Lecture # 12: Simplification of the Conservation Equations

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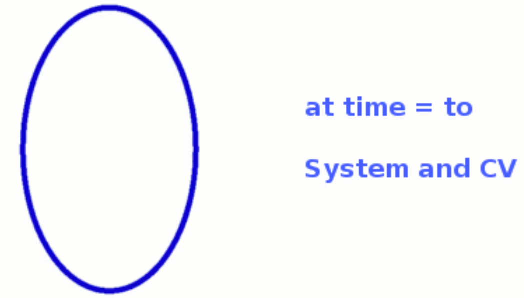
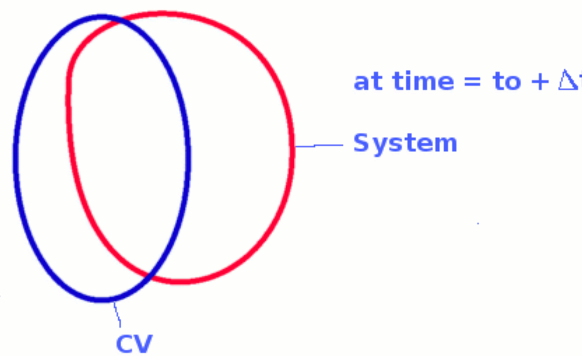
Reynolds Transport Theorem

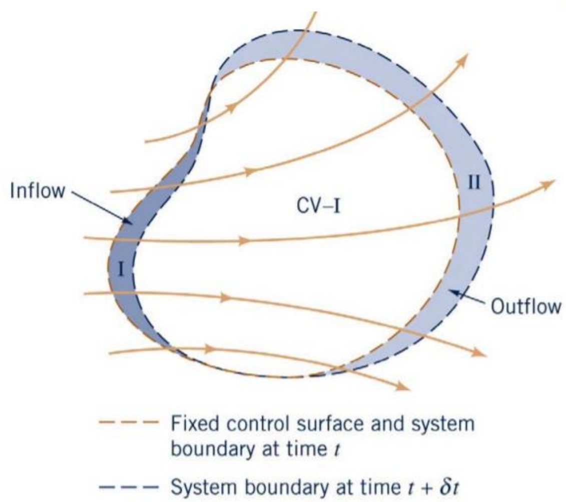
$$\frac{DN_s}{Dt} = \frac{D\int_{\forall} \alpha\,\rho\,d\forall}{Dt} = \frac{\partial}{\partial t}\int_{C.V.} \alpha\,\rho\,d\forall + \int_{C.S.} (\alpha\,\rho\vec{V}) \bullet d\vec{A}$$

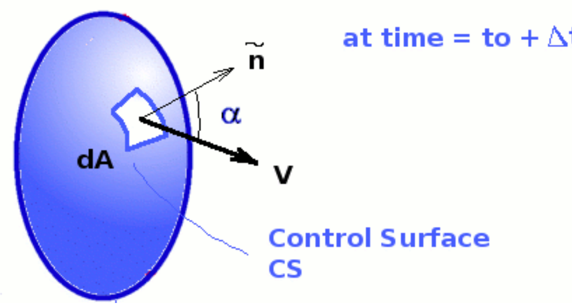


Where α is any intensive property corresponding to N. (i.e., $\alpha = N$ per unit mas
and it can be used for different quantities as follows.



N_s	α
Mass	1
Linear momentum	$\vec{V}$
Angular momentum	$\vec{R}\times\vec{V}$
Energy	e
Entropy	s





❑ Governing Equations for Fluid Flows

Controlling equations:

Continuity equation: $$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \vec{V}) = 0$$

Momentum equation: $$\frac{\partial (\rho \vec{V})}{\partial t} + \nabla \bullet (\rho \vec{V} \vec{V}) + \nabla P - \nabla \bullet \tilde{\tau} - \rho \vec{f} = 0$$

Energy equation: $$\frac{\partial}{\partial t}\left[\rho\left(e + \frac{V^2}{2}\right)\right] + \nabla.\left[\rho\left(e + \frac{V^2}{2}\right)\boldsymbol{V}\right] = \rho\dot{q} - \nabla.(p\boldsymbol{V}) + \rho(\boldsymbol{f}.\boldsymbol{V}) + \dot{\boldsymbol{Q}}'_{viscous} + \dot{\boldsymbol{W}}'_{viscous}$$

State equation: $$P = \rho R T$$

Conservation laws for	Equations	Number of Eqns.	Order of Eqns.	Total order
Mass	$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{V} = 0$	1	1	1
Momentum	$\frac{\partial}{\partial t}\left(\rho \vec{V}\right) + \nabla \cdot (\rho \vec{V} \vec{V}) = -\nabla p + \nabla \cdot \tilde{\tau} + \rho \vec{f}$	3	2	6
Energy	–	1	2	2
Equation of state	$p = f(\rho, T)$	1	0	0
	Total	6	9	9

❑ Isotropic Fluid and Ideal Flow

Isotropic Fluid:

- ***Isotropic, Newtonian are assumed to have linear relationship between stress and rate of strain.***

- ***A fluid is said to be isotropic when the relation between the components of stress and those of rate of train is the same in all directions. It is said to be Newtonian when this relationship is linear, that is when the fluid obeys stokes law of friction.***

$$\rho \frac{D\vec{V}}{Dt} + \nabla P - \nabla \bullet \tilde{\tau} - \rho \vec{f} = 0$$

Stress Tensor

The stress tensor has nine components:

$$\tilde{\tau} = \begin{vmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{vmatrix}$$

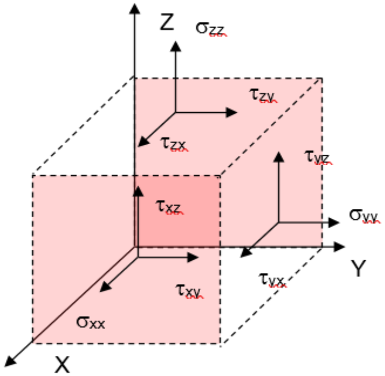
Newtonian fluid,

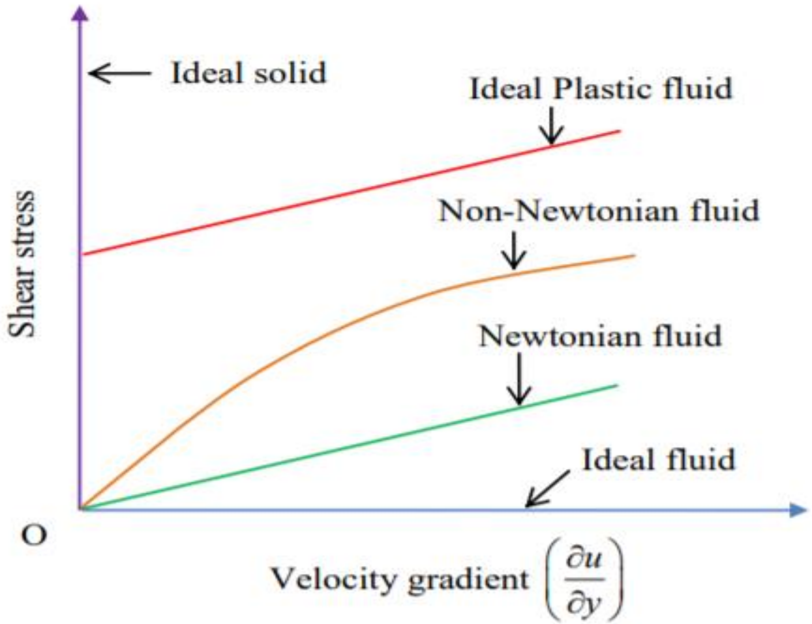
$$\tilde{\tau} = \mu[\nabla\vec{V} + (\nabla\vec{V})^T - \frac{2}{3}(\nabla \bullet \vec{V})\tilde{I}]$$

For incompressible flow, in Cartesian coordinate system

$$\tau_{ij} = \mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$$

$$\tau_{xy} = \tau_{yx}; \qquad \tau_{xz} = \tau_{zx} \qquad \tau_{zy} = \tau_{yz}$$

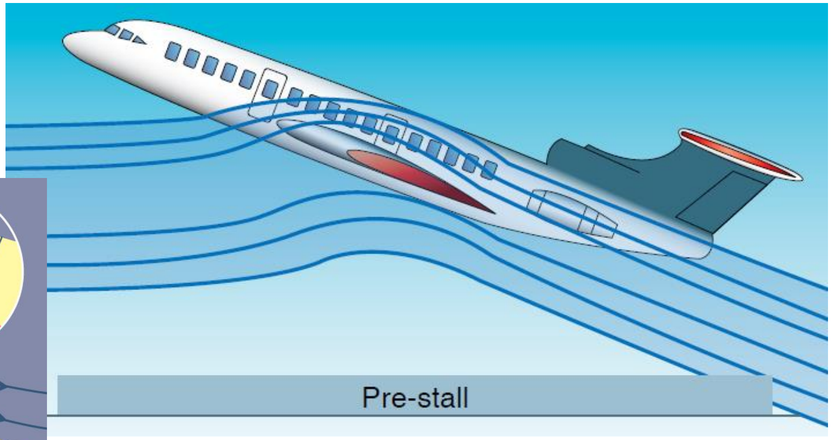


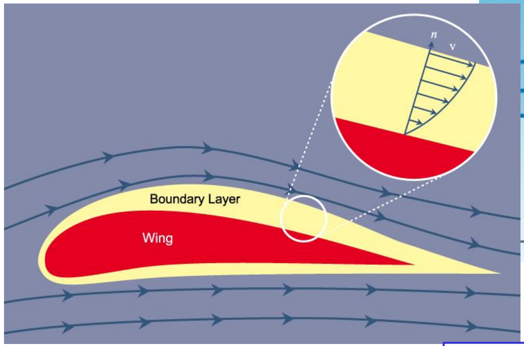


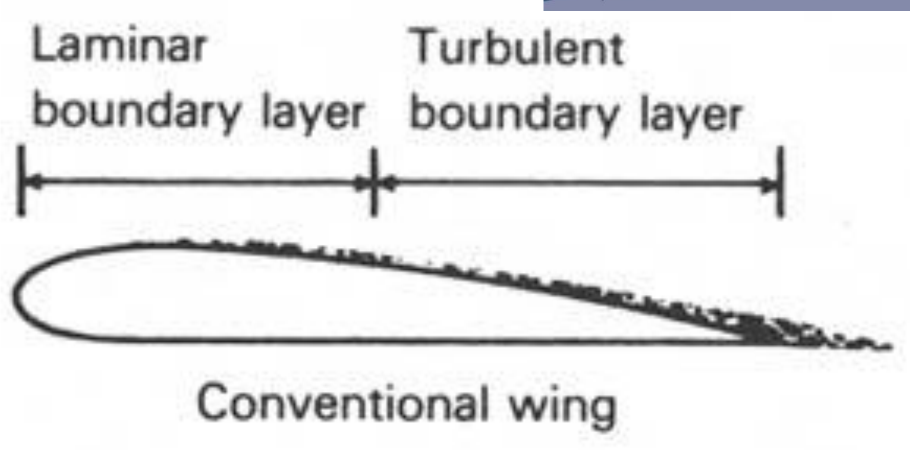
❑ Isotropic Fluid and Ideal Flow

Ideal flow:

- ***Non-heat conducting, inviscid, incompressible, homogeneous fluid is defined as ideal fluid.***
- ***Assumptions used are:***
  - ***Non-heat conductive***
  - ***Homogeneous***
  - ***Incompressible***
  - ***Inviscid flow***







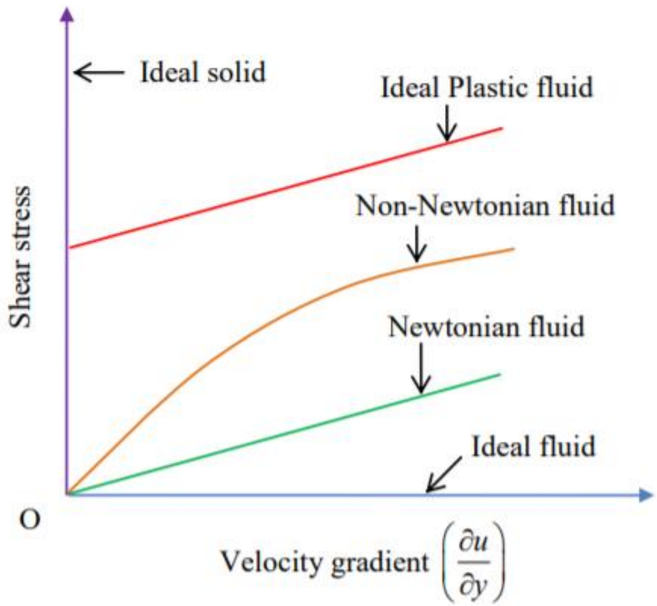
Made with KINEMASTER

[Define ideal fluid (youtube.com)]

❑ Governing Equations for Ideal Fluid Flows

Ideal flow:

- ***Non-heat conductive***
- ***Homogeneous***
- ***Incompressible***
- ***Inviscid flow***



Controlling equations:

Continuity equation: $$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \vec{V}) = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \vec{V}) = 0 \Rightarrow \frac{\partial \rho}{\partial t} + \vec{V} \bullet \nabla \rho + \rho \nabla \bullet \vec{V} = 0$$

$$\Rightarrow \frac{D\rho}{Dt} + \rho \nabla \bullet \vec{V} = 0$$

$$\Rightarrow \nabla \bullet \vec{V} = 0$$

❑ Governing Equations for Ideal Fluid Flows

Momentum equation: $$\frac{\partial(\rho\vec{V})}{\partial t}+\nabla\bullet(\rho\vec{V}\vec{V})+\nabla P-\nabla\bullet\tilde{\tau}-\rho\vec{f}=0$$

$$\frac{\partial(\rho\vec{V})}{\partial t}+\nabla\bullet(\rho\vec{V}\vec{V})+\nabla P-\nabla\bullet\tilde{\tau}-\rho\vec{f}=0$$

$$\Rightarrow\quad \vec{V}\frac{\partial\rho}{\partial t}+\rho\frac{\partial\vec{V}}{\partial t}+\vec{V}\nabla\bullet(\rho\vec{V})+(\rho\vec{V}\bullet\nabla)\vec{V}+\nabla P-\overbrace{\nabla\bullet\tilde{\tau}}^{=0\ due\ to\ inviscid}-\rho\vec{f}=0$$

$$\Rightarrow\quad \vec{V}[\frac{\partial\rho}{\partial t}+\nabla\bullet(\rho\vec{V})]+\rho[\frac{\partial\vec{V}}{\partial t}+(\vec{V}\bullet\nabla)\vec{V}]+\nabla P-\overbrace{\nabla\bullet\tilde{\tau}}^{=0\ due\ to\ inviscid}-\rho\vec{f}=0$$

$$\Rightarrow\quad \rho\frac{D\vec{V}}{Dt}=-\nabla P+\rho\vec{f}$$

$$\boxed{\rho\frac{D\vec{V}}{Dt}=-\nabla P+\rho\vec{f}}$$

is also called Euler equation

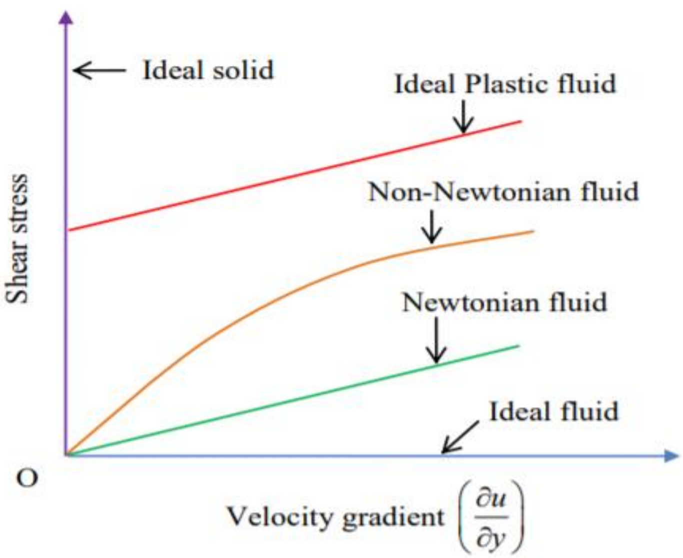
Stress Tensor

The stress tensor has nine components:

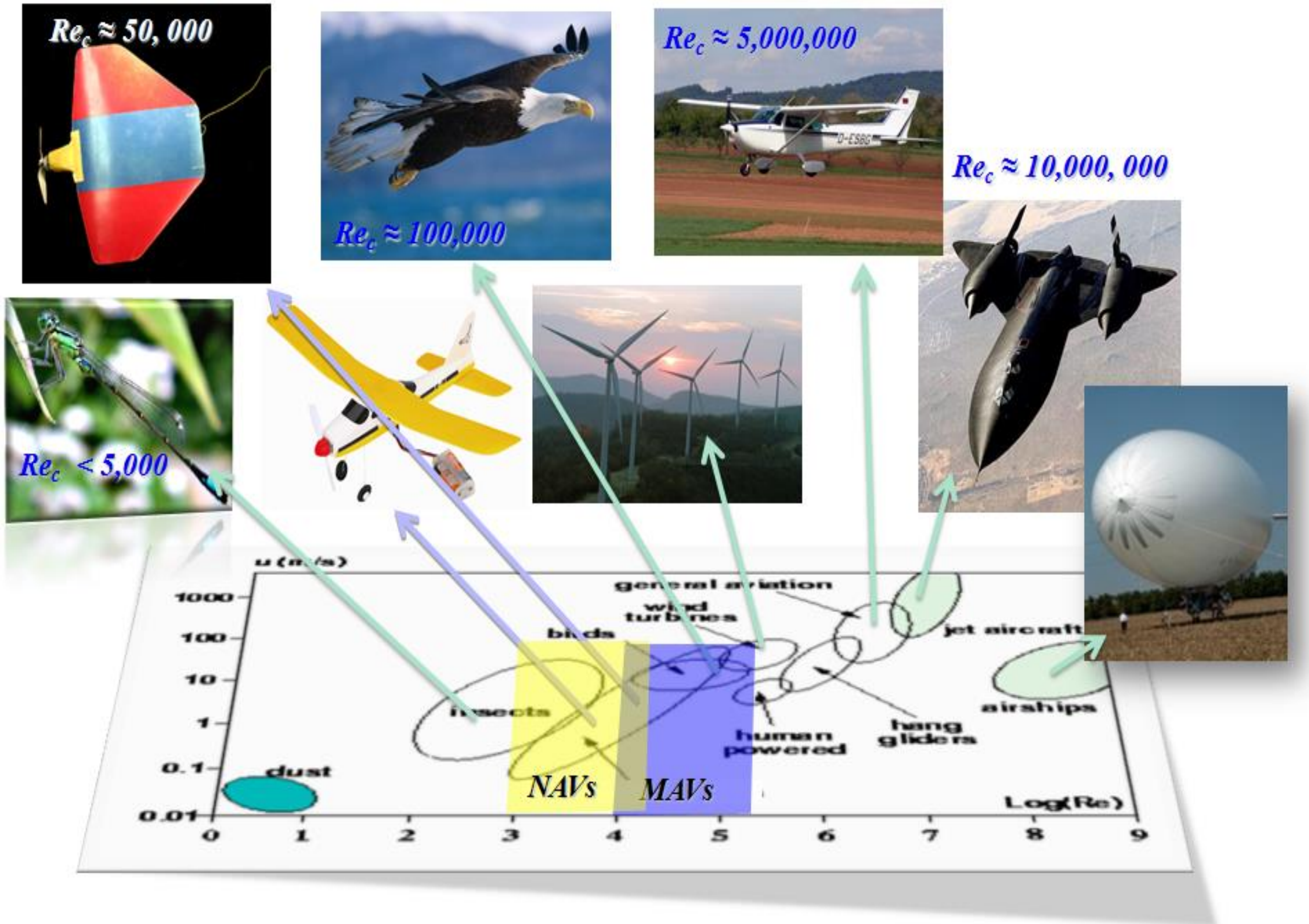
$$\tilde{\tau}=\begin{vmatrix}\sigma_{xx} & \tau_{yx} & \tau_{zx}\\ \tau_{xy} & \sigma_{yy} & \tau_{zy}\\ \tau_{xz} & \tau_{yz} & \sigma_{zz}\end{vmatrix}$$

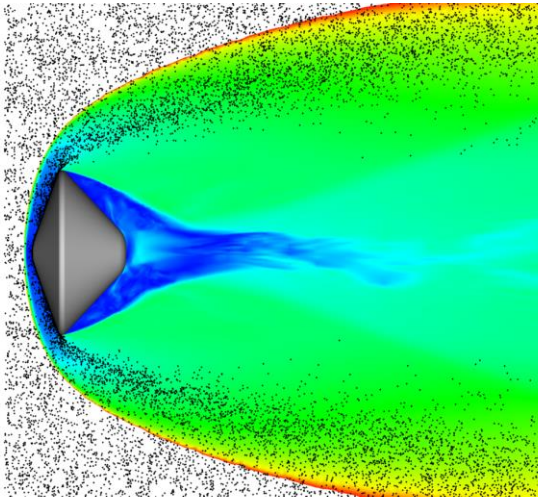
Newtonian fluid,

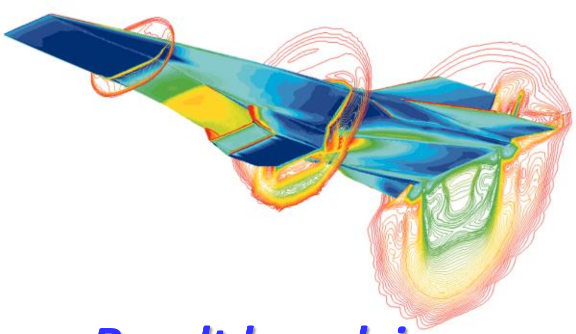
$$\tilde{\tau}=\mu[\nabla\vec{V}+(\nabla\vec{V})^T-\frac{2}{3}(\nabla\bullet\vec{V})\tilde{I}]$$



❑ Flow with different Reynolds numbers





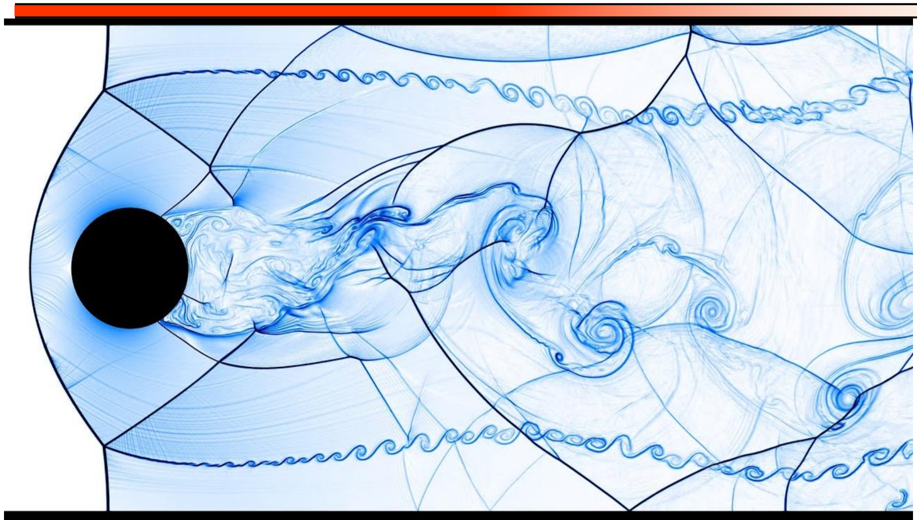


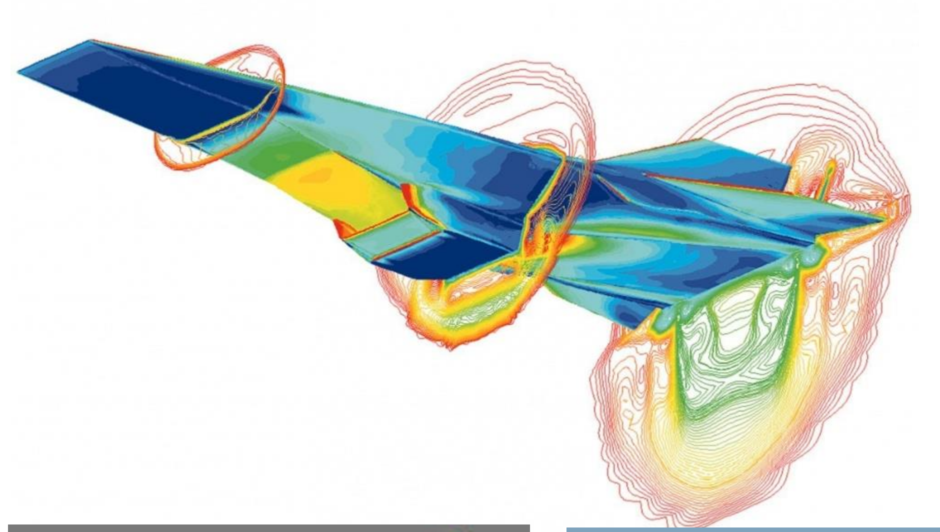
- *Result by solving Euler equation*

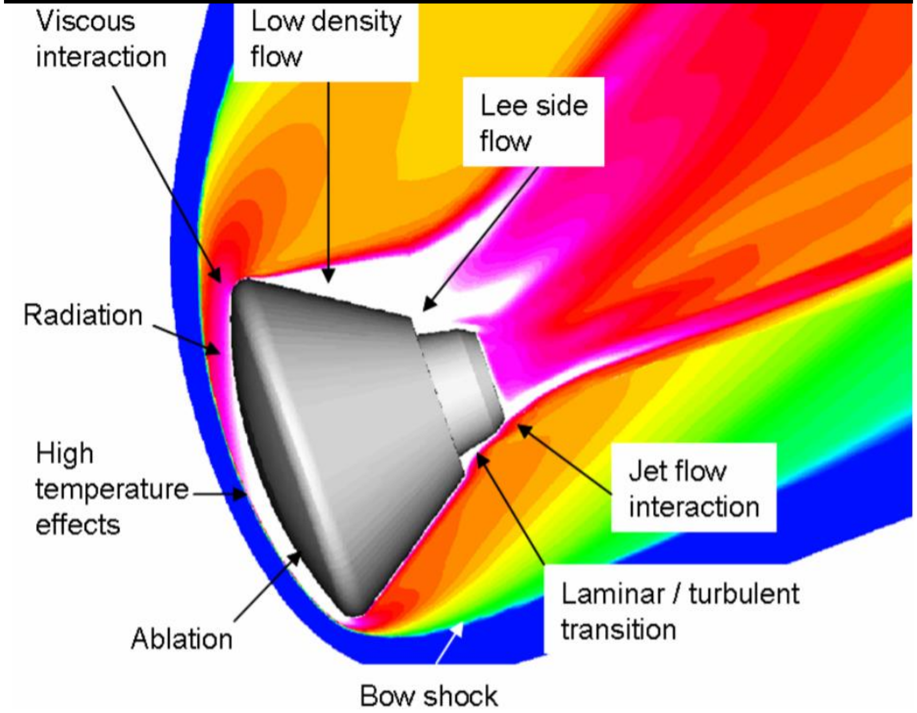
$$Re = \rho UL/\mu$$

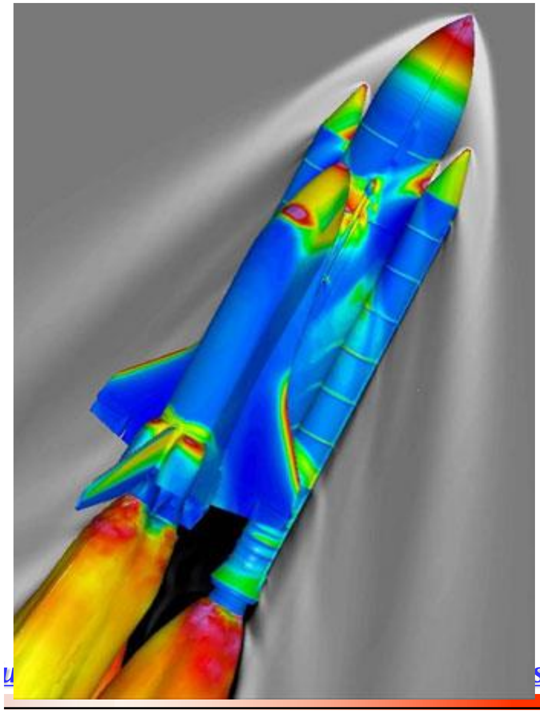
- *Re number is higher for large/fast moving objects*
- *Viscosity can be neglected for high Reynolds number flows*

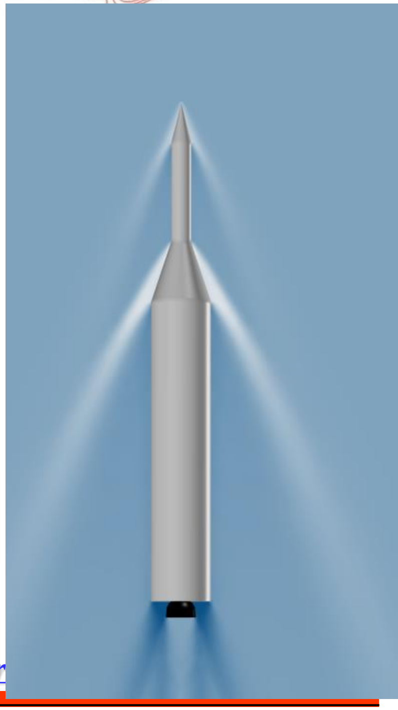
❑ CFD Results by solving Euler equation











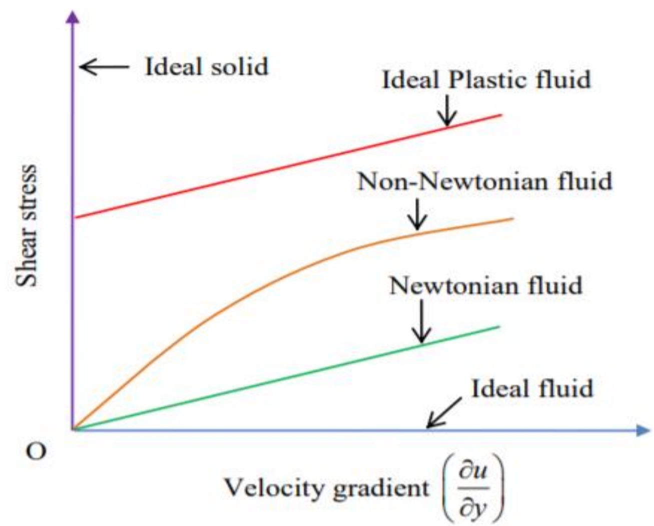
❑ Governing Equations for Ideal Fluid Flows

For ideal fluid, the governing equations are:

1). Continuity equation: $\nabla \bullet \vec{V} = 0$

2). Euler equation $\rho \dfrac{D\vec{V}}{Dt} = -\nabla P + \rho \vec{f}$

or $\dfrac{D\vec{V}}{Dt} = -\nabla(\dfrac{P}{\rho}) + \vec{f}$



- ***For the Euler equation, it can also be re-written as:***

$$\frac{D\vec{V}}{Dt} = -\nabla(\frac{P}{\rho}) + \vec{f} \quad \Rightarrow \quad \frac{\partial \vec{V}}{\partial t} + (\vec{V} \bullet \nabla)\vec{V} = -\nabla(\frac{P}{\rho}) + \vec{f}$$

$$Since \quad (\vec{V} \bullet \nabla)\vec{V} = \nabla(\frac{\vec{V} \bullet \vec{V}}{2}) - \vec{V} \times (\nabla \times \vec{V}) \quad \Rightarrow \quad \frac{\partial \vec{V}}{\partial t} + \nabla(\frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho}) - \vec{V} \times (\nabla \times \vec{V}) = \vec{f}$$

- ***Let us only consider body forces that are conservative only. A necessary and sufficient condition for the body force can be represented as the gradient of a scalar field U, i.e.,***

$$\vec{f} = \nabla U$$

- ***Therefore:***

$$\frac{\partial \vec{V}}{\partial t} + \nabla(\frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho}) - \vec{V} \times (\nabla \times \vec{V}) = \nabla U \quad \Rightarrow \quad \frac{\partial \vec{V}}{\partial t} + \nabla(\frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho} - U) - \vec{V} \times (\nabla \times \vec{V}) = 0$$

❑ Ideal Fluid Flow

https://www.youtube.com/watch?v=oasL7ZWNly8