Lecture # 13: Vorticity and Circulation

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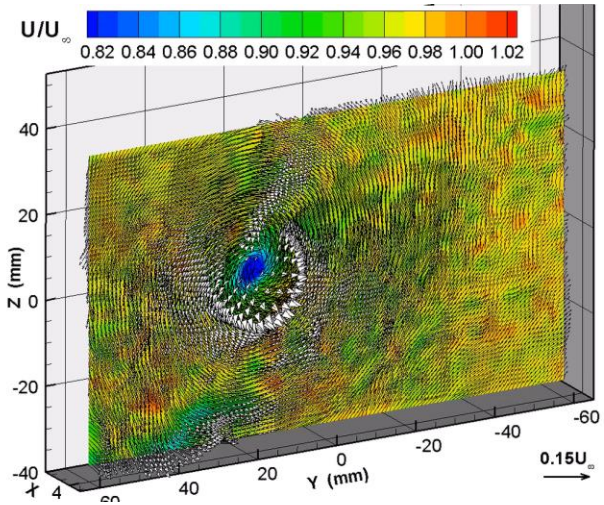
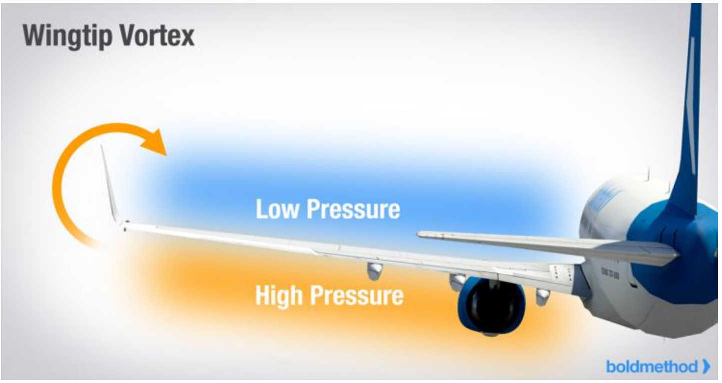
❑ Vorticity and Circulation



Fluid rotation

- ***Definition*: Fluid rotation at a point O (also called the angular velocity at the point) is the average angular velocity of two infinitesimal and mutually perpendicular fluid lines OA and OB instantaneously passing thorough point O.**
- ***A fluid line is a line passing through a set of fluid particles of fixed identity.***
- ***The average fluid rotation at a point is given by:***

$$\vec{\varpi} = \frac{1}{2}(\nabla \times \vec{V})$$



Vorticity

- ***Definition: Vorticity is defined to be:***

$$\vec{\Omega} = 2\vec{\varpi} = \nabla \times \vec{V}$$

- ***In an irrotational velocity field as know as irrotational flow***

$$\vec{\Omega} = 2\vec{\varpi} = \nabla \times \vec{V} = 0$$

❑ Vorticity and Circulation

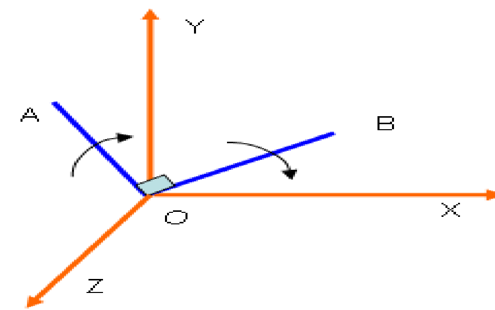
- It is defined as curl of the velocity vector

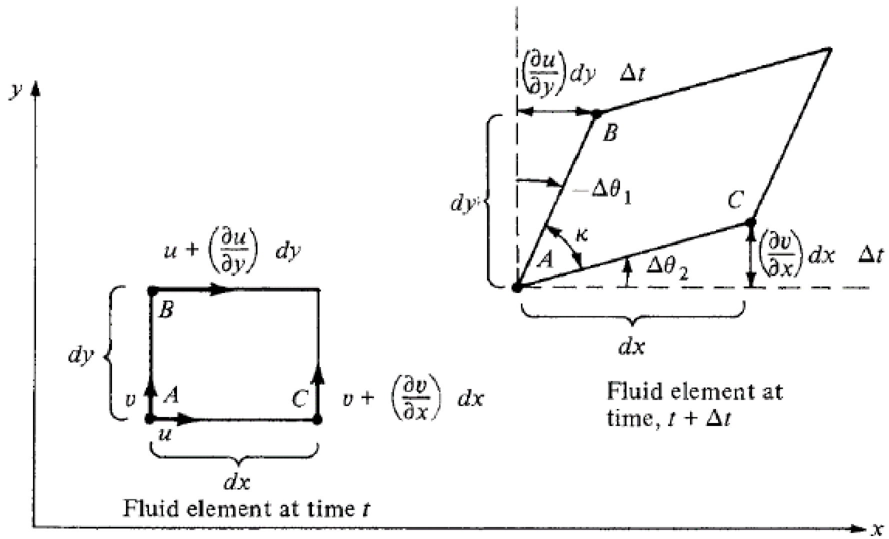
$$\vec{\Omega} = 2\vec{\varpi} = \nabla \times \vec{V}$$

It is a measure of "moment of momentum" of a fluid element around its own center of mass.

- Physically, it is twice the rate of rotation of fluid element when frozen.
- Angular velocity of fluid element

$$= \frac{1}{2}\left(\frac{d\theta_1}{dt} + \frac{d\theta_2}{dt}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\omega_z$$





$$\Delta\theta_1 = -\frac{\partial u}{\partial y}\Delta t$$

$$\Delta\theta_2 = \frac{\partial v}{\partial x}\Delta t$$

❑ Vorticity and Circulation

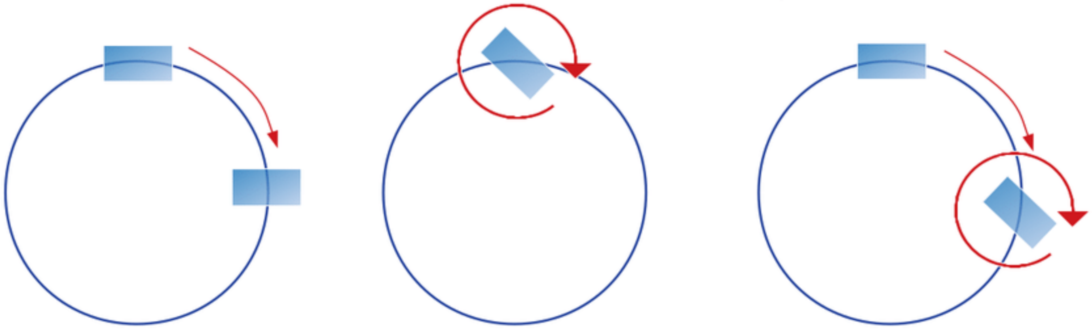
Vorticity and rotational flow

$$\vec{\Omega} = 2\vec{\varpi} = \nabla \times \vec{V}$$

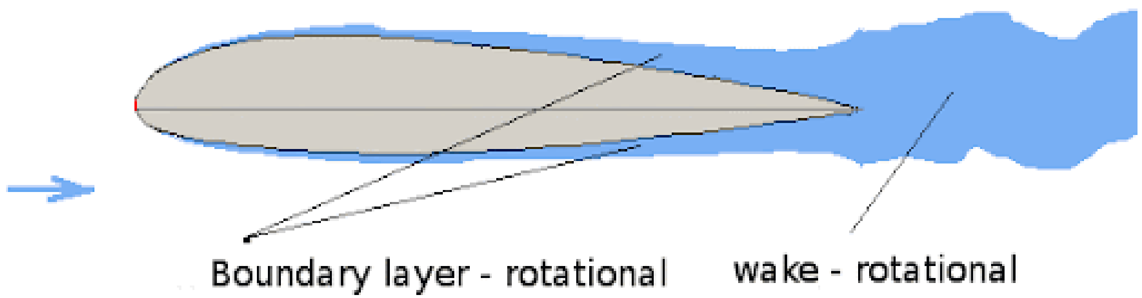
$\vec{\omega} = 0 \rightarrow$ irrotational flow (often means inviscid)

$\vec{\omega} \neq 0 \rightarrow$ rotational flow (viscous flow)

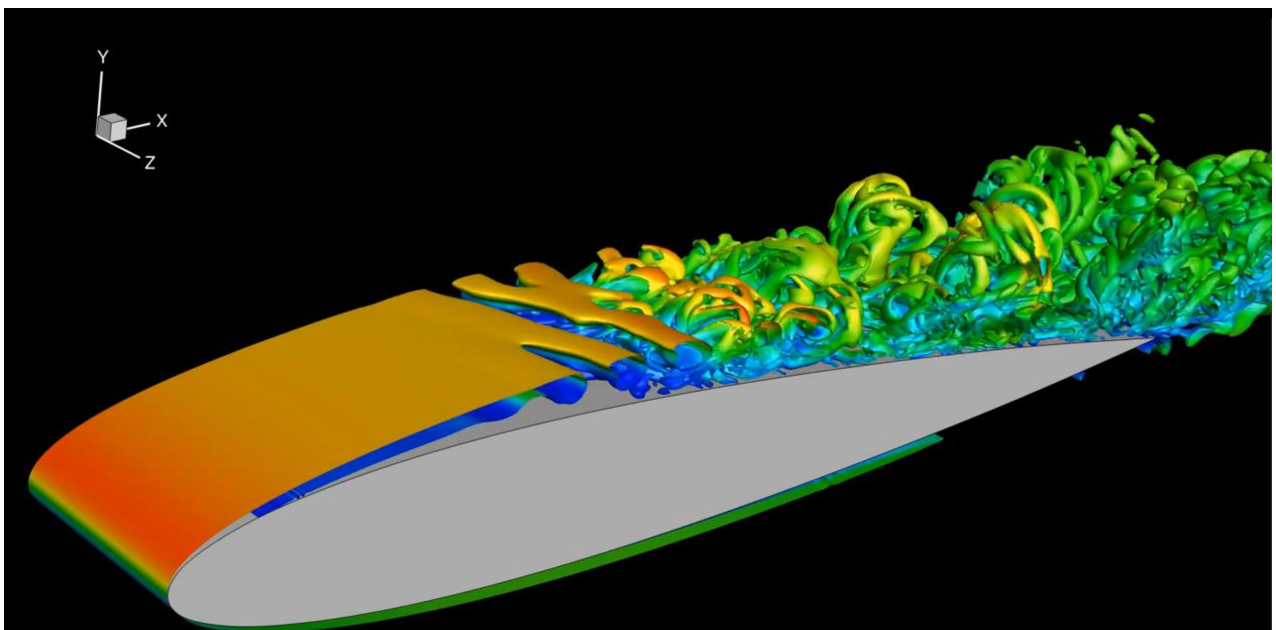
Note: having a circulatory motion does not always mean rotational flow!



❑ Vorticity and Circulation



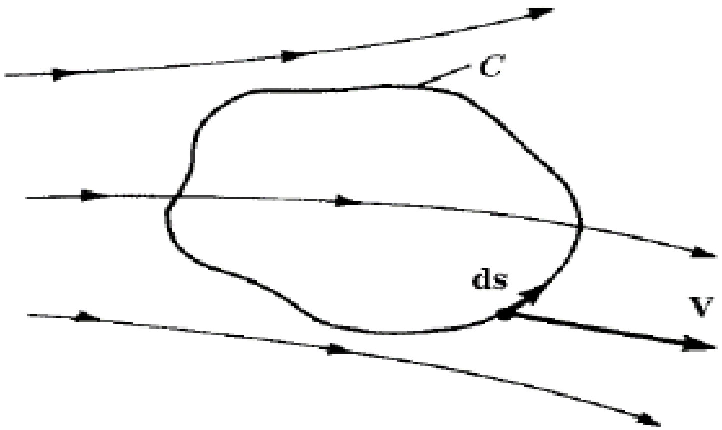
Vortex structures over a NACA0012 airfoil



- ***https://www.youtube.com/watch?v=AfAM6mfuN3c***

Vorticity and Circulation

Circulation



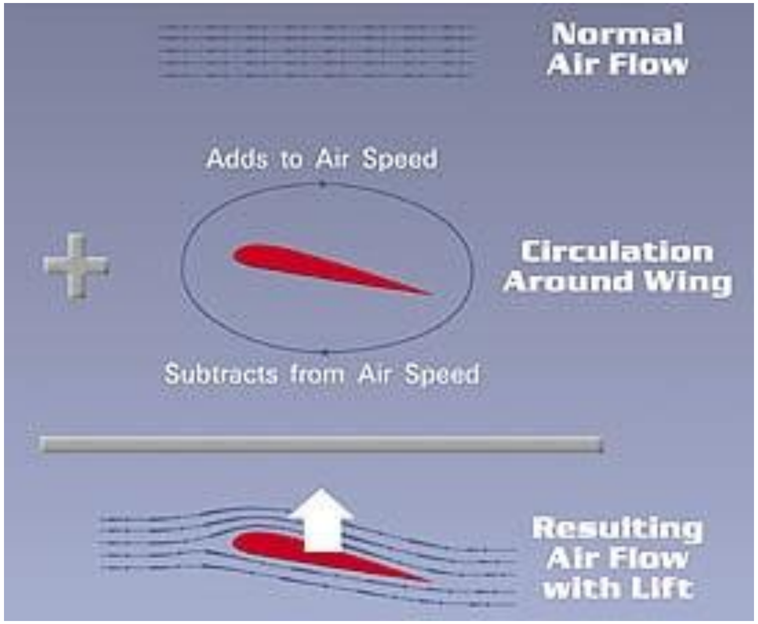
Consider a closed curve C in a flow field.
Circulation Γ is defined as

$$\Gamma = \oint_C \vec{V}.d\vec{s}$$

It is a line integral of velocity around a closed curve in the flow.

Later, you will see Γ is directly related to the lift generated around a body.

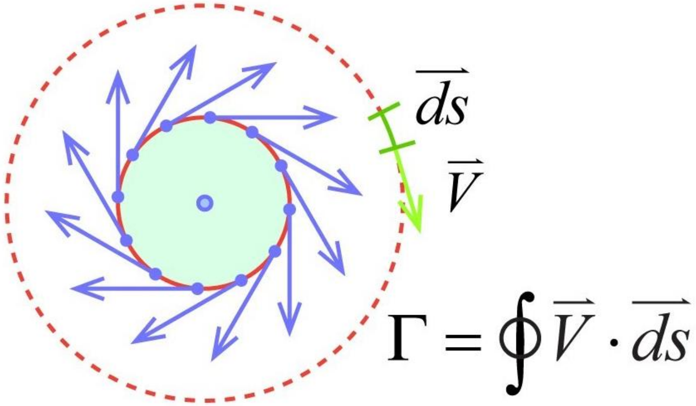
$\Gamma > 0$ counterclockwise and <0 clockwise (note Anderson textbook uses the opposite definition).



❑ Vorticity and Circulation

Circulation is related to the vorticity by Stoke's theorem

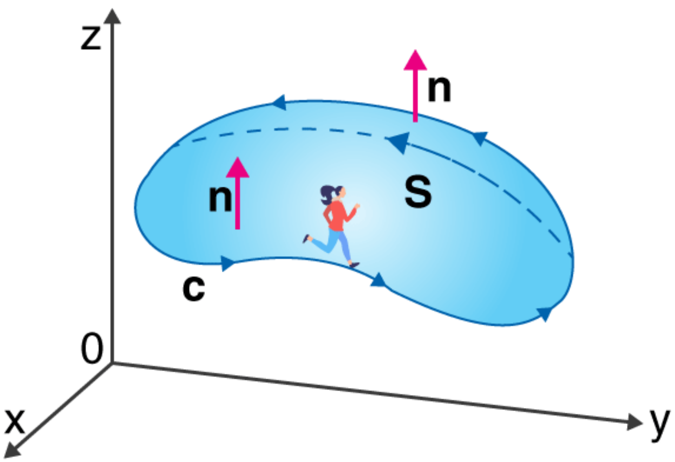
$$\Gamma = \oint_C \vec{V}.d\vec{s} = \iint_S (\vec{\nabla} \times \vec{V}).\hat{n}dS = \iint_S \vec{\Omega}.\hat{n}dS$$

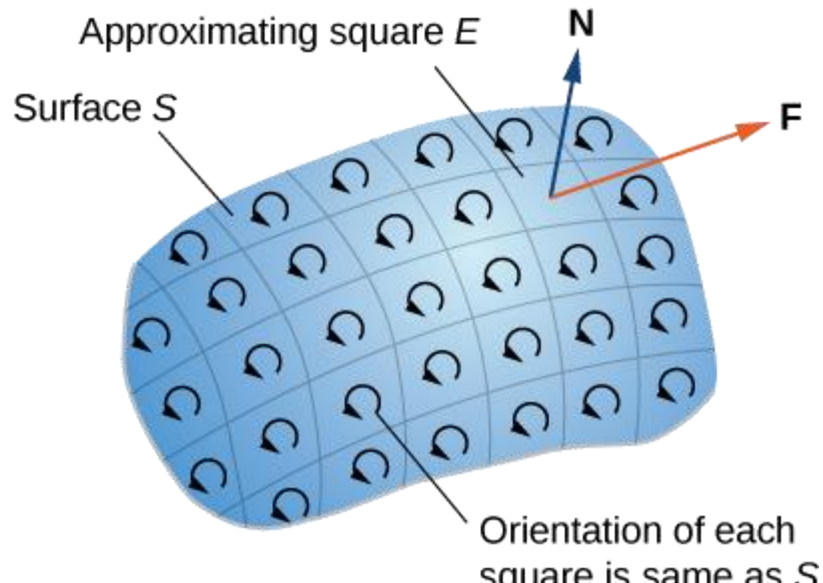


Where S is the area enclosed by curve C.

Also

$$(\vec{\nabla} \times \vec{V}).\hat{n} = \frac{d\Gamma}{dS} \quad \vec{\Omega}.\hat{n} = \frac{d\Gamma}{dS}$$





Vorticity and Circulation

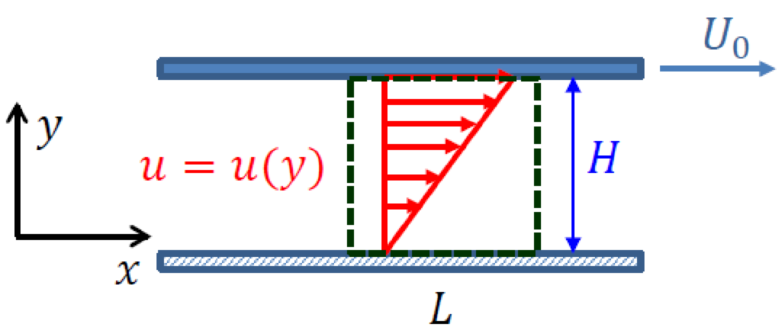
- ***Example 01:***

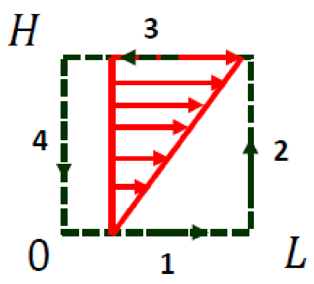
Couette flow

Steady incompressible flow between two large plates. The top plate moves with constant velocity U_0 but the bottom plate is fixed. The velocity distribution in the gap is given as $u(y) = \frac{U_0}{H} y$.

Calculate circulation for the rectangular box shown.

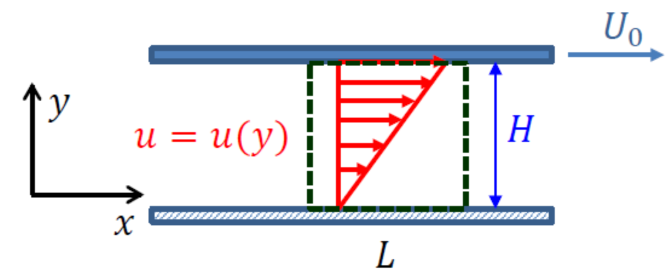
Find the vorticity for this flow and verify circulation-vorticity relationship.





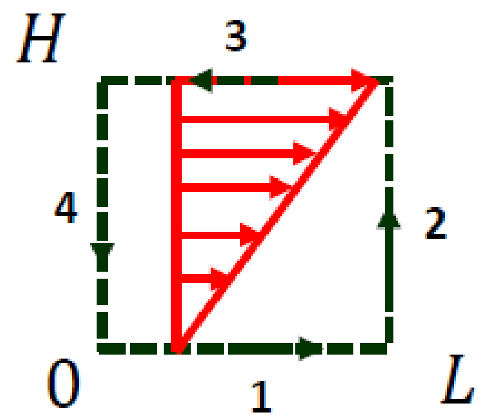
❑ Vorticity and Circulation

- ***Solution of example 01:***



$$\Gamma = \oint_C \vec{V}.\,d\vec{s}$$

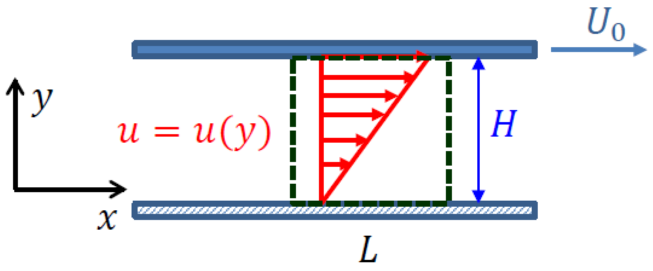
$$= \int_1 0\hat{\imath}.\hat{\imath}\,dx + \int_2 u(y)\hat{\imath}.\hat{\jmath}dy + \int_3 U_0\hat{\imath}.(-\hat{\imath})dx + \int_4 u(y)\hat{\imath}.(-\hat{\jmath})dy$$



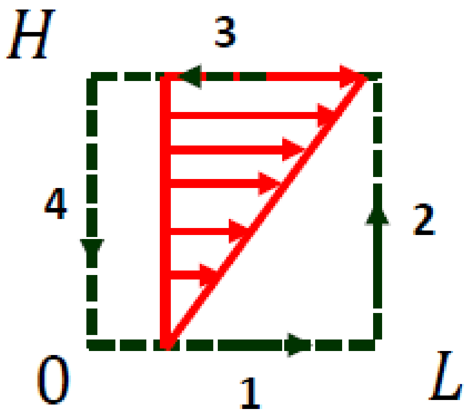
$$\Gamma = \int_0^L -U_0\,dx = -U_0 L$$

❑ Vorticity and Circulation

- ***Solution of example 01:***



$$\vec{\Omega} = 2\varpi_z \hat{k} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k} = -\frac{U_0}{H}\hat{k}$$

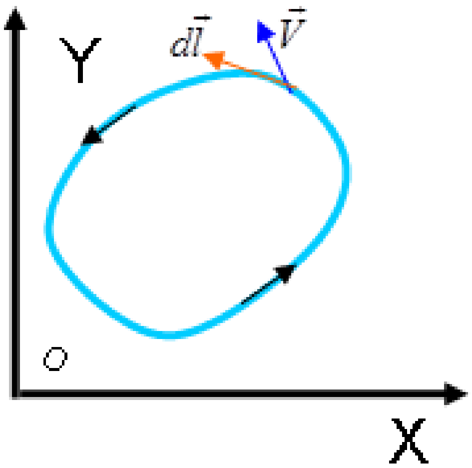


$$\Gamma = \iint_S \vec{\Omega}.\hat{n}dS = \int_0^L \int_0^H -\frac{U_0}{H}\hat{k}.\hat{k}dy\,dx = \int_0^L -\frac{U_0}{H}Hdx = -U_0 L$$

Which is the same result obtained using definition of the circulation

❑ Vorticity and Circulation

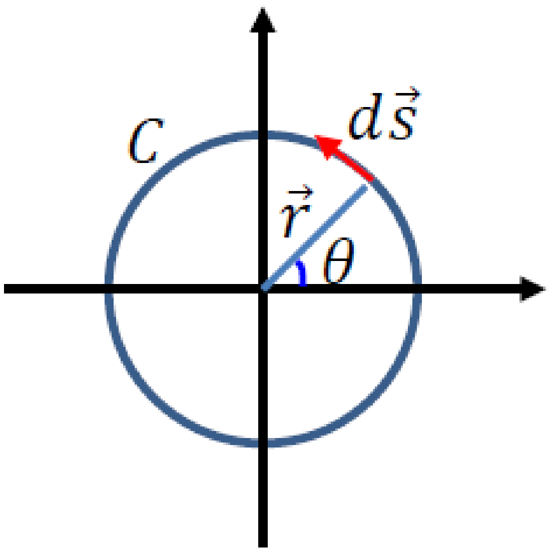
- *Example 02:*



- Velocity field for 'vortex flow' is given as

$$u = \frac{cy}{x^2 + y^2}, \qquad v = -\frac{cx}{x^2 + y^2}$$

Where c is a constant.



Find circulation around a circular curve with radius

Calculate vorticity field and verify circulation and vorticity relationship.

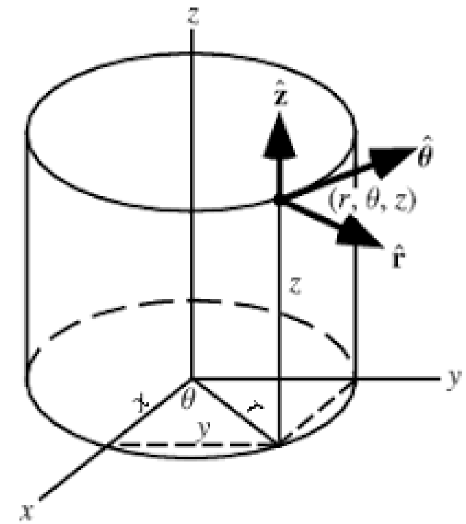
❑ Vorticity and Circulation

Example - solution

It is easier to solve this problem in cylindrical (polar) coordinate system

In polar coordinates:

$$x = r\cos\theta \ , \qquad y = r\sin\theta$$



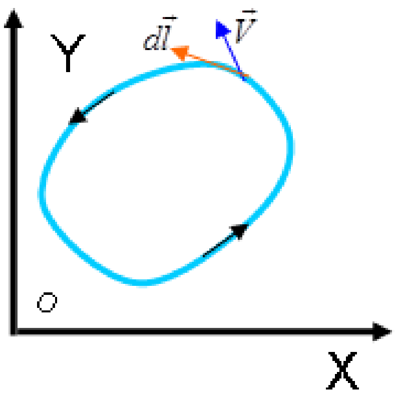
$$v_r = u\cos\theta + v\sin\theta$$

$$v_r = \frac{cr\sin\theta}{r^2}\cos\theta - \frac{cr\cos\theta}{r^2}\sin\theta = 0$$

$$v_\theta = -u\sin\theta + v\cos\theta$$

$$v_\theta = -\frac{cr\sin\theta}{r^2}\sin\theta - \frac{cr\cos\theta}{r^2}\cos\theta = -\frac{c}{r}(\sin^2\theta + \cos^2\theta) = -\frac{c}{r}$$

❑ Vorticity and Circulation



Consider a circle with radius r as C

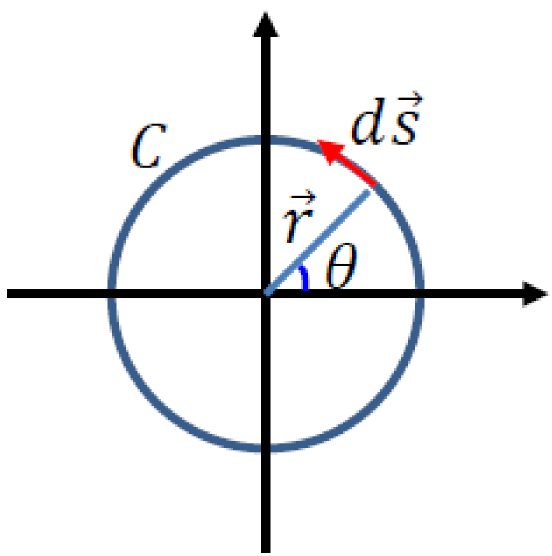
$$d\vec{s} = dr\hat{e}_r + rd\theta\hat{e}_\theta$$

$$\Gamma = \oint_C \vec{V}.d\vec{s} = \oint_C (v_r\hat{e}_r + v_\theta\hat{e}_\theta).(dr\hat{e}_r + rd\theta\hat{e}_\theta)$$

$$\Gamma = \oint_C (v_r dr + rv_\theta d\theta) = \oint_C \left(0 + r\left(-\frac{c}{r}\right)d\theta\right)$$

$$\Gamma = -\oint_C c d\theta = -c\int_0^{2\pi} d\theta = -c(2\pi)$$

$$\Gamma = -2\pi c$$

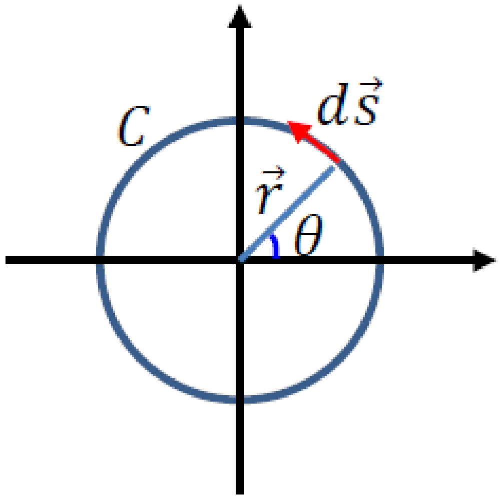


Vorticity and Circulation

Example - continued

Note for a 2D flow, the only component of vorticity that can exist is normal to the plane of motion, here ω_z

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial r}(-c) - \frac{1}{r}\frac{\partial}{\partial \theta}(0) = 0\,,$$
$$r \neq 0$$

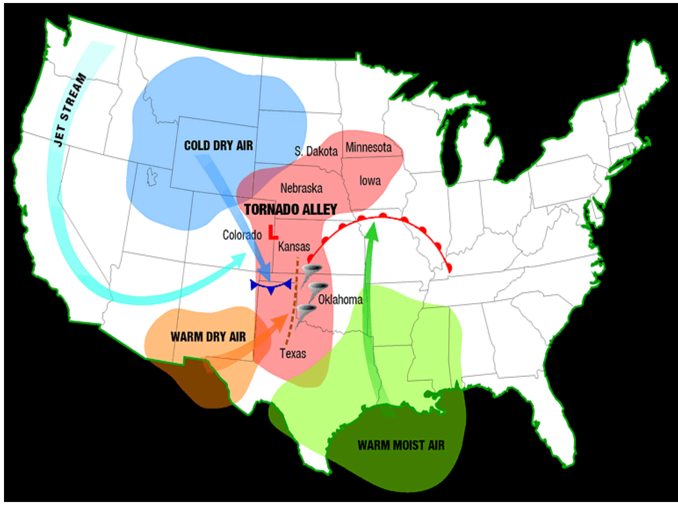


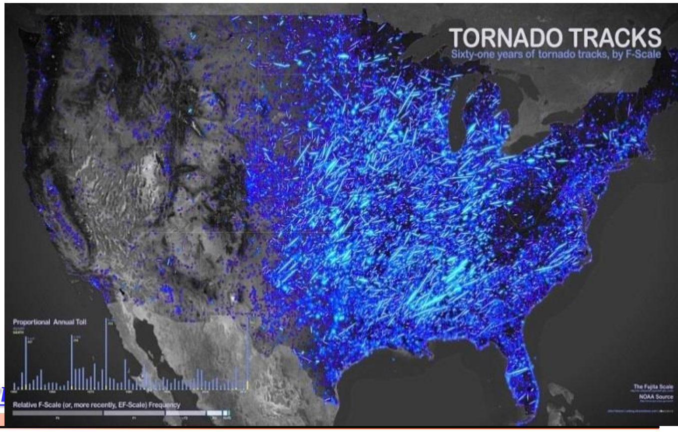
$$\Gamma = \iint_S \vec{\Omega}.\hat{n}dS = \iint_S (0)dS = const.$$

We can confirm Γ is a constant but can't obtain the constant this way.

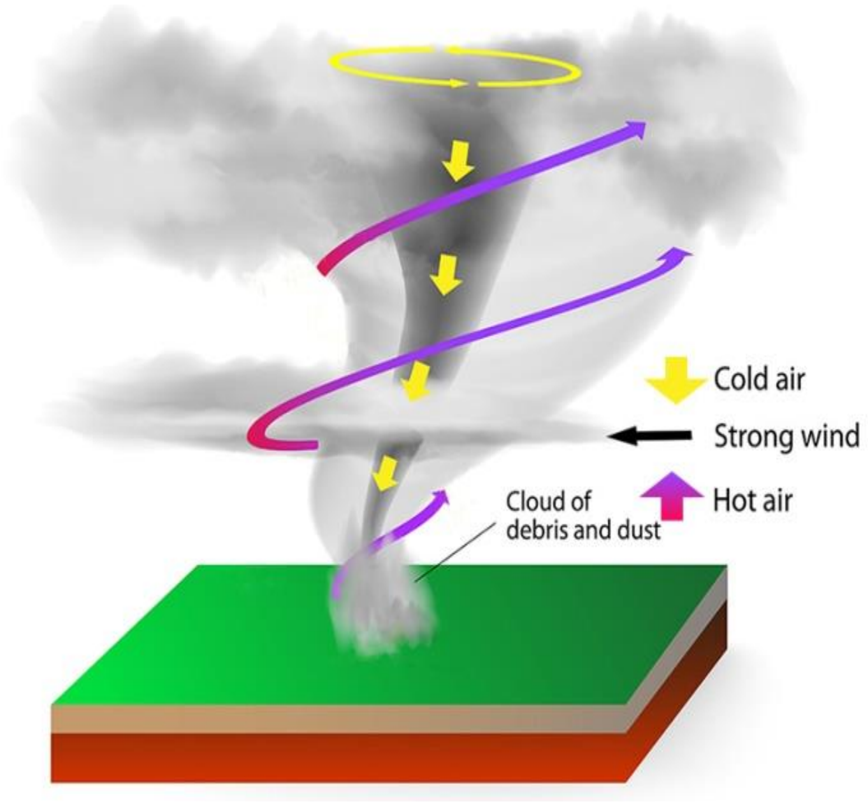
Tornadoes in USA

- *The most violent storms on earth.*
- *Diameter of 1 mile and travel up-to 50 miles. Wind speed ranges from 20 to 135 m/s.*
- *800 ~ 1000 tornados occur each year in the U.S.*
- *Annual averaged data of 1500 injuries and 80 deaths. ~ $1 billion worth of damage.*





Characteristics of Tornado-like Winds and Their Induced Wind Loadings Acting on Wind Turbines



- https://www.youtube.com/watch?v=aacHWoB7cmY

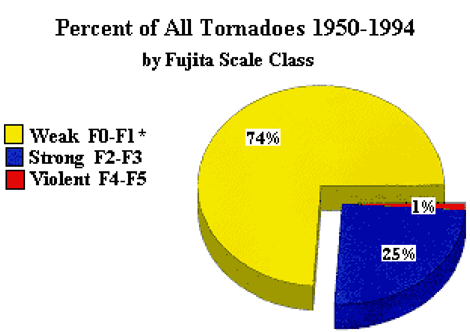
Introduction – Tornado Classifications

EF3 tornadoes struck Woodward, Iowa (Nov. 11, 2005)

EF5 tornado hits Parkersburg, Iowa on Sunday 05/27/2007 and killed seven.

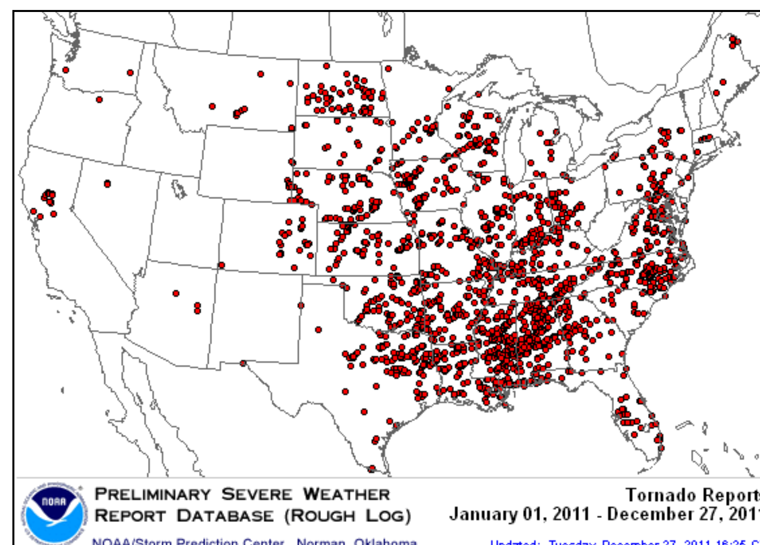
2016, April 27 tornado in Omaha

Fujita Scale



F-Scale	Intensity Phrase	Wind Speed
F0	Gale tornado	40-72 mph
F1	Moderate tornado	73-112 mph
F2	Significant tornado	113-157 mph
F3	Severe tornado	158-206 mph
F4	Devastating tornado	207-260 mph
F5	Incredible tornado	261-318 mph
F6	Inconceivable tornado	319-379 mph

- *Occurrence of tornadoes in 2011 alone*



Tornado-like Vortex and ISU Tornado Simulator

Swirl ratio: The ratio of angular to radial momentum

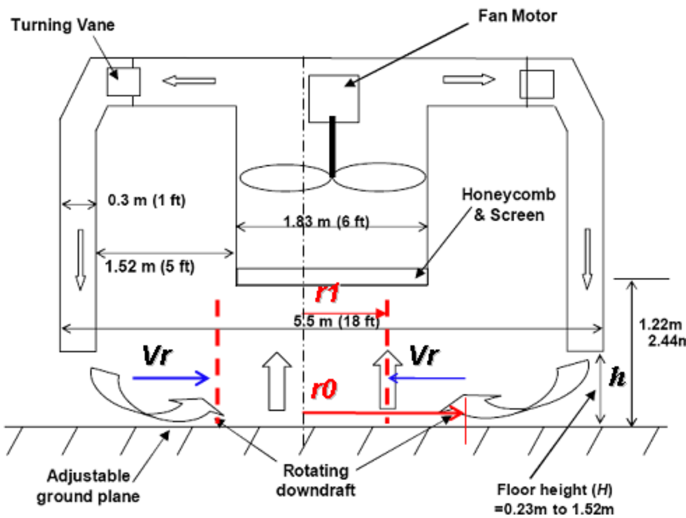
$$S = \frac{r_1 \Gamma}{2Q} = \frac{V_\theta}{2V_r a}$$

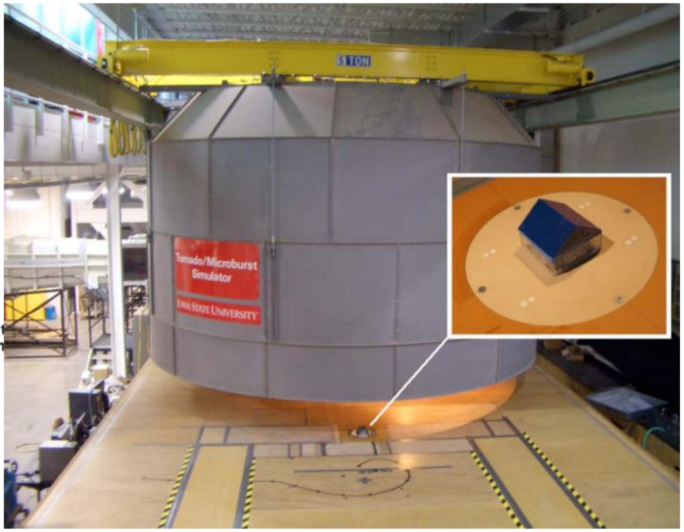
Γ- Circulation,
V_θ – Tangential velocity,
Vr – Radial velocity,
a - **Aspect ratio.**

$$a = \frac{h}{r_1}$$

h – Inflow depth,
r1 – Radius of the inflow domain

Core Radius, R_0: the distance where the maximum V_0 is found





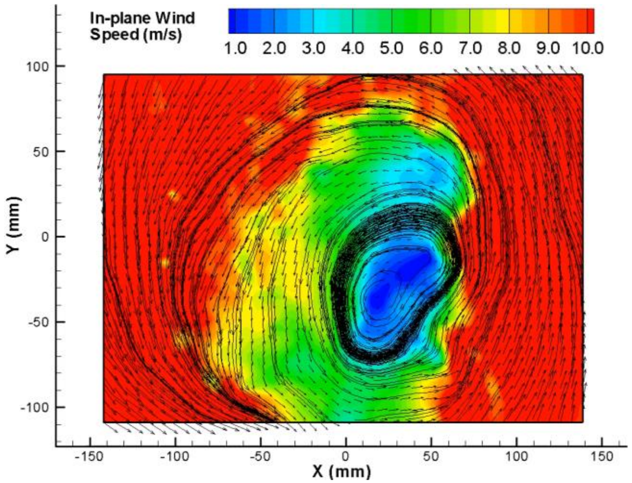
In the present study:

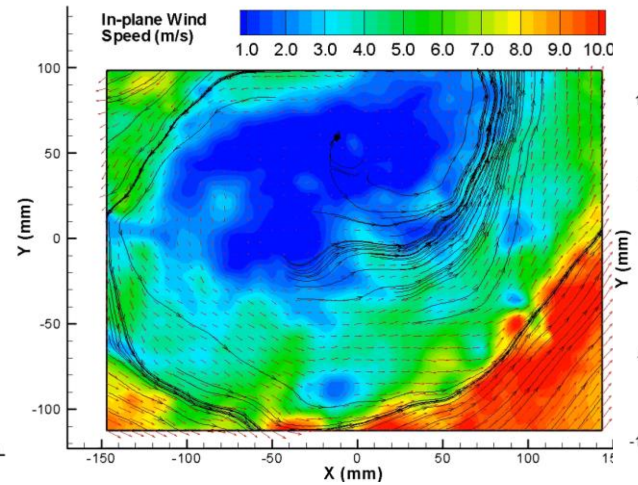
- V_0 = 15.0 m/s,
- D_0 = 0.50 m,
- S = 0.1
- a =1.6

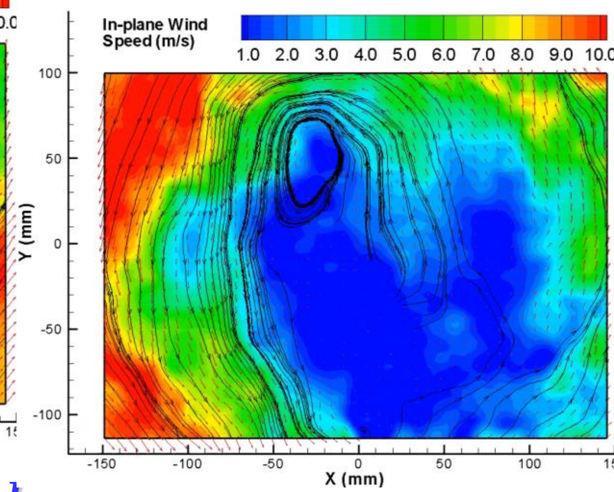
The world largest moving Tornado/Microburst Simulator
$D_{tornado}$ = 0. 5m ~ 2.0 m

Tornado-like Vortex at different Elevations

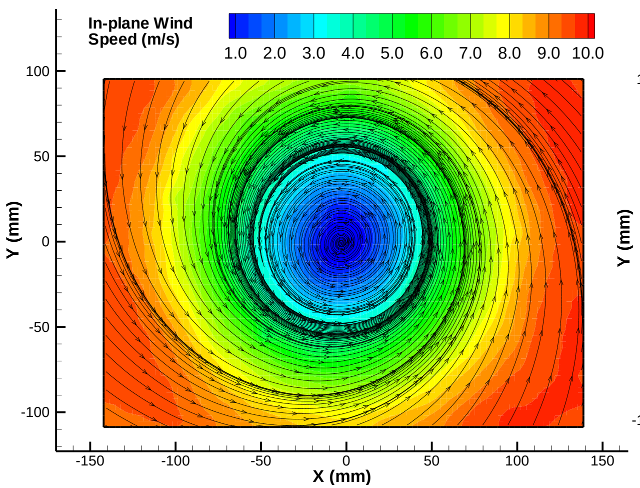
$Z/R_O \approx 0.10$ | $Z/R_O \approx 0.40$ | $Z/R_O \approx 0.70$

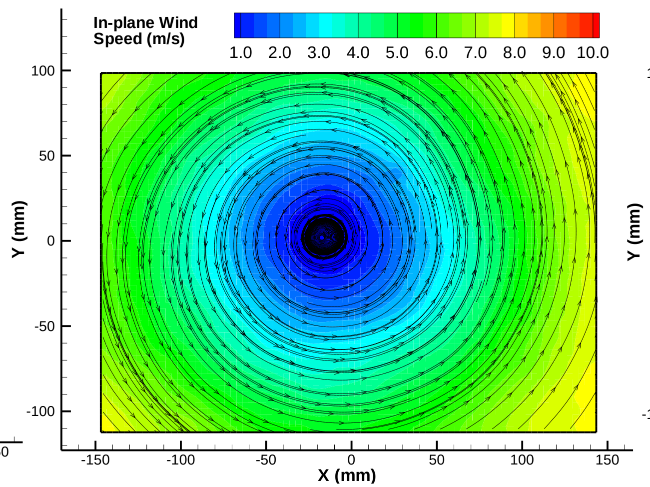


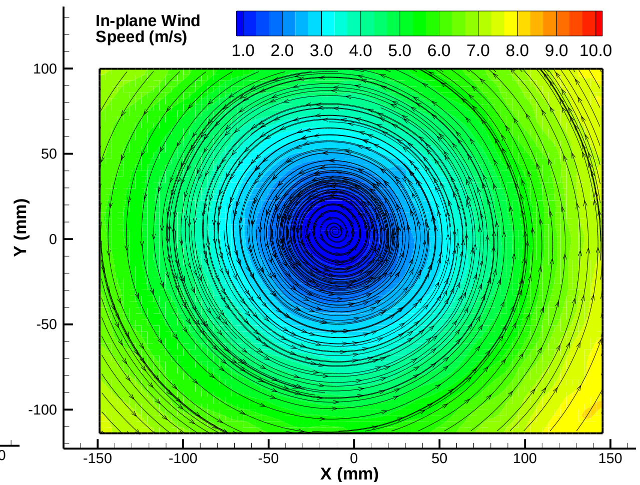




Instantaneous flow fields

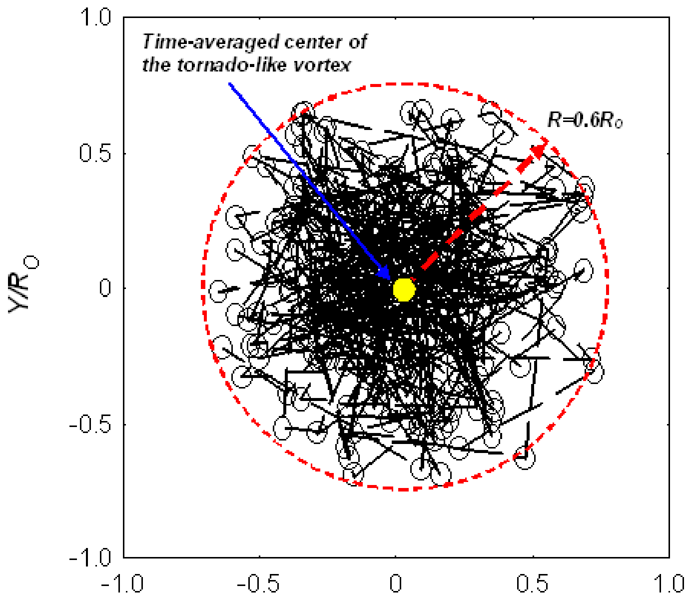


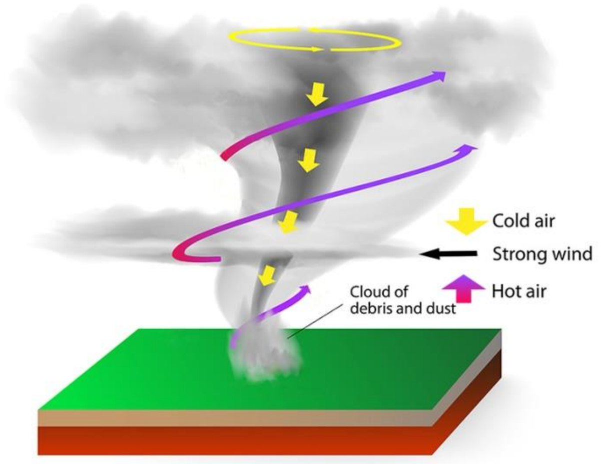


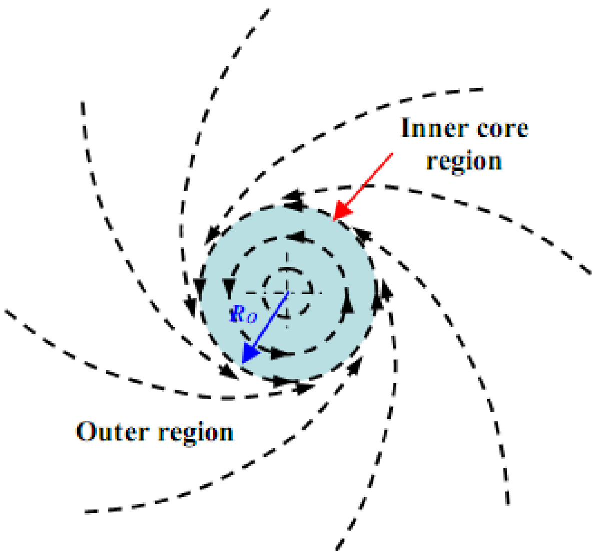


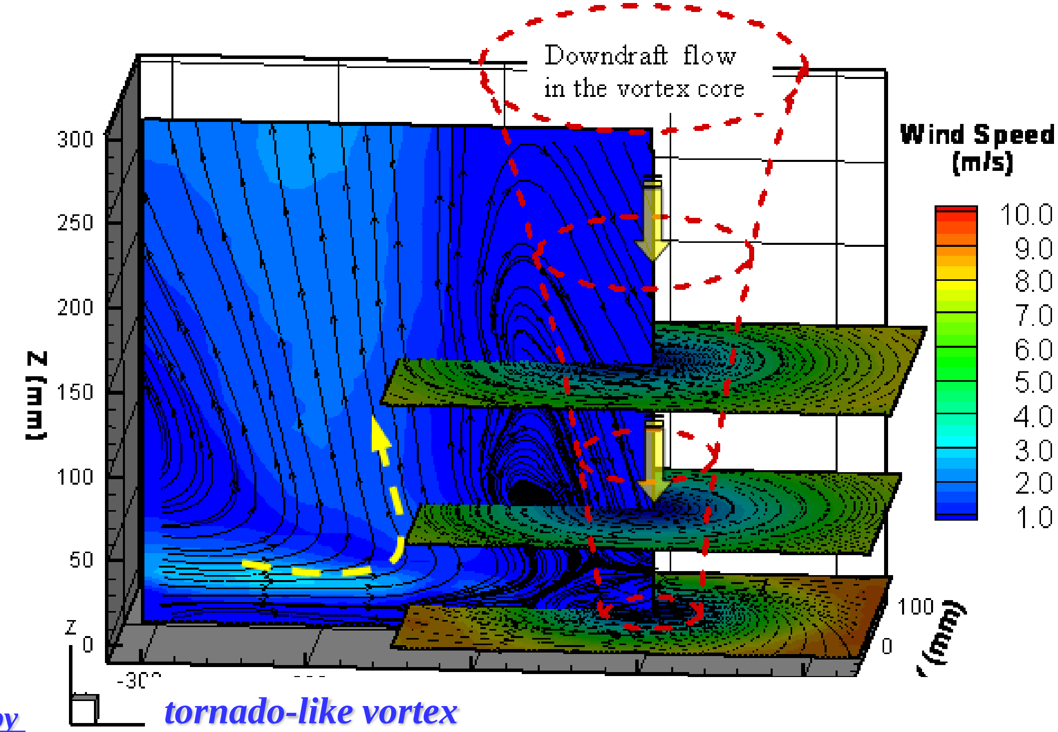
Time-averaged flow fields

Flow Characteristics of a Tornado-like Vortex









Planar view of a tornado-like vortex

© by

tornado-like vortex

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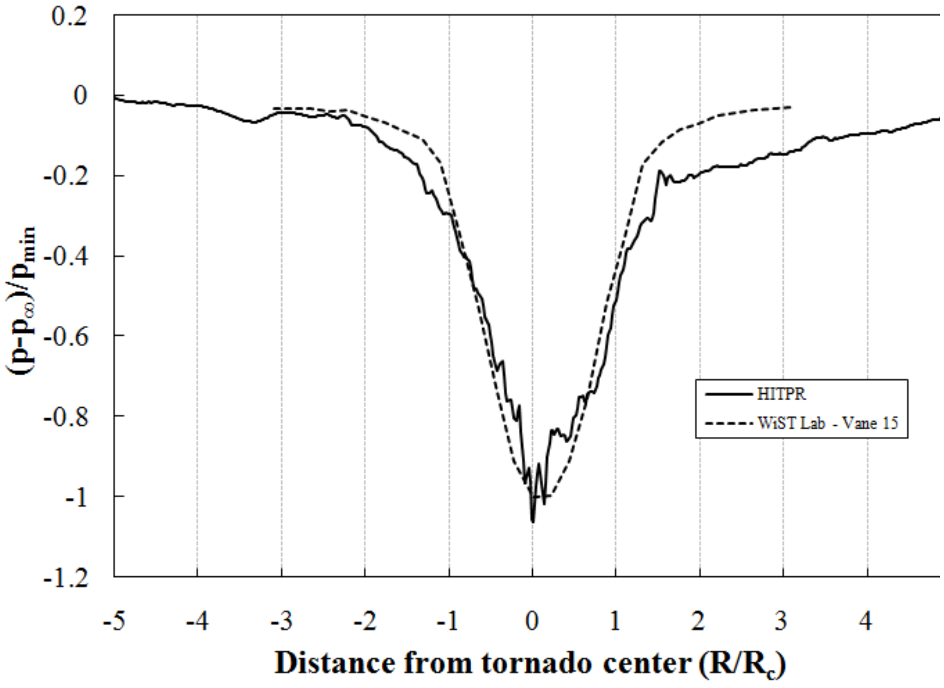
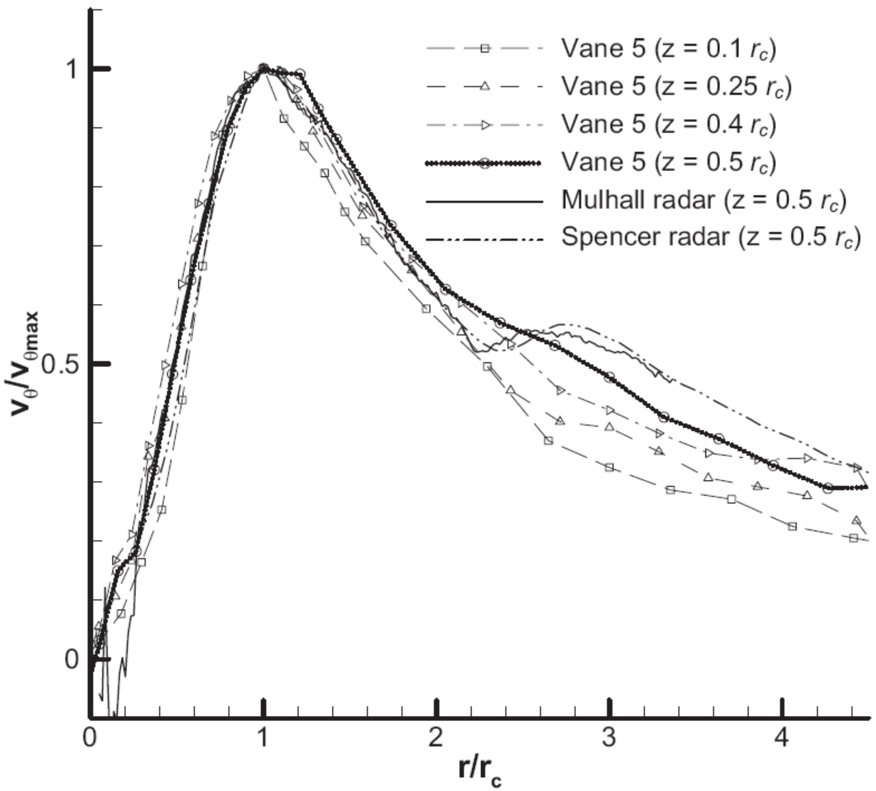
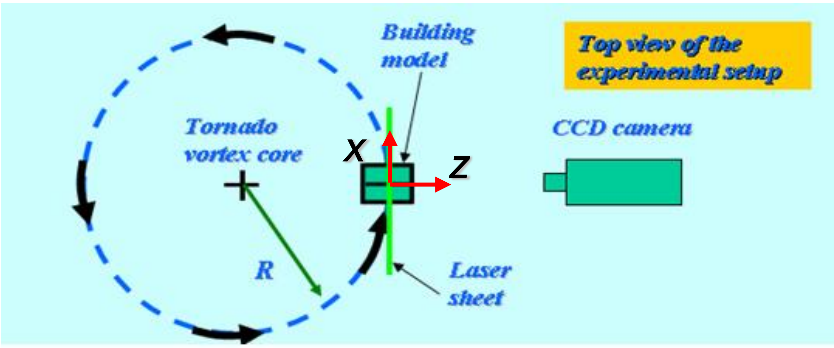
Tornado-like Vortex Generated in Lab vs. Tornadoes in Nature

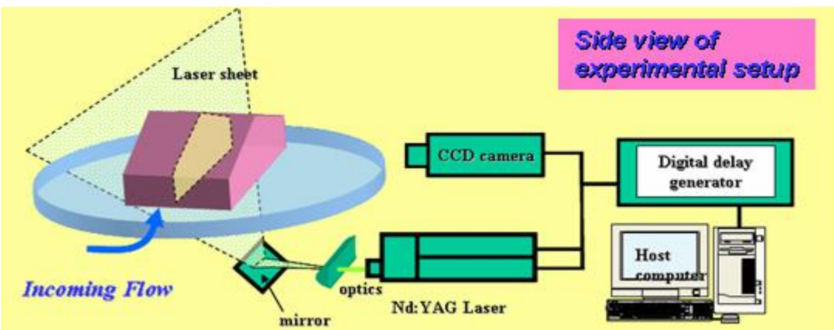


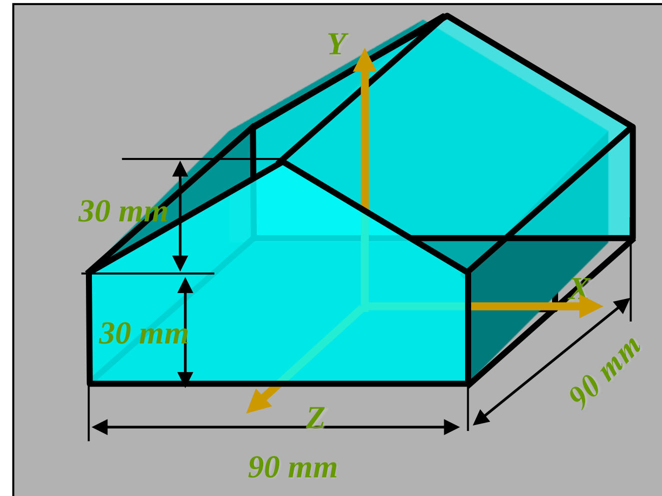


The tornado-like vortex vs. Mulhall and Spencer tornadoes found in nature

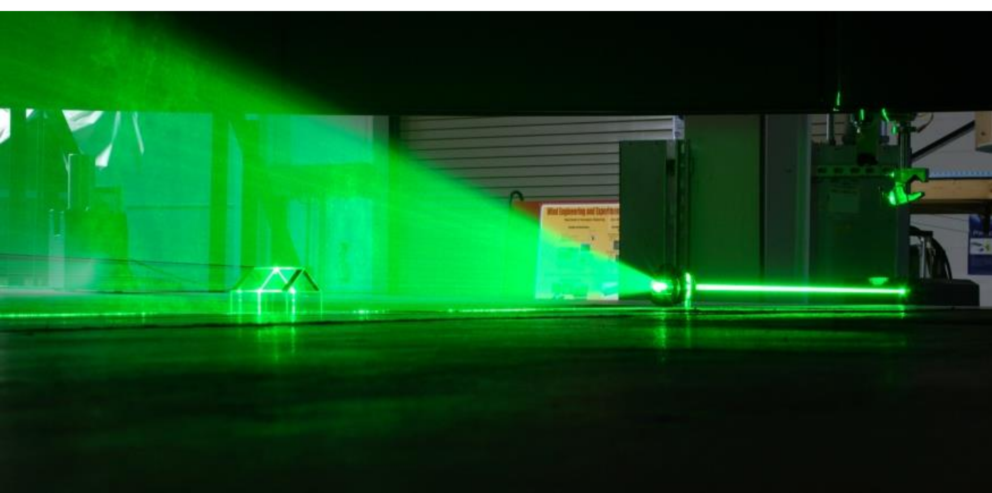
Experimental setup



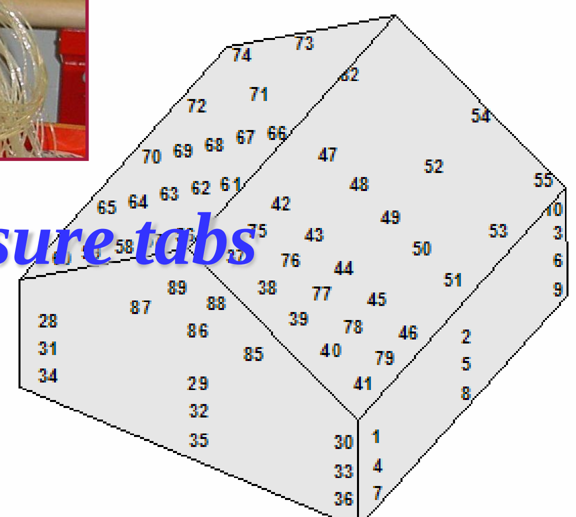




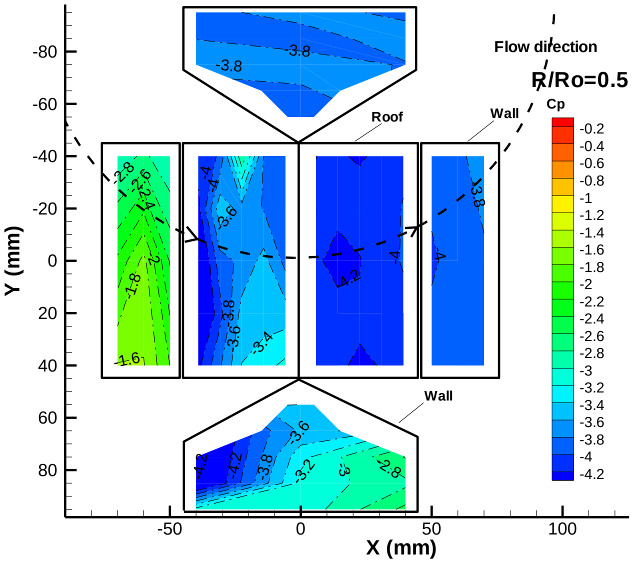
1:100 scaled, low-rise gable-roofed building model

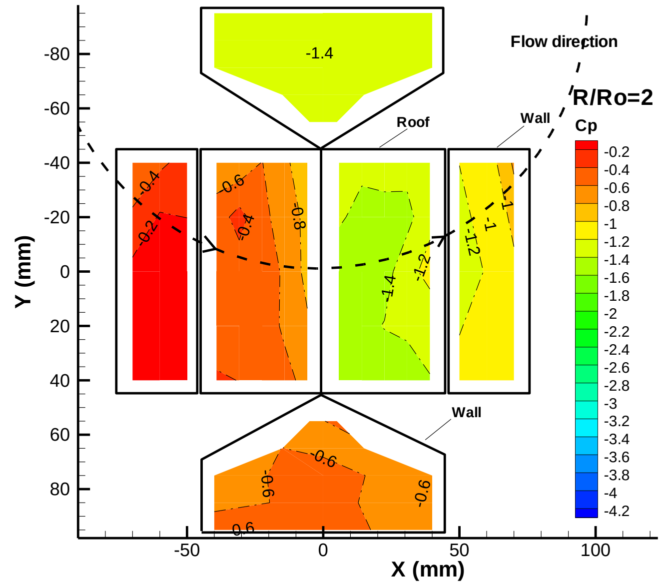


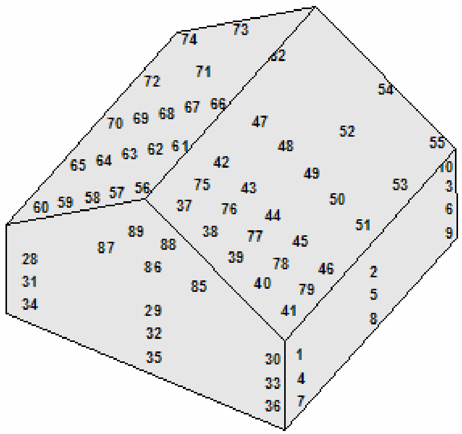
128 pressure tabs



Wind Load Measurement Results

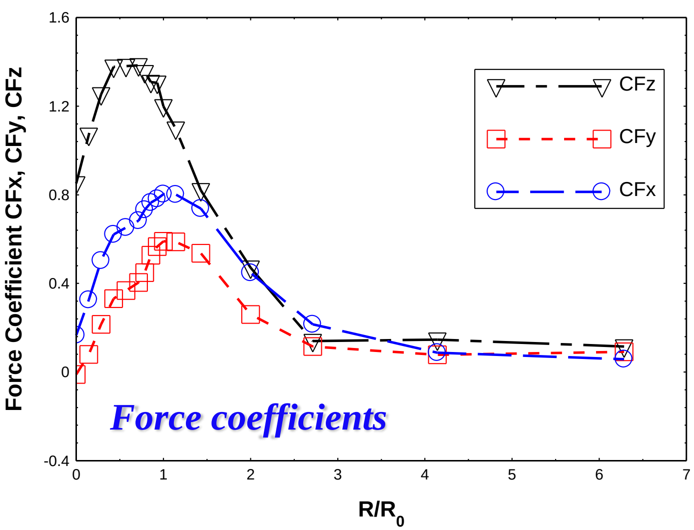


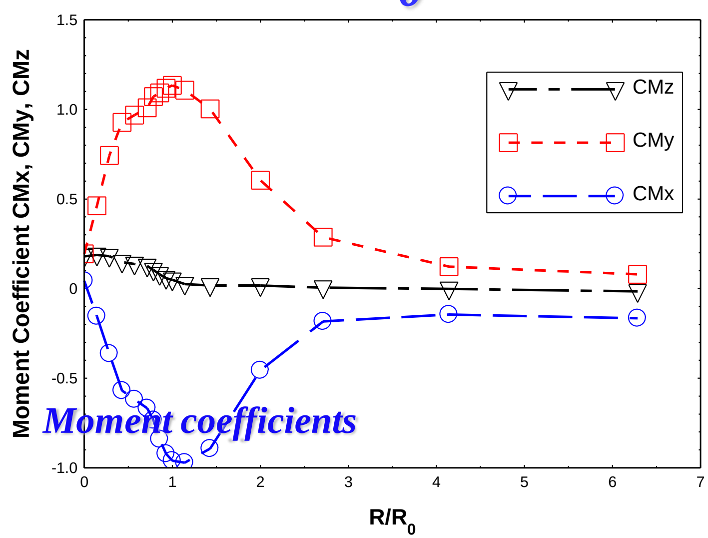


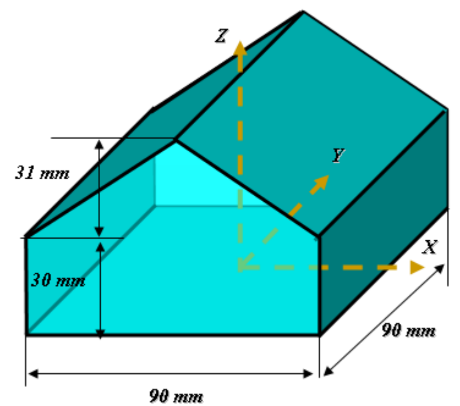


R/R_0=0.50

R/R_0=2.00



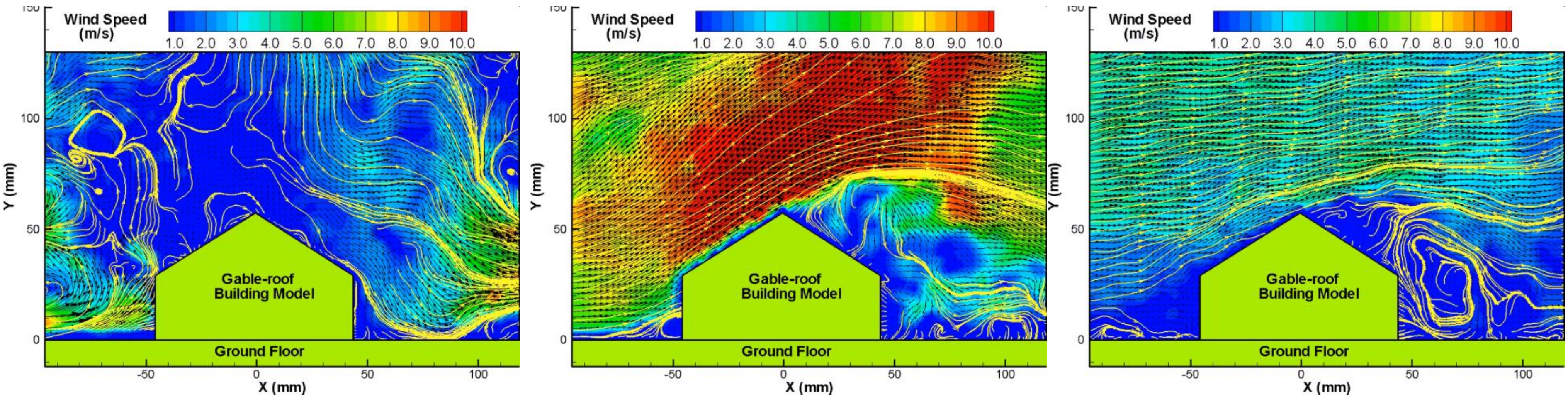




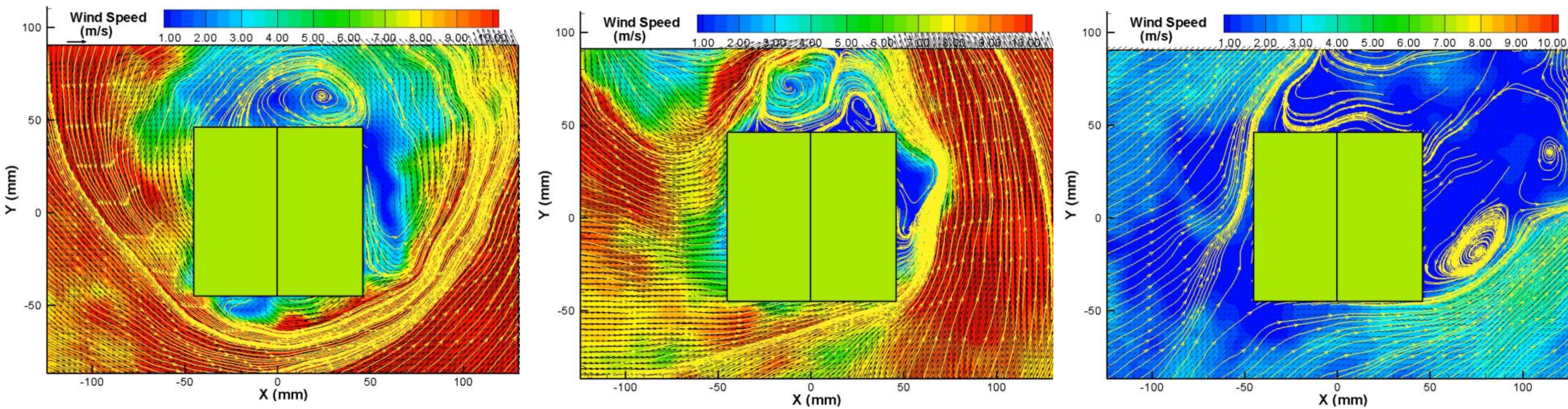
Aerospace Engineering

Flow Structures around a Low-Rise Building Model in Tornado-like Winds

$R/R_0 \approx 0.00$ $R/R_0 \approx 1.00$ $R/R_0 \approx 4.00$



Side view

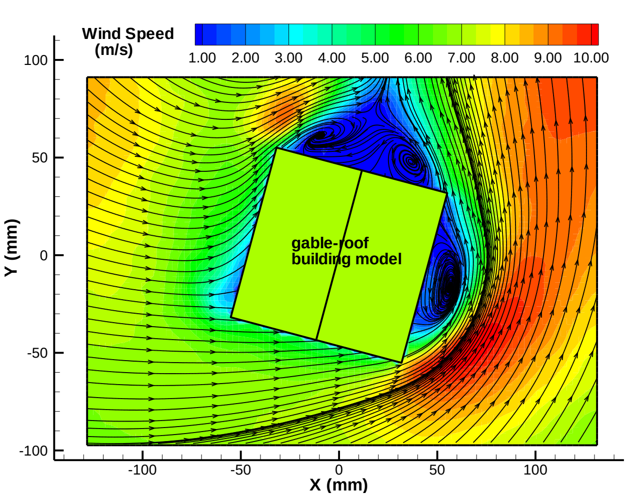


Top view

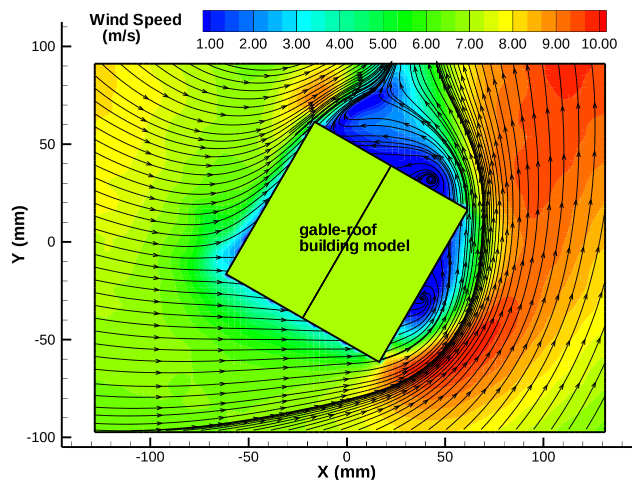
IOWA STATE UNIVERSITY

Aerospace Engineering

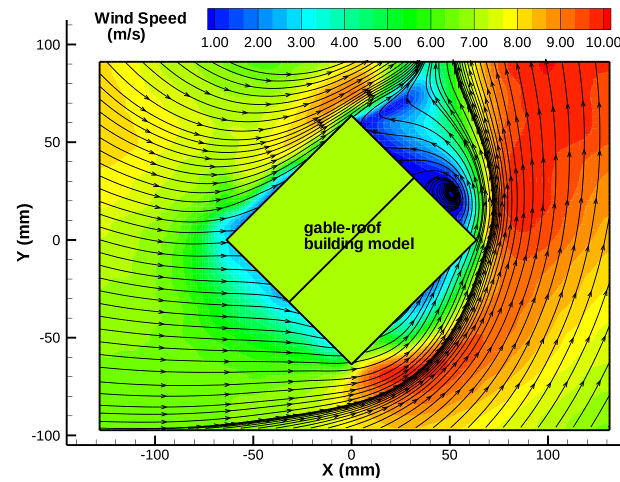
Effects of Orientation Angle on the flow structures



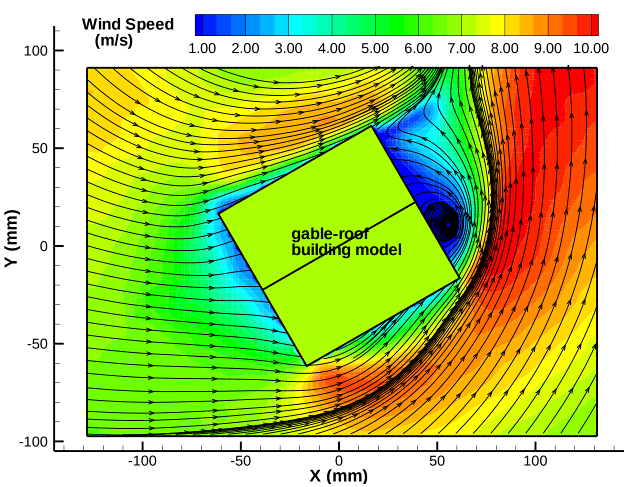
OA=15 deg



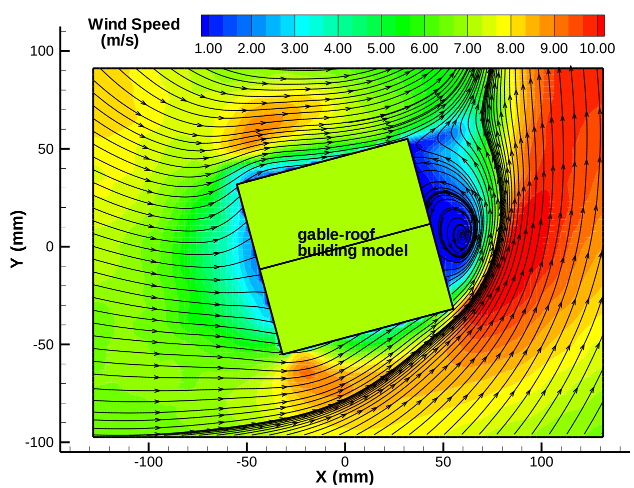
OA=30 deg



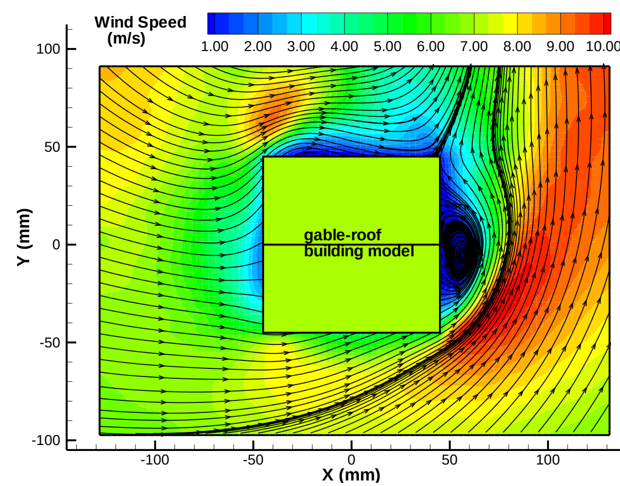
OA=45 deg



OA=60 deg

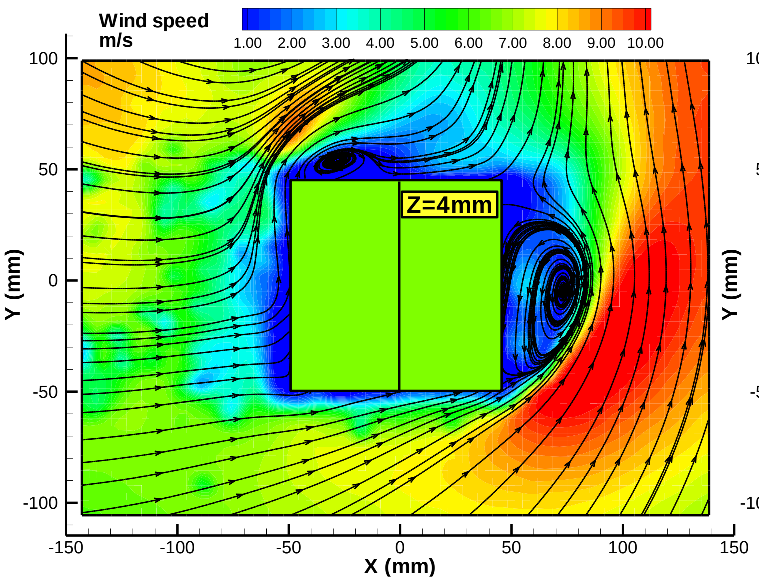


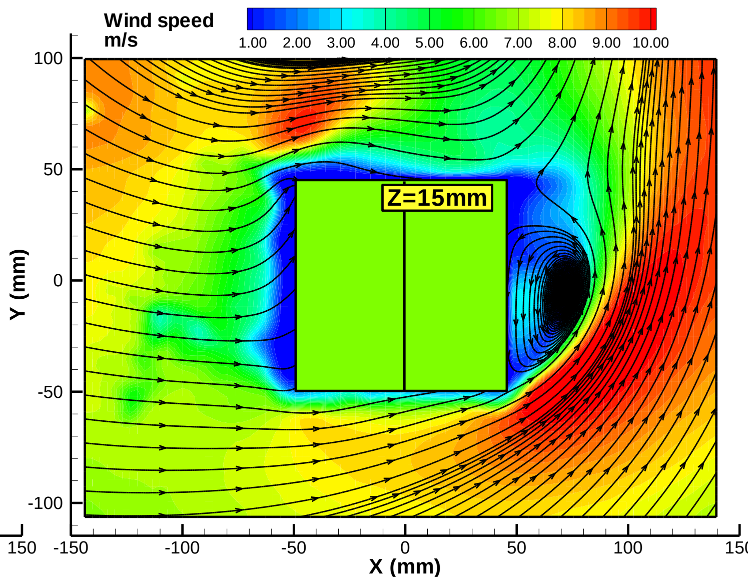
OA=75 deg

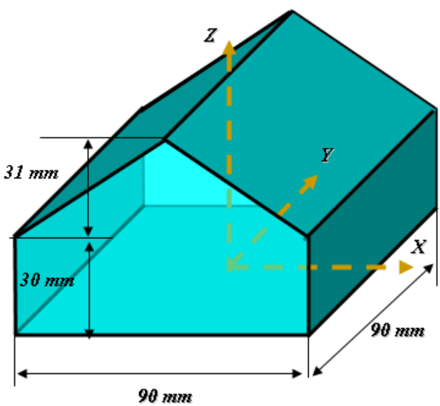


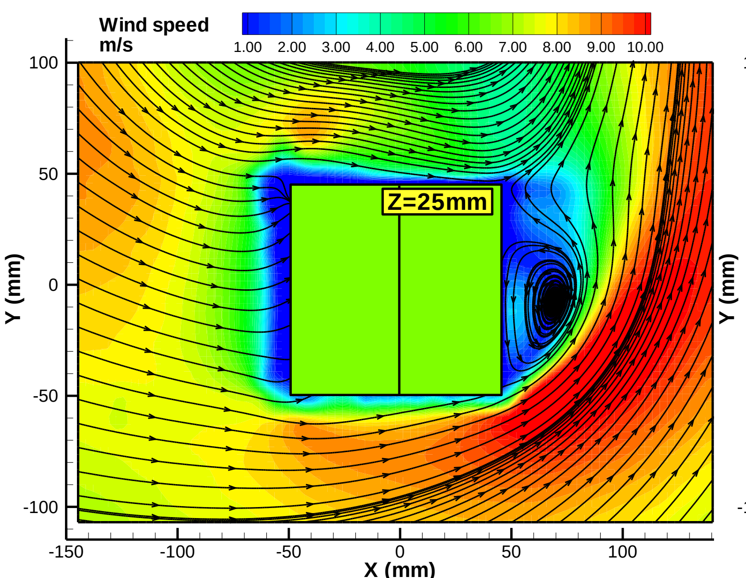
OA=90 deg

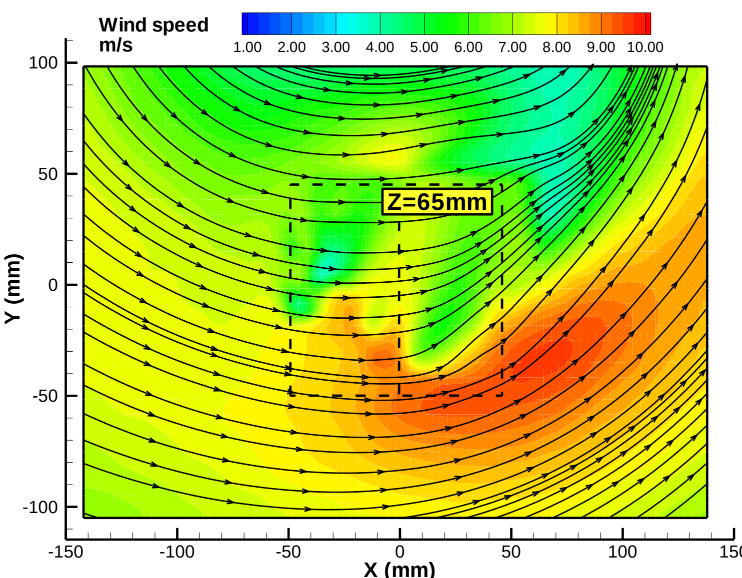
Flow field around the Building Model at different elevations



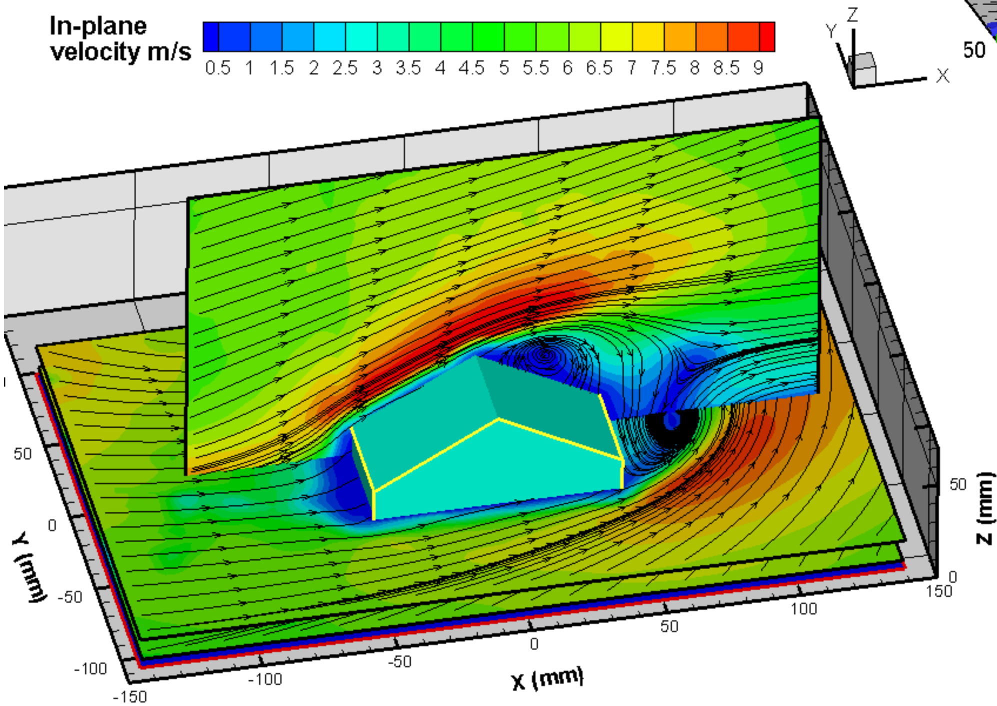


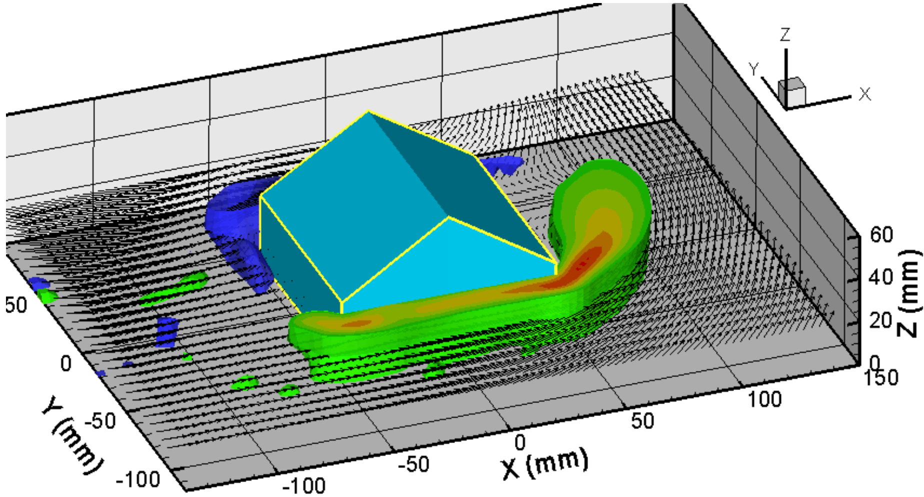


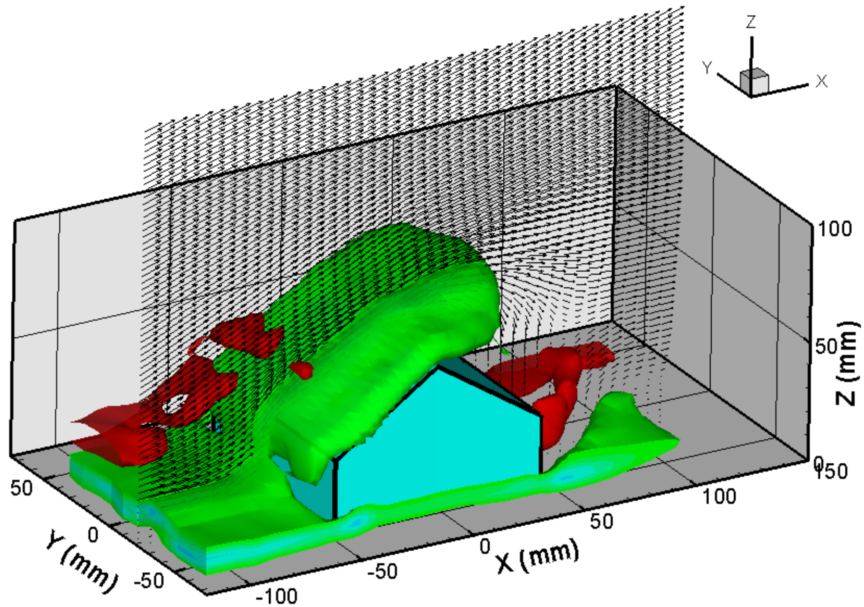




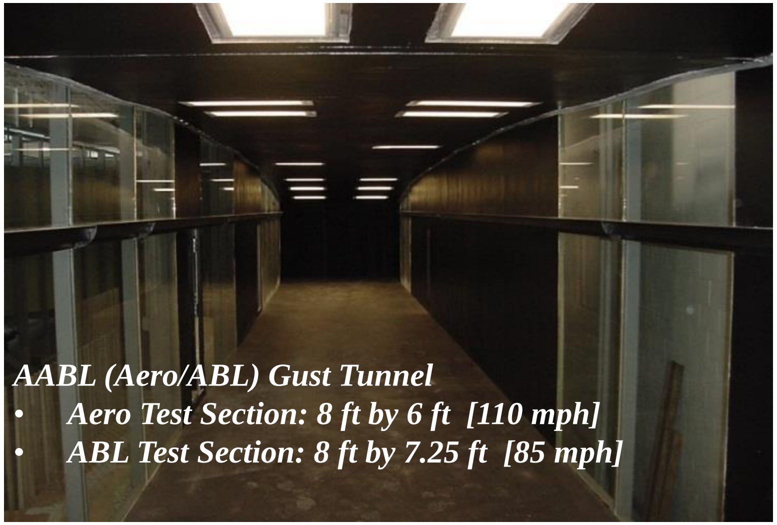
Reconstructed 3-D flow structures around the building model



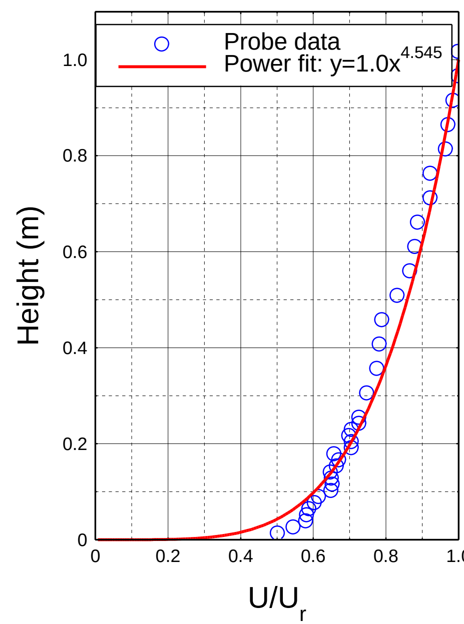




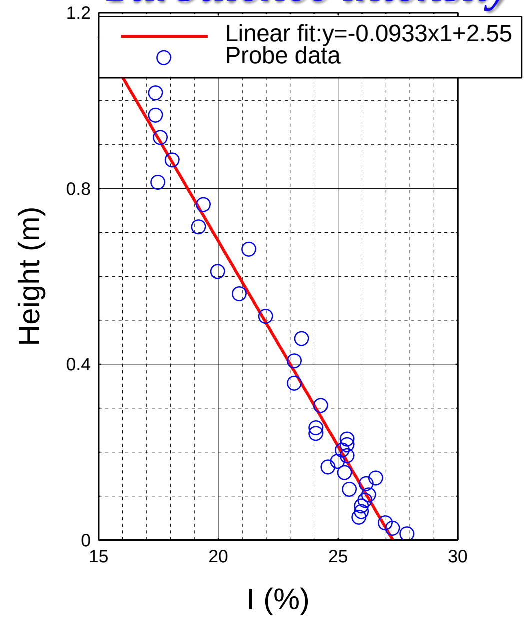
Gable-Roof building Model in Straight-line Winds



Velocity profile



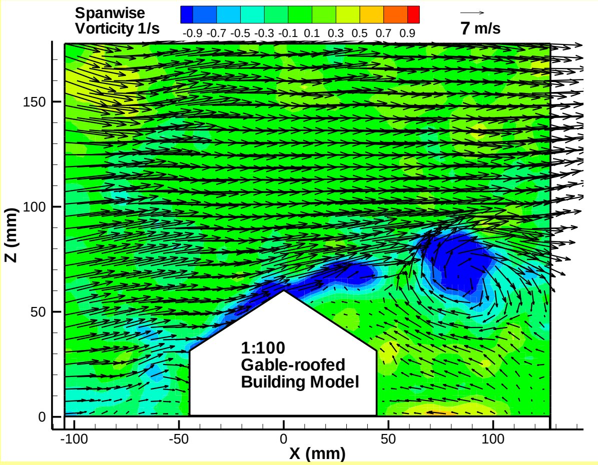
Turbulence intensity

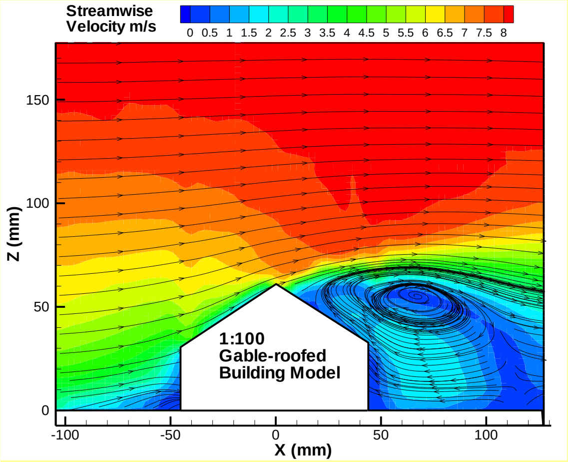


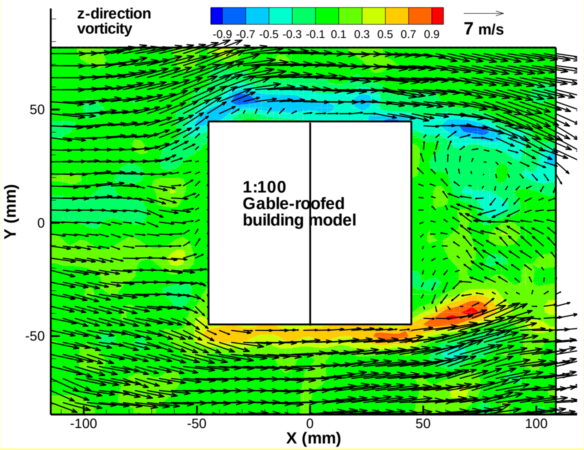
Aerodynamic/Atmospheric Boundary Layer (AABL) Straight-line wind tunnel

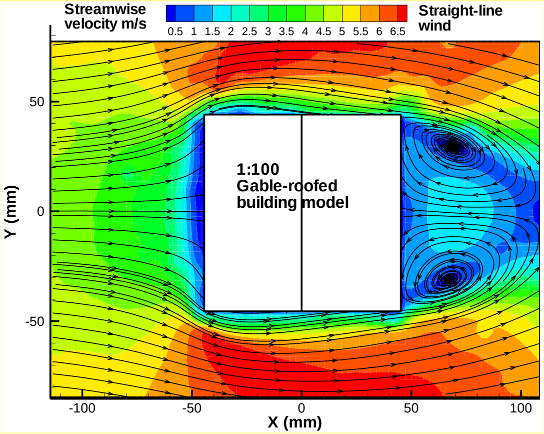
Gable Roof Building Model in Straight-line winds vs. Tornado-like Winds

Straight-line winds

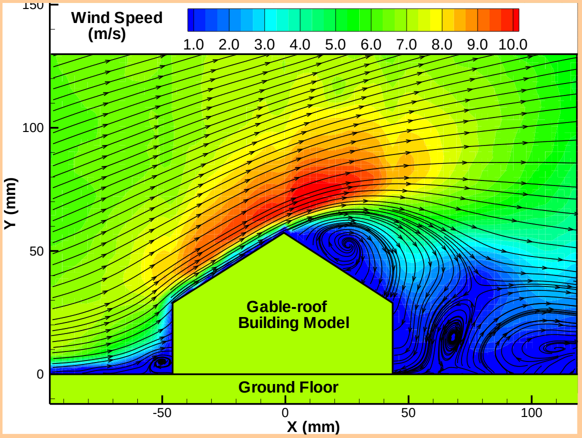


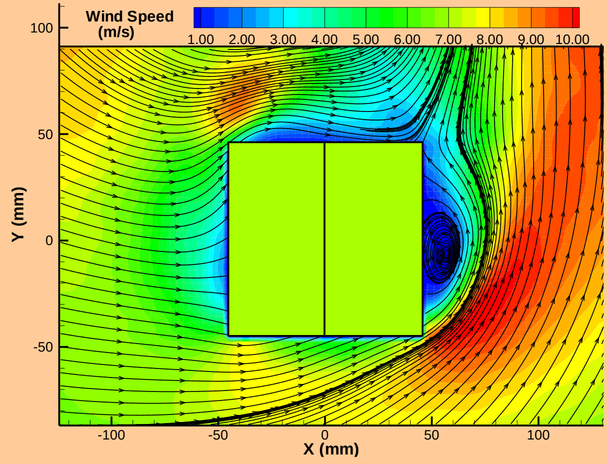




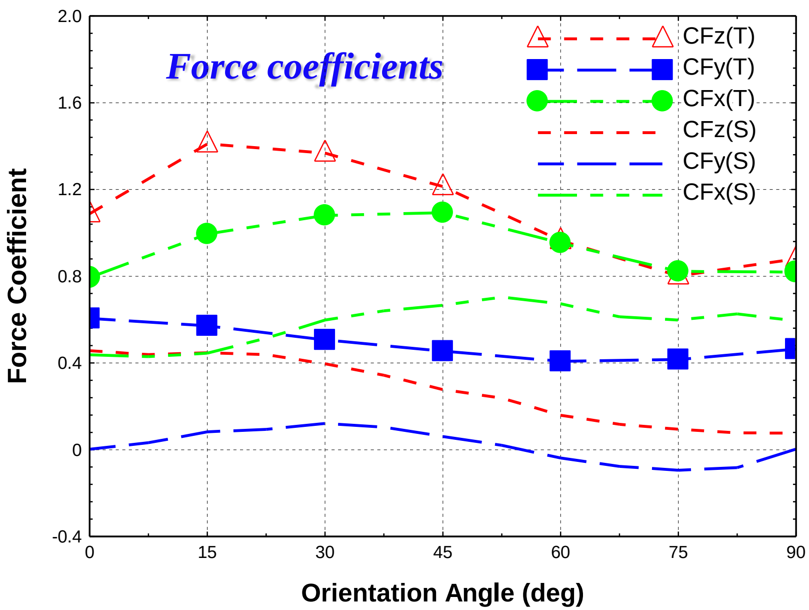


Tornado-like winds

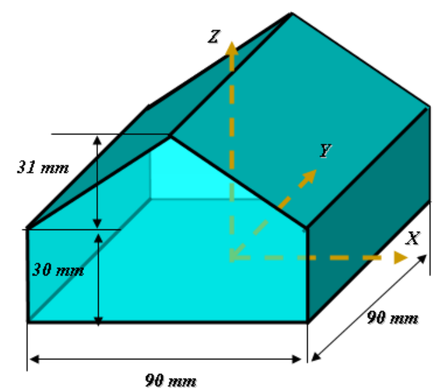


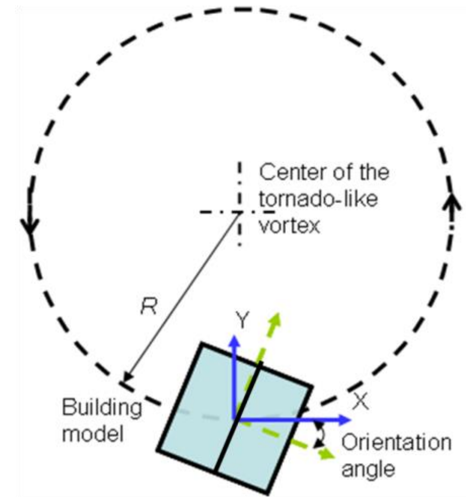


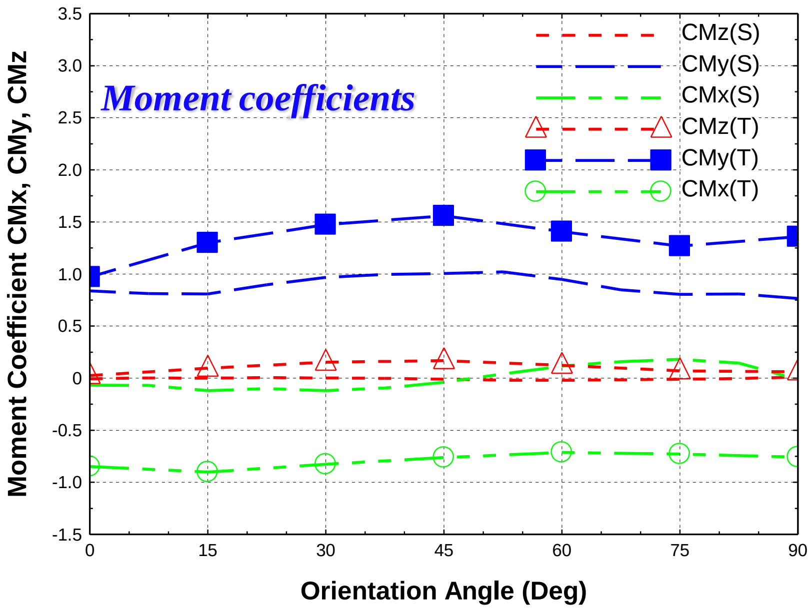
Effects of Orientation Angle on Wind Load



Straight-line wind







- ***At the same wind speed at the eave height of the building, tornado-induced wind loadings would be about 2~3 times higher than those of straight-like winds.***