Lecture # 11: Conservation Equations of Mass and Momentum in Differential Form-P3

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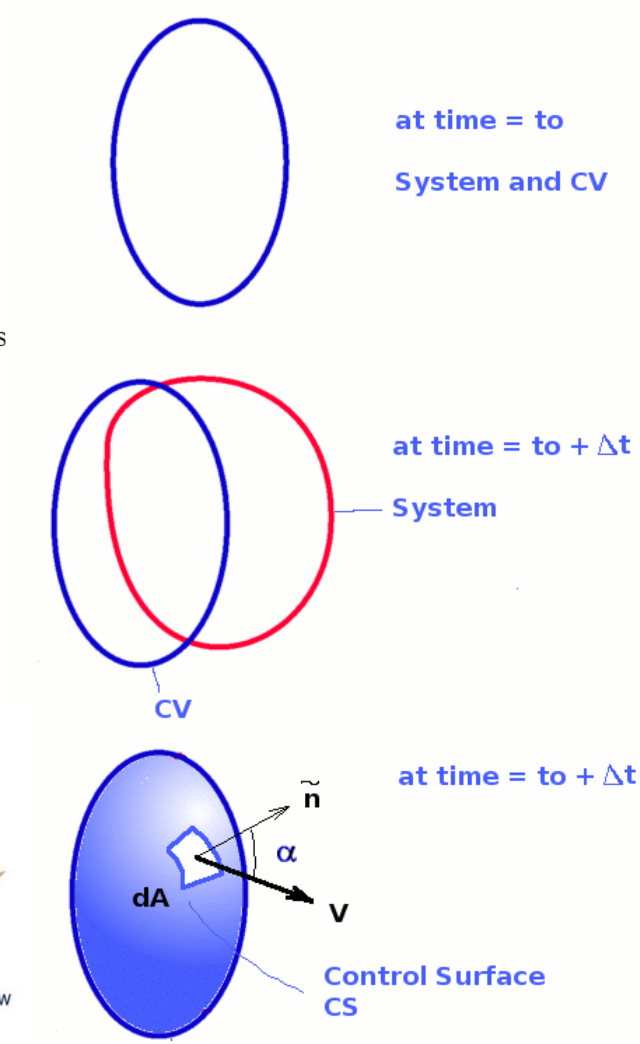
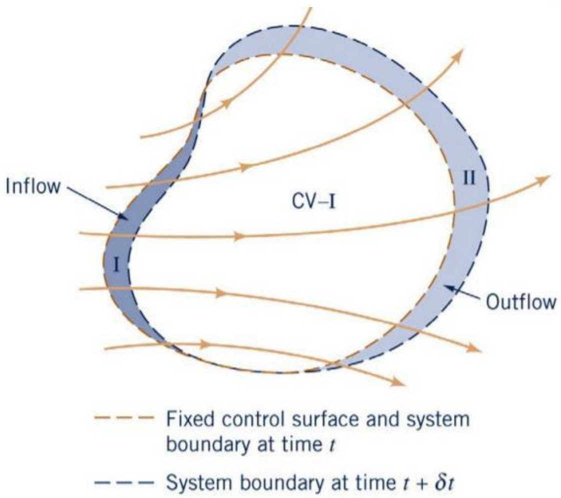
Reynolds Transport Theorem

$$\frac{DN_s}{Dt} = \frac{D\int_{V\!\!\!\!-} \alpha\,\rho\,dV\!\!\!\!-}{Dt} = \frac{\partial}{\partial t}\int_{C.V.} \alpha\,\rho\,dV\!\!\!\!- + \int_{C.S.} (\alpha\,\rho\vec{V})\bullet d\vec{A}$$



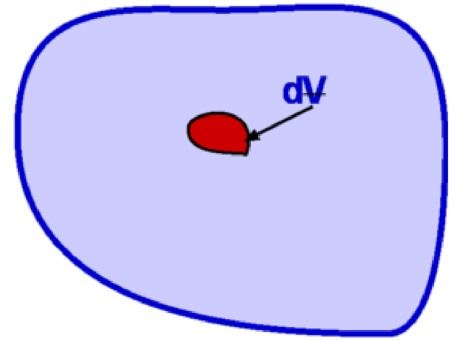
Where α is any intensive property corresponding to N. (i.e., $\alpha = N$ per unit mas and it can be used for different quantities as follows.

N_s	α
Mass	1
Linear momentum	$\vec{V}$
Angular momentum	$\vec{R}\times\vec{V}$
Energy	e
Entropy	s



Conservation of Mass

- *Physical principle: Mass can be neither created or destroyed.*



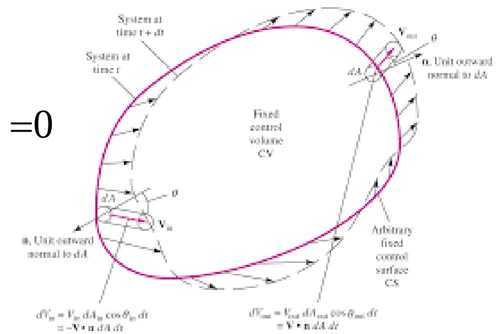
System

- *Integral form:*

$$\int_{C.\forall.} \frac{\partial \rho}{\partial t}\, d\forall + \int_{C.S.} (\rho \vec{V}) \bullet d\vec{A} = 0$$

- *Applying Guess divergence theorem, we convert the surface integral to volume integral to obtain:*

$$\int_{C.\forall.} \frac{\partial \rho}{\partial t}\, d\forall + \int_{C.S.} (\rho \vec{V}) \bullet d\vec{A} = \int_{C.V.} \frac{\partial \rho}{\partial t}\, d\forall + \int_{C.V.} \nabla \bullet (\rho \vec{V})\, d\forall = \int_{C.V.} [\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \vec{V})]\, d\forall = 0$$



- *Differential form of the mass conservation equation (or continuity equation):*

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \vec{V}) = \frac{\partial \rho}{\partial t} + \vec{V} \bullet \nabla \rho + \rho \nabla \bullet \vec{V}$$
$$= \frac{D\rho}{Dt} + \rho \nabla \bullet \vec{V} = 0$$

Simplifications:
Form incompressible flows:

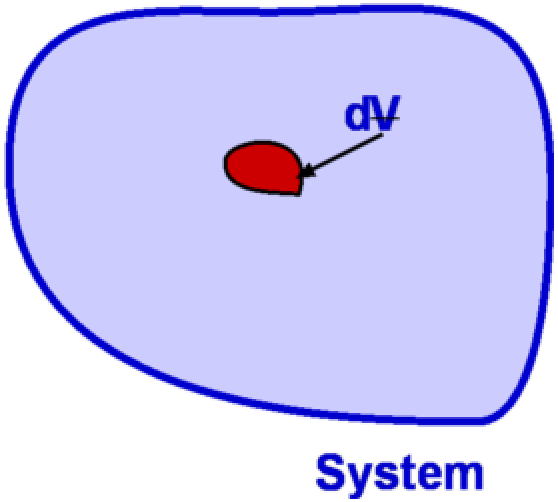
ρ is constant, then $\quad \frac{\partial \rho}{\partial t} = 0; \qquad \nabla \rho = 0$

Therefore, $\quad \nabla \bullet \vec{V} = 0$

Conservation of Momentum

- ***Newton's second law states that:***
  [Time change rate of momentum of a system] = [Resultant external force acting on the system]

$$\frac{d\vec{M}_S}{dt} = \sum \vec{F}_S = \sum \vec{F}_{Surface} + \sum \vec{F}_{body}$$



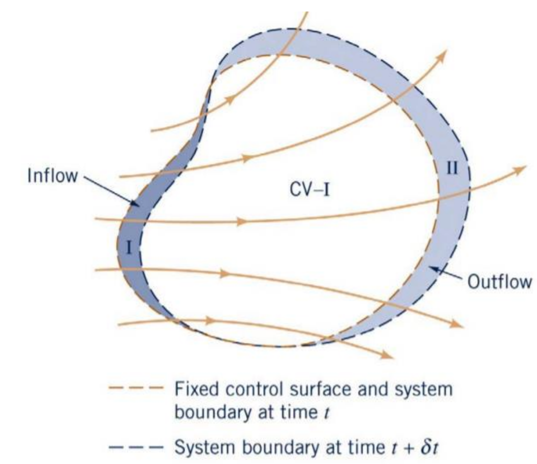
Reynolds Transport Theorem:

$$\left.\frac{dN_s}{dt}\right|_{system} = \frac{d\int_{V} \alpha\, \rho\, dV}{Dt} = \frac{\partial}{\partial t}\int_{C.V.} \alpha\, \rho\, dV + \int_{C.S.} (\alpha\, \rho \vec{V}) \bullet d\vec{A}$$

$$Make: \quad \alpha = \vec{V} \qquad \left.\frac{DM_s}{Dt}\right|_{system} = \frac{\partial}{\partial t}\int_{C.V.} \rho \vec{V}\, dV + \int_{C.S.} (\vec{V}\, \rho \vec{V}) \bullet d\vec{A} = 0$$

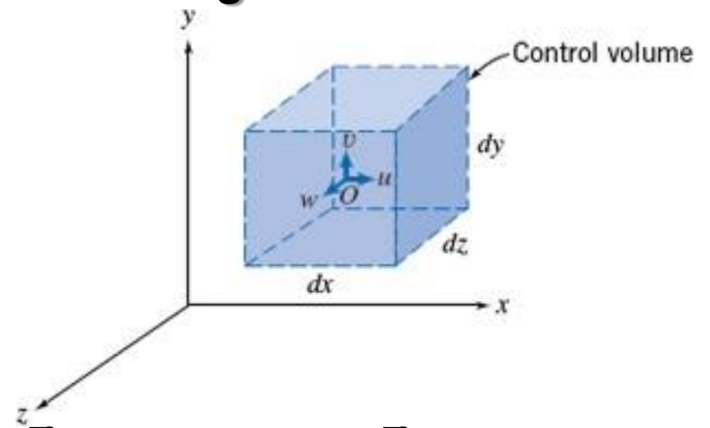
- ***Integral form of the Mass Conservation Equation:***

$$\frac{\partial}{\partial t}\int_{C.V.} \vec{V}\, \rho\, dV + \int_{C.S.} (\vec{V}\, \rho \vec{V}) \bullet d\vec{A} = \vec{F}_{surface} + \vec{F}_{body}$$



❑ Conservation of Momentum

- ***Using divergence theorem for the control surface integrals, we obtained following equation after noting that the limits do not change.***



$$\frac{\partial}{\partial t}\int_{C.V.}\rho\vec{V}\,d\forall+\int_{C.S.}\nabla\bullet(\rho\vec{V}\vec{V})\,d\forall=\int_{C.S.}\nabla\bullet\tilde{P}\,d\forall+\int_{C.V.}\rho\,\vec{f}\,d\forall$$

$$\Rightarrow\int_{C.V.}\Big[\frac{\partial(\rho\vec{V})}{\partial t}+\nabla\bullet(\rho\vec{V}\vec{V})-\nabla\bullet\tilde{P}-\rho\,\vec{f}\,\Big]d\forall=0$$

$$\Rightarrow\frac{\partial(\rho\vec{V})}{\partial t}+\nabla\bullet(\rho\vec{V}\vec{V})-\nabla\bullet\tilde{P}-\rho\,\vec{f}=0$$

- ***Expand the above equation using*** $\nabla\bullet(\phi\vec{A})=(\vec{A}\bullet\nabla)\phi+\phi\nabla\bullet\vec{A}$

$$\frac{\partial(\rho\vec{V})}{\partial t}+\nabla\bullet(\rho\vec{V}\vec{V})-\nabla\bullet\tilde{P}-\rho\,\vec{f}=0$$

$$\Rightarrow\vec{V}\frac{\partial\rho}{\partial t}+\rho\frac{\partial\vec{V}}{\partial t}+\vec{V}\nabla\bullet(\rho\vec{V})+(\rho\vec{V}\bullet\nabla)\vec{V}-\nabla\bullet\tilde{P}-\rho\,\vec{f}=0$$

$$\Rightarrow\vec{V}\Big[\frac{\partial\rho}{\partial t}+\nabla\bullet(\rho\vec{V})\Big]+\rho\frac{\partial\vec{V}}{\partial t}+(\rho\vec{V}\bullet\nabla)\vec{V}-\nabla\bullet\tilde{P}-\rho\,\vec{f}=0$$

$$\Rightarrow\rho\frac{\partial\vec{V}}{\partial t}+(\rho\vec{V}\bullet\nabla)\vec{V}-\nabla\bullet\tilde{P}-\rho\,\vec{f}=0$$

$$\Rightarrow\rho\frac{D\vec{V}}{Dt}-\nabla\bullet\tilde{P}-\rho\,\vec{f}=0$$

- ***The differential form of the momentum equation is:***

$$\rho\frac{D\vec{V}}{Dt}-\nabla\bullet\tilde{P}-\rho\,\vec{f}=0$$

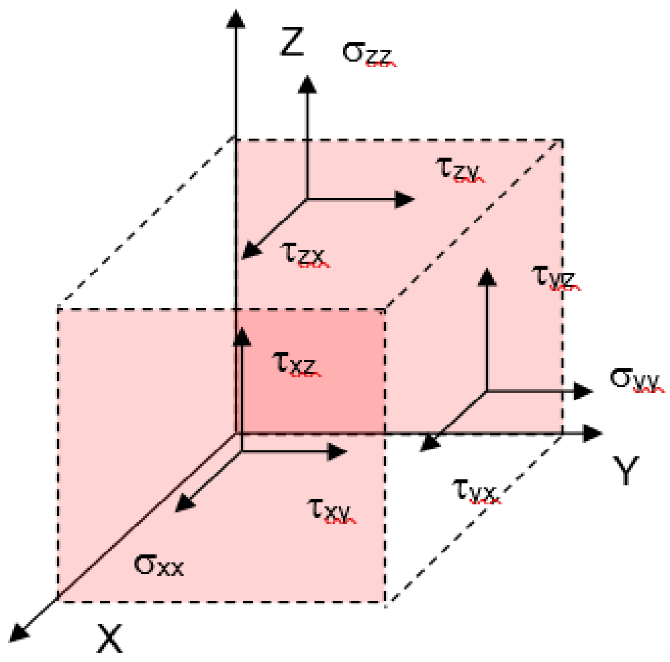
The Navier-Stokes Equations

$$\rho \frac{D\vec{V}}{Dt} + \nabla P - \nabla \bullet \tilde{\tau} - \rho \vec{f} = 0$$

Stress Tensor

The stress tensor has nine components:

$$\tilde{\tau} = \begin{vmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{vmatrix}$$



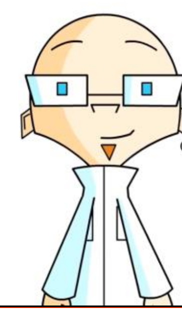
Newtonian fluid,

$$\tilde{\tau} = \mu[\nabla\vec{V} + (\nabla\vec{V})^T - \frac{2}{3}(\nabla \bullet \vec{V})\tilde{I}]$$

For incompressible flow, in Cartesian coordinate system

$$\tau_{ij} = \mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$$

$$\tau_{xy} = \tau_{yx}; \qquad \tau_{xz} = \tau_{zx} \qquad \tau_{zy} = \tau_{yz}$$



Reynolds Transport Theorem

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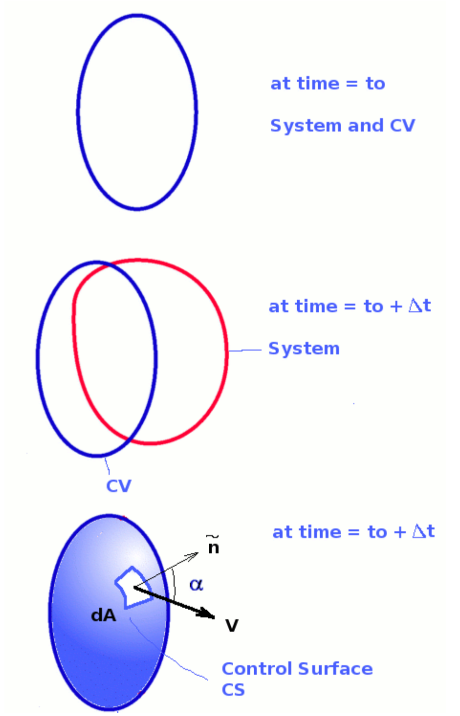

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N_s	α
Mass	1
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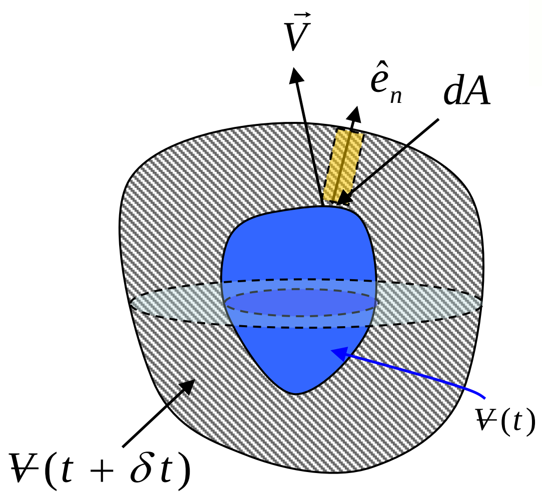
Energy	e
Entropy	s

- ***e is the total energy per unit mass of the fluid.***

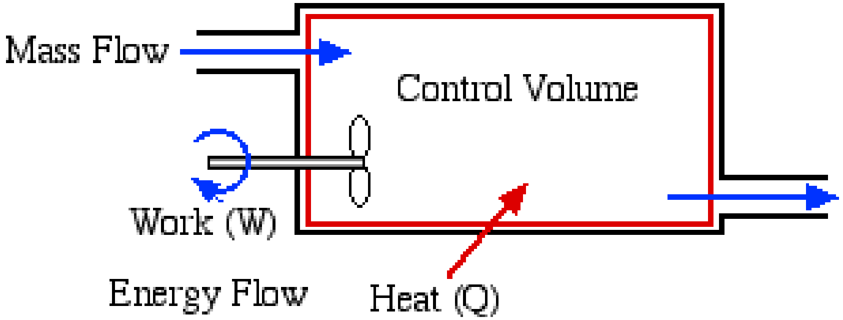


❑ Conservation of Energy

$$\dot{Q} + \dot{W} = \left.\frac{dE}{dt}\right|_{system}$$

$$E_{system} = \int_{M(sys)} e\,dm = \int_{V(sys)} e\rho\,dV$$

System Equation



- *e is the total energy per unit mass of the fluid.*

$$Q + W = \Delta E$$

heat — work — Change of internal energy

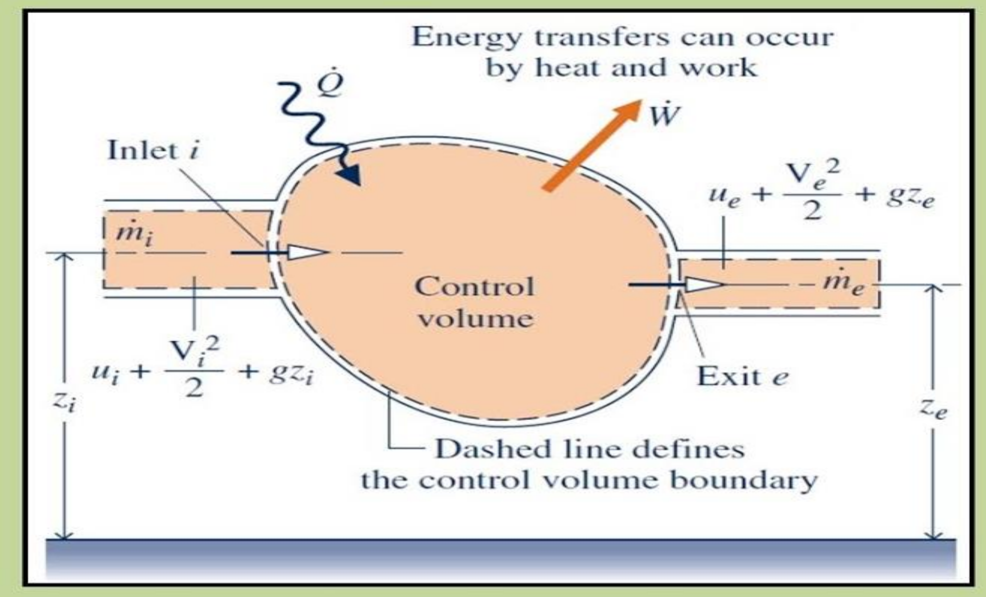
Or in rate term

$$\dot{Q} + \dot{W} = \frac{\Delta E}{\Delta t}$$

Rate of heat addition to CV

$$\dot{Q} = \iiint_V \rho \dot{q}\, dV + \dot{Q}_{vis}$$

Heat source



Note about work

Work= $\vec{F}.d\vec{r}$, rate of work=$\vec{F}.\frac{d\vec{r}}{dt} = \vec{F}.\vec{V}$

❑ Conservation of Energy

Rate of work done on a fixed CV

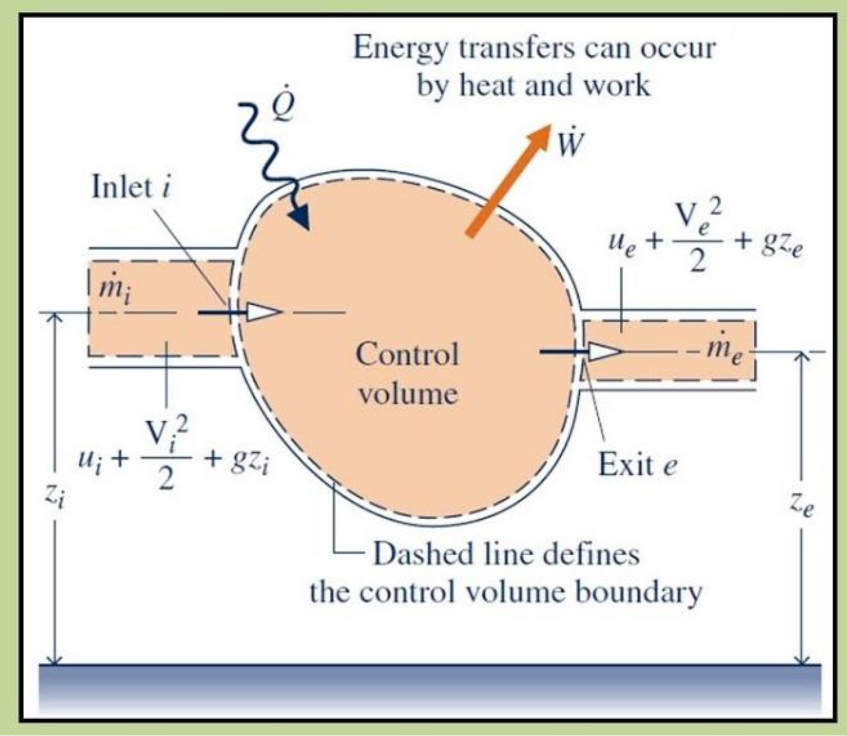
$$\dot{W} = \iiint_V \rho(\vec{f}.\vec{V})dV - \iint_S p\vec{V}.\hat{n}dS + \dot{W}_{vis} + \dot{W}_{shaft}$$

Rate of change of energy in CV

$$\frac{\partial}{\partial t}\iiint_V \rho\left(e+\frac{V^2}{2}\right)dV$$

Net flux of energy within CV

$$\iint_S \rho\left(e+\frac{V^2}{2}\right)\vec{V}.\hat{n}dS$$

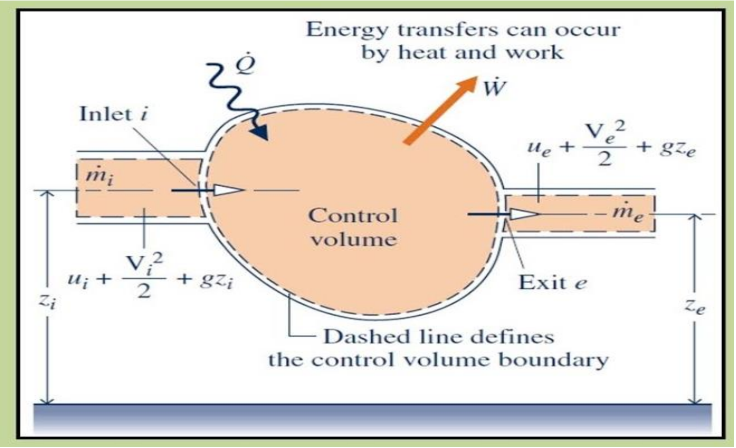


❑ Conservation of Energy

Conservation of Energy

$$\frac{\partial}{\partial t}\iiint\limits_{V} \rho\left(e+\frac{V^2}{2}\right)dV + \iint\limits_{S} \rho\left(e+\frac{V^2}{2}\right)\vec{V}.\hat{n}dS$$

$$= \iiint\limits_{V} \rho\dot{q}dV + \iiint\limits_{V} \rho(\vec{f}.\vec{V})dV - \iint\limits_{S} p\vec{V}.\hat{n}dS + \dot{Q}_{vis} + \dot{W}_{vis} + \dot{W}_{shaft}$$



And the differential form

$$\rho\frac{D\left(e+\frac{V^2}{2}\right)}{Dt} = \rho\dot{q} - \vec{\nabla}.(p\vec{V}) + \rho(\vec{f}.\vec{V}) + \dot{Q}'_{vis} + \dot{W}'_{vis}$$

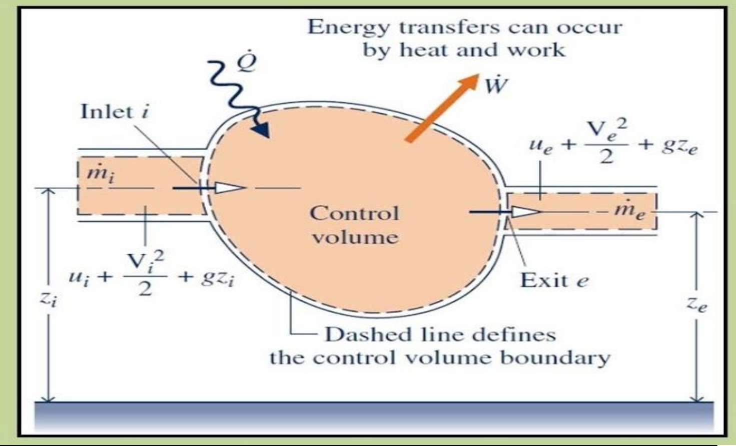
❑ Conservation of Energy

Alternative form with Enthalpy

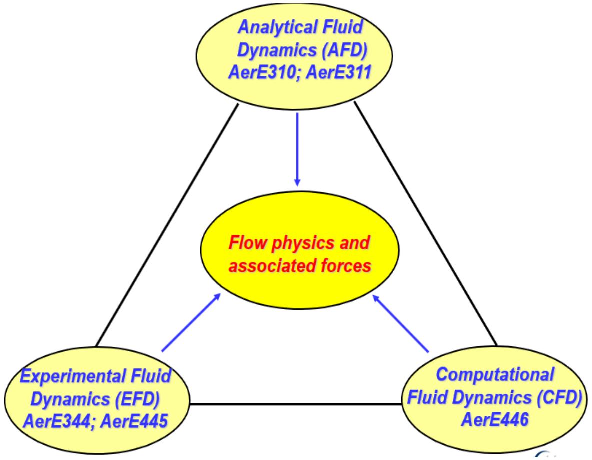
- Enthalpy is defined as $h = e + p/\rho$
- It is common to combine pressure work and internal energy flux into enthalpy flux.
- Also, often gravity is the only body force acting on a CV. It is then included as part of internal energy terms.

$$\frac{\partial}{\partial t}\iiint_{V} \rho\left(e+\frac{V^2}{2}+gz\right)d\mathrm{V} + \iint_{S} \rho\left(h+\frac{V^2}{2}+gz\right)(\vec{V}.\hat{n})dS$$

$$= \iiint_{V} \rho\dot{q}d\mathrm{V} + \dot{Q}_{vis} + \dot{W}_{vis} + \dot{W}_{shaft}$$



Discussion of Flow Governing Equations



The properties and the flow pattern of a moving fluid are governed by the fundamental laws of physics expression:

- Conservation of mass
- Conservation of momentum
- Conservation of energy
- Equation of state

When the mathematical equations expression these laws are solved satisfying the approximate initial and boundary conditions, the fluid properties and the flow pattern results.

These conservation equations involved three scalar fields (i.e., ρ, P, T) and one vector field (i.e., $\vec{V}$) as the unknown functions.

Conservation laws for	Equations	Number of Eqns.	Order of Eqns.	Total order
Mass	$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{V} = 0$	1	1	1
Momentum	$\frac{\partial}{\partial t}\left(\rho \vec{V}\right) + \nabla \cdot (\rho \vec{V} \vec{V}) = -\nabla p + \nabla \cdot \tilde{\tau} + \rho \vec{f}$	3	2	6
Energy	–	1	2	2
Equation of state	$p = f(\rho, T)$	1	0	0
	Total	6	9	9

- Independent variables: q_1, q_2, q_3, t
- Dependent variables: $\rho, P, T, V_1, V_2, V_3$
- Prescribed quantities: $\vec{f}, \mu(T), c_p(T), R$, etc.
- There are six equations and six dependent variables $\Rightarrow$ Equations can be solved.
- The sum of the order of the differential equations is equal to nine and we need nine boundary conditions.
- The conservation equations are nonlinear, that is coefficients of some of the derivatives are dependent variables. Need an interactive solution.
- All equations are coupled and hence must be solved for simultaneously.