+0+0+0]+0$$

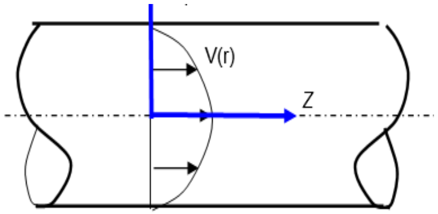
$$\Rightarrow 0=0$$



Z-direction:

$$\frac{\partial(\rho V_Z)}{\partial t}+\frac{\partial(\rho V_r V_Z)}{\partial r}+\frac{\partial(\rho V_\theta V_Z)}{r\partial\theta}+\frac{\partial(\rho V_Z V_Z)}{\partial Z}=-\frac{\partial P}{\partial Z}+[\frac{\partial(r\tau_{rZ})}{r\partial r}+\frac{\partial\tau_{rZ}}{r\partial\theta}+\frac{\partial\sigma_{zz}}{\partial Z}]+\rho f_Z$$

$$\Rightarrow 0+0+0+0=-\frac{\partial P}{\partial Z}+[\frac{\partial(r\mu\frac{\partial V_Z}{\partial r})}{r\partial r}+0+0]+0$$



$$\Rightarrow \underbrace{\frac{\partial P}{\partial Z}}_{\text{Only a function of Z}}=\underbrace{\mu\frac{\partial(r\frac{\partial V_Z}{\partial r})}{r\partial r}}_{\text{Only a function of r}}$$

The only way the above equation is true if the two side are both equal to a constant.

Therefore:

$$\frac{\partial p}{\partial Z}=C$$

$$\mu\frac{\partial(r\frac{\partial V_Z}{\partial r})}{r\partial r}=C$$

where C is a constant.

❑ Hagan-Poiseuille Flow

Boundary conditions:

- At the wall, $r = r_0,\qquad V_Z = 0$.
- At the centerline, slope of velocity profile is zero. $r = 0,\quad \dfrac{\partial V_Z}{\partial r} = 0$

$$\mu \frac{\partial (r \frac{\partial V_Z}{\partial r})}{r \partial r} = C \quad \Rightarrow \frac{\partial (r \frac{\partial V_Z}{\partial r})}{\partial r} = \frac{Cr}{\mu}$$

$$\Rightarrow \int \partial (r \frac{\partial V_Z}{\partial r}) = \int \frac{Cr}{\mu} \partial r \frac{Cr}{\mu} \partial r \quad \Rightarrow r \frac{\partial V_Z}{\partial r} = \frac{Cr^2}{2\mu} + B_1$$

Since $r = 0,\qquad \dfrac{\partial V_Z}{\partial r} = 0 \qquad \Rightarrow \qquad B_1 = 0$

$$r \frac{\partial V_Z}{\partial r} = \frac{Cr^2}{2\mu} \quad \Rightarrow \frac{\partial V_Z}{\partial r} = \frac{Cr}{2\mu}$$

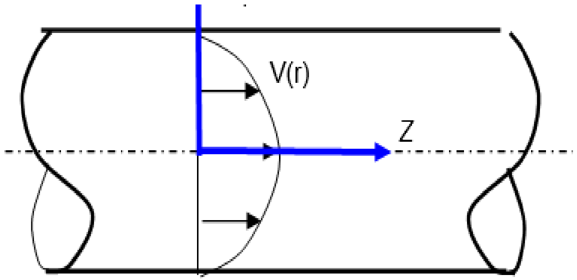
$$\Rightarrow \int \partial V_Z = \int \frac{Cr}{2\mu} \partial r \quad \Rightarrow V_Z = \frac{Cr^2}{4\mu} + B_2$$

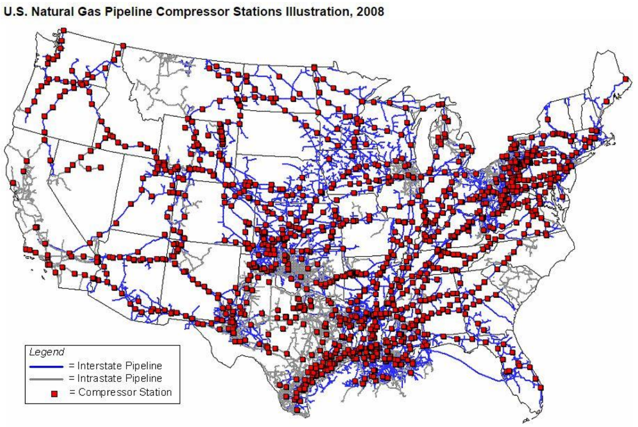
Since $r = r_0,\qquad V_Z = 0 \qquad \Rightarrow \qquad B_2 = -\dfrac{Cr_0^{\,2}}{4\mu}$

Therefore: $V_Z = \dfrac{C}{4\mu}(r^2 - r_0^{\,2})$

Since $\dfrac{\partial p}{\partial Z} = C$ and $P = P(Z)$

Therefore: $V_Z = \dfrac{1}{4\mu}\dfrac{dP}{dZ}(r^2 - r_0^{\,2})$





❑ Hagan-Poiseuille Flow

Mass flow rate:

$$\dot{m} = \int \rho V_Z dA = \int_0^{r_0} \rho \frac{1}{4\mu}\frac{dP}{dZ}(r^2 - r_0^2)(2\pi r dr)$$

$$= \rho \frac{\pi}{2\mu}\frac{dP}{dZ}\left(\frac{r^4}{4} - \frac{r_0^2 r^2}{2}\right)\Big|_0^{r_0}$$

$$= \rho \frac{\pi}{2\mu}\frac{dP}{dZ}\left(-\frac{r_0^4}{4}\right)$$

$$= -\rho \frac{\pi}{8\mu}\frac{dP}{dZ} r_0^4$$

Or

$$\frac{dP}{dZ} = -\frac{8\mu\dot{m}}{\rho\pi r_0^4}$$