Lecture # 14: BERNOULLI EQUATION

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❑ Ideal Flow

Ideal flow:

- ***Non-heat conducting, inviscid, incompressible, homogeneous fluid is defined as ideal fluid.***
- ***Assumptions used are:***
  - ***Non-heat conductive***
  - ***Homogeneous***
  - ***Incompressible***
  - ***Inviscid flow***

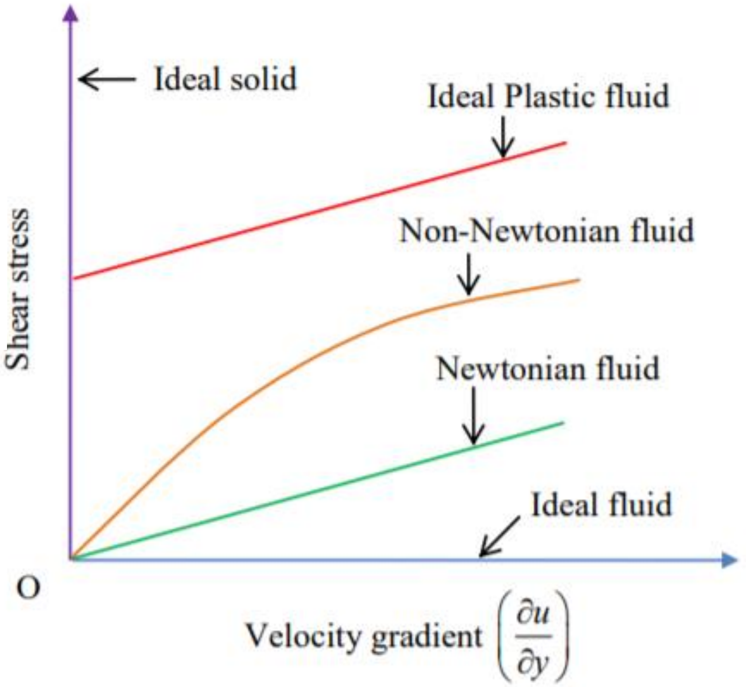
Controlling equations:

Continuity equation: $\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \vec{V}) = 0$

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \vec{V}) = 0 \Rightarrow \frac{\partial \rho}{\partial t} + \vec{V} \bullet \nabla \rho + \rho \nabla \bullet \vec{V} = 0$$

$$\Rightarrow \frac{D\rho}{Dt} + \rho \nabla \bullet \vec{V} = 0$$

$$\Rightarrow \nabla \bullet \vec{V} = 0$$



❑ Governing Equations for Ideal Fluid Flows

Momentum equation: $\frac{\partial(\rho\vec{V})}{\partial t}+\nabla\bullet(\rho\vec{V}\vec{V})+\nabla P-\nabla\bullet\tilde{\tau}-\rho\vec{f}=0$

$$\frac{\partial(\rho\vec{V})}{\partial t}+\nabla\bullet(\rho\vec{V}\vec{V})+\nabla P-\nabla\bullet\tilde{\tau}-\rho\vec{f}=0$$

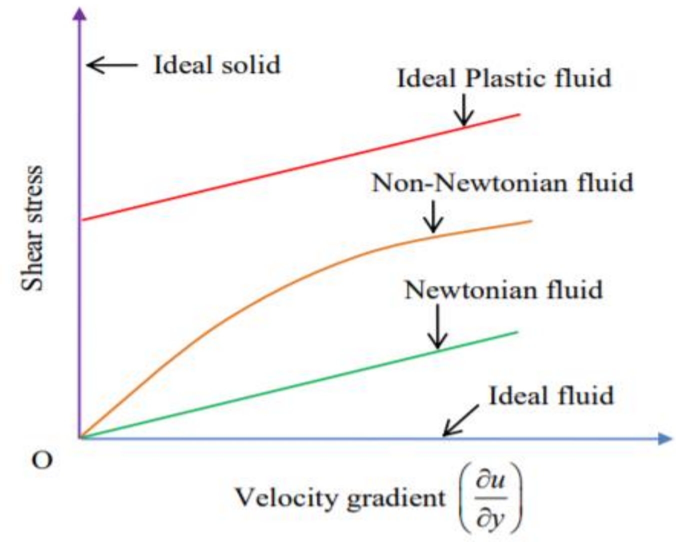
$$\Rightarrow \quad \vec{V}\frac{\partial\rho}{\partial t}+\rho\frac{\partial\vec{V}}{\partial t}+\vec{V}\nabla\bullet(\rho\vec{V})+(\rho\vec{V}\bullet\nabla)\vec{V}+\nabla P-\overbrace{\nabla\bullet\tilde{\tau}}^{=0\ due\,to\,inviscid}-\rho\vec{f}=0$$

$$\Rightarrow \quad \vec{V}[\frac{\partial\rho}{\partial t}+\nabla\bullet(\rho\vec{V})]+\rho[\frac{\partial\vec{V}}{\partial t}+(\vec{V}\bullet\nabla)\vec{V}]+\nabla P-\overbrace{\nabla\bullet\tilde{\tau}}^{=0\ due\,to\,inviscid}-\rho\vec{f}=0$$

$$\Rightarrow \quad \rho\frac{D\vec{V}}{Dt}=-\nabla P+\rho\vec{f}$$

$$\rho\frac{D\vec{V}}{Dt}=-\nabla P+\rho\vec{f}$$

is also called Euler equation



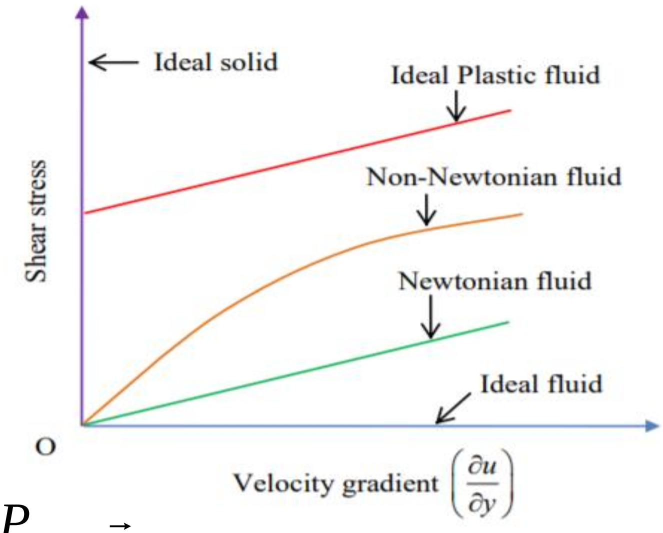
❑ Governing Equations for Ideal Fluid Flows

For ideal fluid, the governing equations are:

1). Continuity equation: $\nabla \bullet \vec{V} = 0$

2). Euler equation $\rho \dfrac{D\vec{V}}{Dt} = -\nabla P + \rho \vec{f}$

or $\dfrac{D\vec{V}}{Dt} = -\nabla(\dfrac{P}{\rho}) + \vec{f}$



- ***For the Euler equation, it can also be re-written as:***

$$\frac{D\vec{V}}{Dt} = -\nabla(\frac{P}{\rho}) + \vec{f} \quad \Rightarrow \quad \frac{\partial \vec{V}}{\partial t} + (\vec{V} \bullet \nabla)\vec{V} = -\nabla(\frac{P}{\rho}) + \vec{f}$$

$$Since \quad (\vec{V} \bullet \nabla)\vec{V} = \nabla(\frac{\vec{V} \bullet \vec{V}}{2}) - \vec{V} \times (\nabla \times \vec{V}) \quad \Rightarrow \quad \frac{\partial \vec{V}}{\partial t} + \nabla(\frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho}) - \vec{V} \times (\nabla \times \vec{V}) = \vec{f}$$

- ***Let us only consider body forces that are conservative only. A necessary and sufficient condition for the body force can be represented as the gradient of a scalar field U, i.e.,***

$$\vec{f} = \nabla U$$

- ***Therefore:***

$$\frac{\partial \vec{V}}{\partial t} + \nabla(\frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho}) - \vec{V} \times (\nabla \times \vec{V}) = \nabla U \quad \Rightarrow \quad \frac{\partial \vec{V}}{\partial t} + \nabla(\frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho} - U) - \vec{V} \times (\nabla \times \vec{V}) = 0$$

Integral of Euler Equation in Irrotational Flows

$$\frac{\partial \vec{V}}{\partial t}+\nabla(\frac{\vec{V}\bullet\vec{V}}{2}+\frac{P}{\rho})-\vec{V}\times(\nabla\times\vec{V})=\nabla U \quad \Rightarrow \quad \frac{\partial \vec{V}}{\partial t}+\nabla(\frac{\vec{V}\bullet\vec{V}}{2}+\frac{P}{\rho}-U)-\vec{V}\times(\nabla\times\vec{V})=0$$

- ***If the flow is irrotational, then*** $\nabla\times\vec{V}=0$

$$\vec{f}=\nabla U$$

- ***The above equation can be simplified as:*** $\frac{\partial \vec{V}}{\partial t}+\nabla(\frac{\vec{V}\bullet\vec{V}}{2}+\frac{P}{\rho}-U)=0$

- ***If the flow is steady:*** $\Rightarrow \quad \nabla(\frac{\vec{V}\bullet\vec{V}}{2}+\frac{P}{\rho}-U)=0 \qquad \vec{f}=\nabla U$

- ***By the definition of directional directive along any arbitrary path :*** $\nabla()\bullet d\vec{l}=d()$

- ***Therefore:*** $\nabla(\frac{\vec{V}\bullet\vec{V}}{2}+\frac{P}{\rho}-U)\bullet d\vec{l}=d(\frac{\vec{V}\bullet\vec{V}}{2}+\frac{P}{\rho}-U)=0$

- ***Integrating the above equation leads to the unsteady Bernoulli equation:***

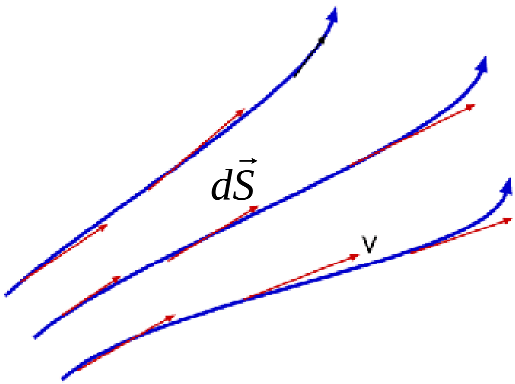
$$\int d(\frac{\vec{V}\bullet\vec{V}}{2}+\frac{P}{\rho}-U)=0 \quad \Rightarrow \quad \frac{\vec{V}\bullet\vec{V}}{2}+\frac{P}{\rho}-U=const.$$

❑ Integral of Euler Equation along a Streamline

$$\frac{\partial \vec{V}}{\partial t} + \nabla\left(\frac{\vec{V}\bullet\vec{V}}{2} + \frac{P}{\rho}\right) - \vec{V}\times(\nabla\times\vec{V}) = \nabla U \quad \Rightarrow \quad \frac{\partial \vec{V}}{\partial t} + \nabla\left(\frac{\vec{V}\bullet\vec{V}}{2} + \frac{P}{\rho} - U\right) - \vec{V}\times(\nabla\times\vec{V}) = 0$$

- ***For the steady flow:*** $\frac{\partial \vec{V}}{\partial t} = 0$

- ***Then:*** $\nabla\left(\frac{\vec{V}\bullet\vec{V}}{2} + \frac{P}{\rho} - U\right) = \vec{V}\times(\nabla\times\vec{V})$



- ***Multiple (dot product) both sides of the above equation by,*** $d\vec{S}$ ***which is an element length alone a streamline.***

- ***Therefore:*** $\nabla\left(\frac{\vec{V}\bullet\vec{V}}{2} + \frac{P}{\rho} - U\right)\bullet d\vec{S} = \vec{V}\times(\nabla\times\vec{V})\bullet d\vec{S}$

- ***Since:*** $\vec{V}\times(\nabla\times\vec{V})\bullet d\vec{S} = d\vec{S}\bullet[\vec{V}\times(\nabla\times\vec{V})] = (\nabla\times\vec{V})\bullet(d\vec{S}\times\vec{V}) = 0$

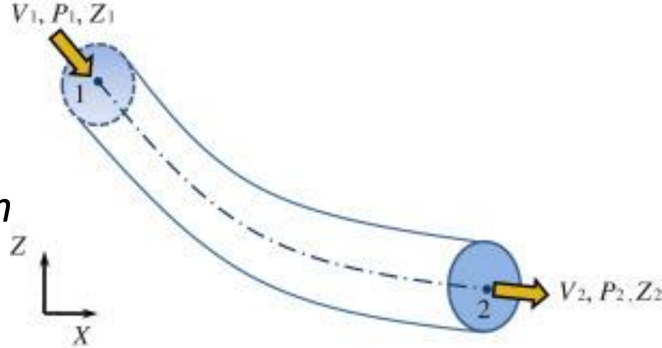
- ***Therefore:*** $\nabla\left(\frac{\vec{V}\bullet\vec{V}}{2} + \frac{P}{\rho} - U\right)\bullet d\vec{S} = 0 \quad \Rightarrow \quad d\left(\frac{\vec{V}\bullet\vec{V}}{2} + \frac{P}{\rho} - U\right) = 0$

- ***Bernoulli equation along a streamline:*** $\frac{\vec{V}\bullet\vec{V}}{2} + \frac{P}{\rho} - U = const\ C$

❑ Integral of Euler Equation in Irrotational Flows

- *If we choose a Cartesian coordinate system with the Z-axial to be positive when pointing upward and normal to the surface of the earth,. The force on a body of mass "m" is given by (0,0,-mg). Thus, the body force vector per unit mass becomes:* $\vec{f} = (f_x, f_y, f_z) = (0, 0, -g)$

$$\frac{\partial U}{\partial Z} = -g \qquad \Rightarrow \qquad U = -g\,Z$$



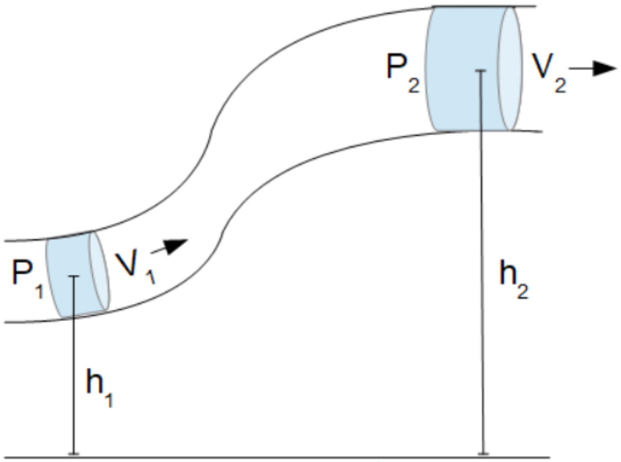
  - *Therefore, **Bernoulli equation** for irrotational flow becom*

$$\frac{\vec{V}\bullet\vec{V}}{2} + \frac{P}{\rho} - U = Const.$$

$$\Rightarrow \quad \frac{\vec{V}\bullet\vec{V}}{2} + \frac{P}{\rho} + gz = Const.$$

- ***Stagnation point:***
  - ***the point on a streamline with the flow velocity becoming zero.***

- ***Total pressure:***
  - ***The pressure measured at the stagnation point.***

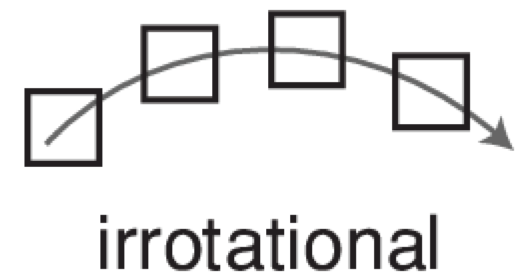


Bernoulli's Equation

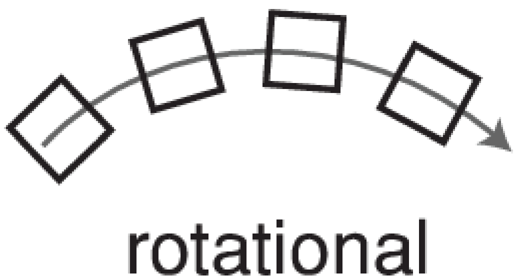
❑ BERNOULLI EQUATION

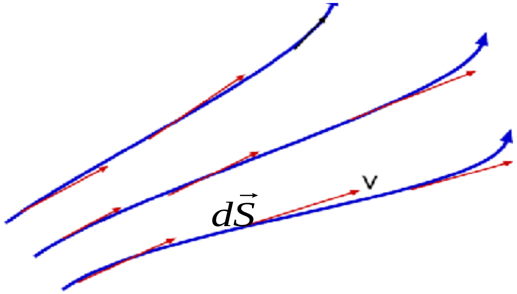
- ***Bernoulli equation:***

$$\frac{\vec{V}\bullet\vec{V}}{2}+\frac{P}{\rho}+gz=Const.$$



- ***Applicable everywhere in irrotational flows***





- ***Along the streamline in rotational flows***

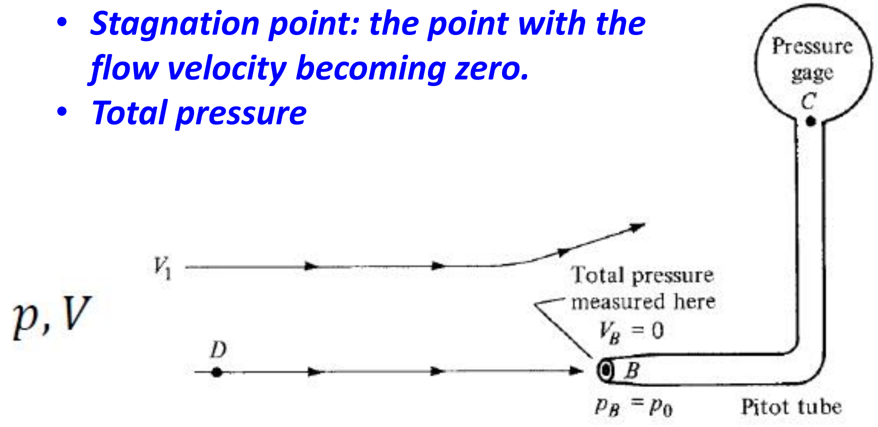
- ***We can drive Bernoulli's equation for an irrotational flow without having to limit the equation to a streamline.***
- ***If flow is irrotational and steady (above case), Bernoulli's equation applies, and the constant is the same everywhere in the flow.***
- ***If flow is rotational but steady, Bernoulli's equation holds along a streamline. The value of constant is different for each streamline.***
- ***If flow is irrotational and unsteady, a modified version of Bernoulli's equation applies, and the constant is the same throughout the flow (We don't cover this in AERE 310).***
- ***If flow is rotational and unsteady, Bernoulli's equation is no longer valid (Full Euler's equation must be solved).***

❑ BERNOULLI EQUATION FOR IRROTATIONAL FLOW

- The most common instrument for measuring airspeed in airplanes.

Consider a flow with pressure p and velocity V approaching a pitot tube:

- ***Stagnation point: the point with the flow velocity becoming zero.***
- ***Total pressure***



Bernoulli's equation between point D and B

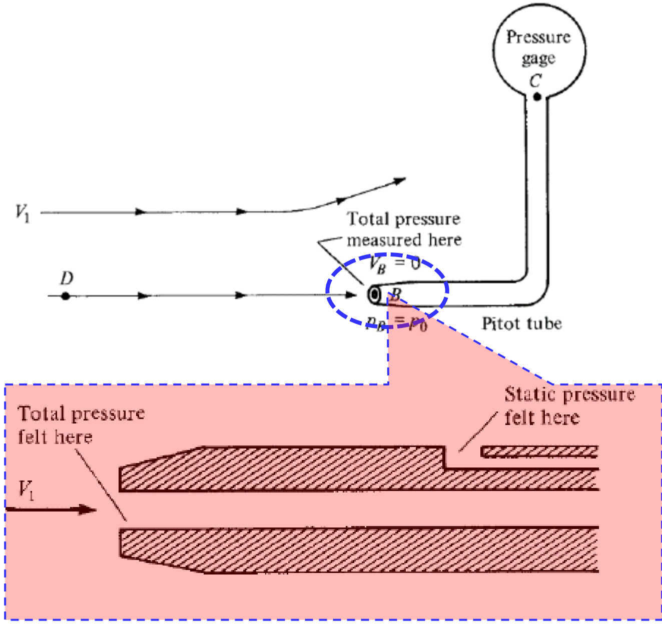
❑ Bernoulli equation for irrotational flow

Pitot tube

- Bernoulli's equation between point D and B

$$p + \rho\frac{V^2}{2} = p_B + \rho\frac{V_B^2}{2}$$

p, V

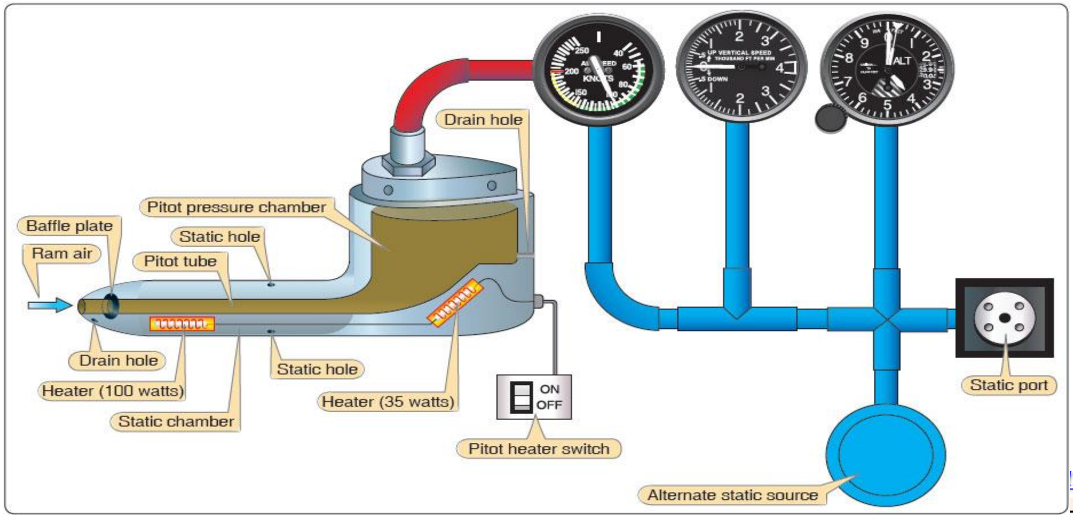


But $V_B = 0$ and by definition $p_B = p_0$ or stagnation pressure

Therefore:

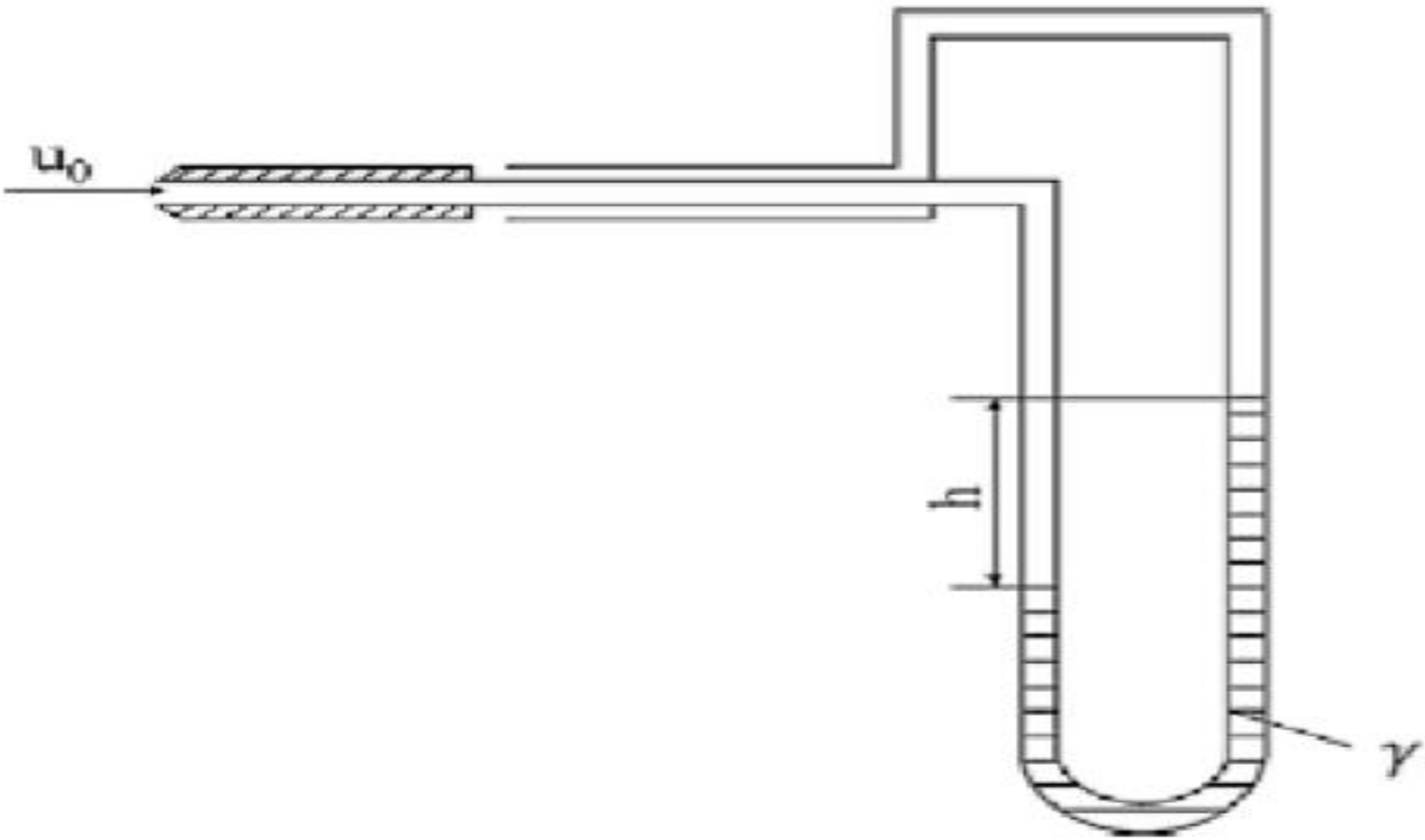
$$V = \sqrt{\frac{2(p_0 - p)}{\rho}}$$

- By measuring the difference between total (stagnation) and static pressure, we can calculate the flow velocity.
- Often a Pitot-static probe is used to measure both pressures



❑ Bernoulli equation for irrotational flow

A Pitot tube connected to a water manometer is used to measure the air velocity in the pipe, as shown in Fig. 3.35. The reading of the manometer is $h = 150\,\text{mm}\ H_2O$. The air density is $\rho_a = 1.20\,\text{kg/m}^3$, and the water density is $\rho = 1000\,\text{kg/m}^3$. The velocity coefficient of Pitot tube is $c = 1$. Neglecting the energy loss, try to determine the air velocity u_0.



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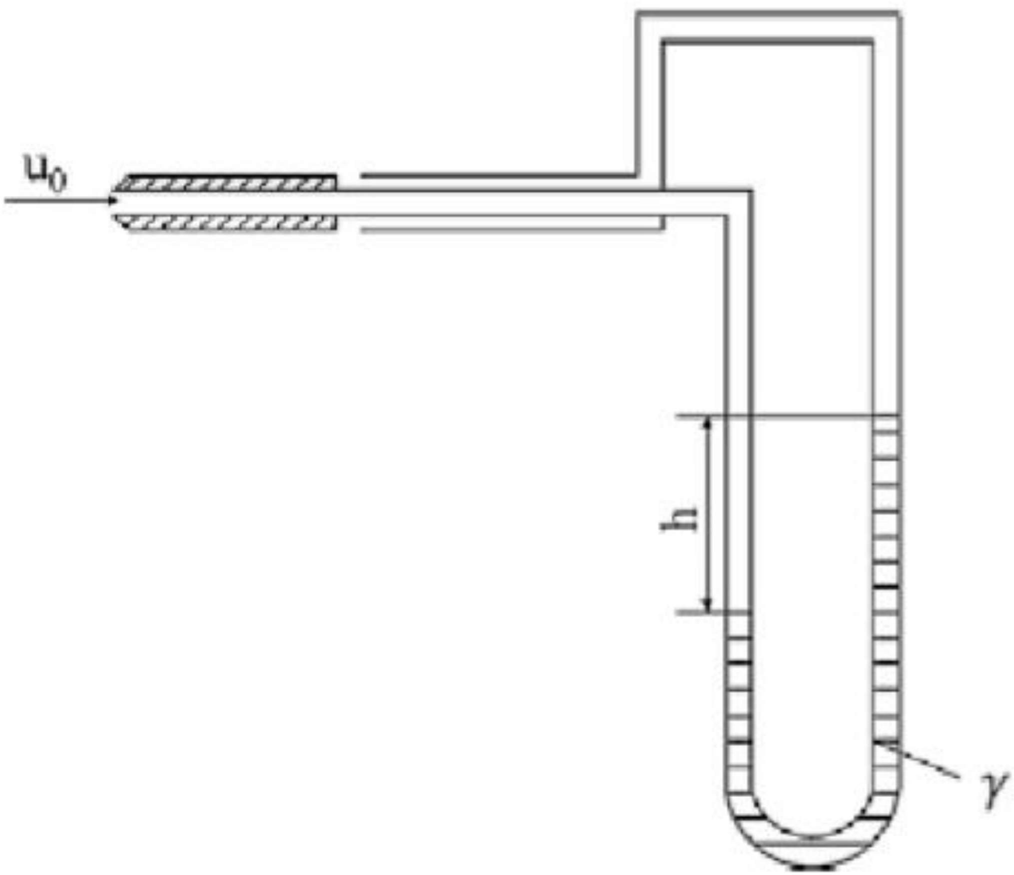


Based on Bernoulli equation:

$$\frac{\vec{V}\bullet\vec{V}}{2}+\frac{P}{\rho}=Constant$$

$$\Rightarrow \frac{V_0^{\,2}}{2}+\frac{P_0}{\rho_{air}}=\frac{V_1^{\,2}}{2}+\frac{P_1}{\rho_{air}}$$

$$\Rightarrow \frac{V_0^{\,2}}{2}+\frac{P_0}{\rho_{air}}=\frac{P_1}{\rho_{air}}$$

$$\Rightarrow V_0=\sqrt{\frac{2(P_1-P_0)}{\rho_{air}}}=\sqrt{\frac{2H\rho_{water}g}{\rho_{air}}}$$

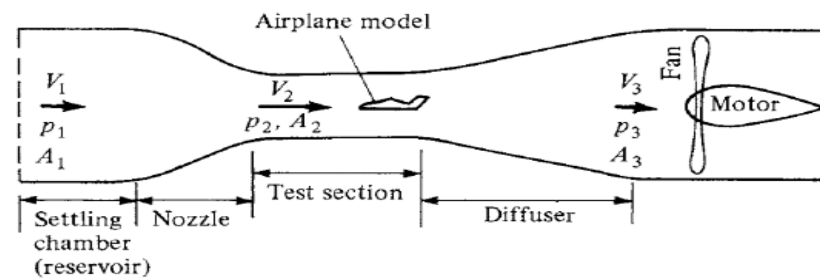
$$=\sqrt{\frac{2*0.15*1000*9.8}{1.20}}\approx 49.5\quad m/s$$

❑ BERNOULLI EQUATION FOR IRROTATIONAL FLOW

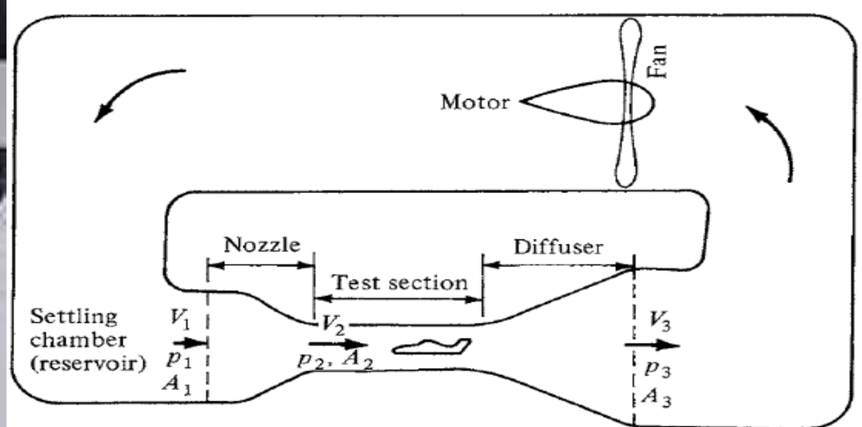
The flow velocity in the test section of a low-speed wind tunnel is 100 mph. The test section is vented to the atmosphere, where atmospheric pressure is 1.01×10^5 N/m^2. The air density is 1.23 kg/m^3. The contraction ratio of the nozzle is 10 to 1.

(a) Calculate the reservoir pressure in atmosphere.

(b) By how much must the reservoir pressure be increased to achieve 200 mph in the test section?



(a) Open-circuit tunnel



(b) Closed-circuit tunnel

❑ Bernoulli equation for irrotational flow

Conservation of mass

$$\rho V_1 A_1 = \rho V_2 A_2 = \rho V_3 A_3$$

Therefore

$$V_2 = \frac{A_1}{A_2} V_1 \text{ and } V_3 = \frac{A_2}{A_3} V_2$$

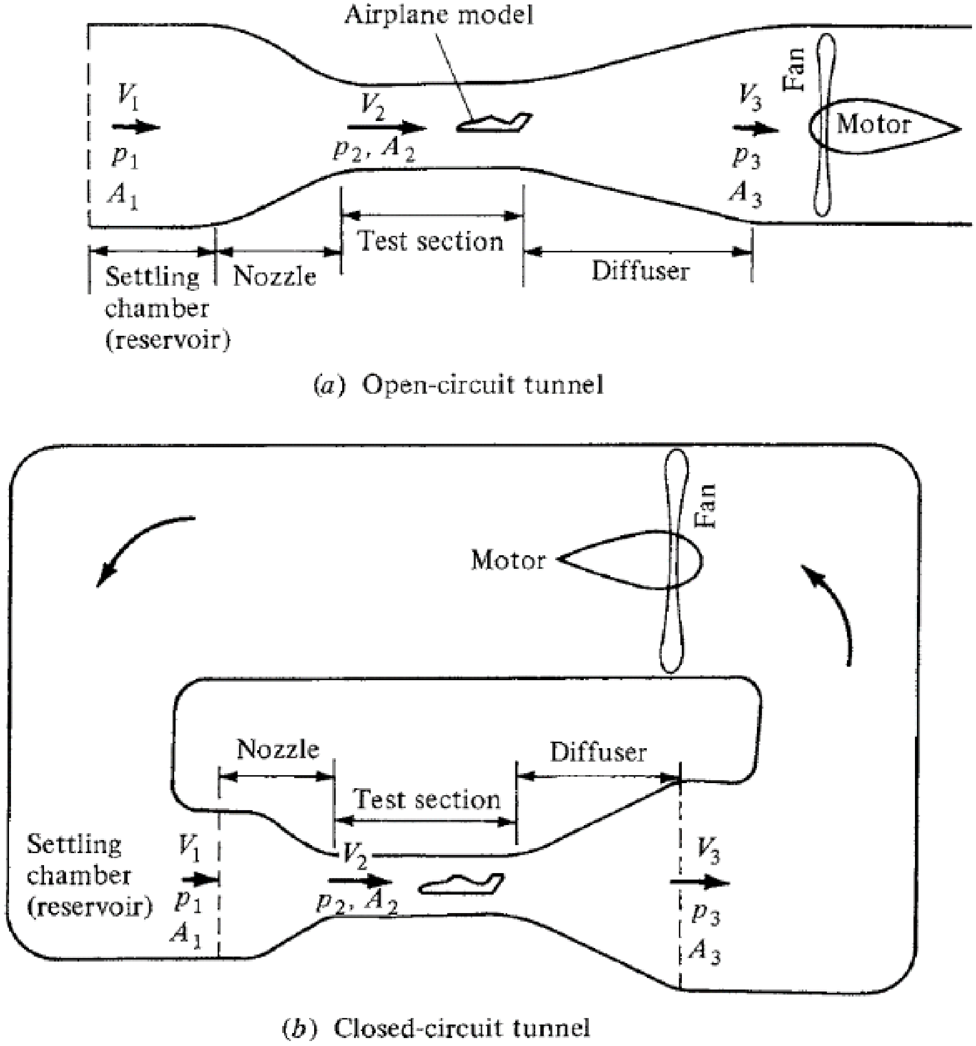
Bernoulli's equation

$$p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2$$

$$= p_3 + \frac{1}{2}\rho V_3^2$$

$$V_2^2 = \frac{2}{\rho}(p_1 - p_2) + V_1^2$$

$$V_2 = \sqrt{\frac{2(p_1 - p_2)}{\rho\left[1 - \left(\frac{A_2}{A_1}\right)^2\right]}}$$



❑ Bernoulli Equation for Irrotational Flow

(a).

We have $V_2 = 100\ mph$, $\frac{A_1}{A_2} = 10$, $p_2 = 1.01 \times 10^5\ Pa$

Need to find p_1

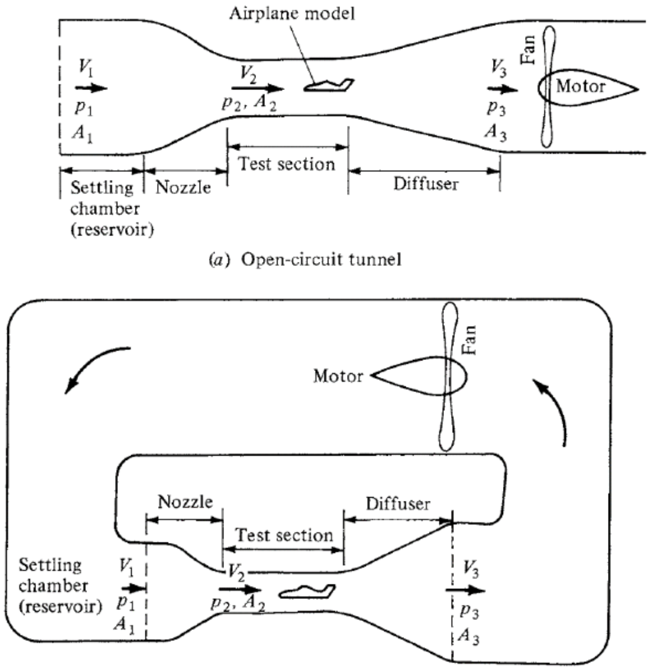
$$p_1 - p_2 = \frac{1}{2}\rho V_2^2 \left[1 - \left(\frac{A_2}{A_1}\right)^2\right]$$

Note $V_2 = 100\ mph = 100\ \times \frac{0.447\ m/s}{1\ mph} = 44.7\ m/s$

$$p_1 - p_2 = \frac{1.23}{2}(44.7)^2 \left[1 - \left(\frac{1}{10}\right)^2\right] = 0.01217 \times 10^5\ Pa$$

$$p_1 = 1.01 \times 10^5 + 0.01217 \times 10^5\ Pa$$

$$p_1 = 1.022 \times 10^5 Pa = 1.01\ atm$$



(a) Open-circuit tunnel

(b) Closed-circuit tunnel

(b).

$$V_2 = 200\ mph = 89.4\ m/s$$

$$p_1 - p_2 = \frac{1.23}{2}(89.4)^2 \left[1 - \left(\frac{1}{10}\right)^2\right] = 0.0487 \times 10^5\ Pa$$

$$p_1 = 1.01 \times 10^5 + 0.0487 \times 10^5 = 1.059 \times 10^5\ Pa$$

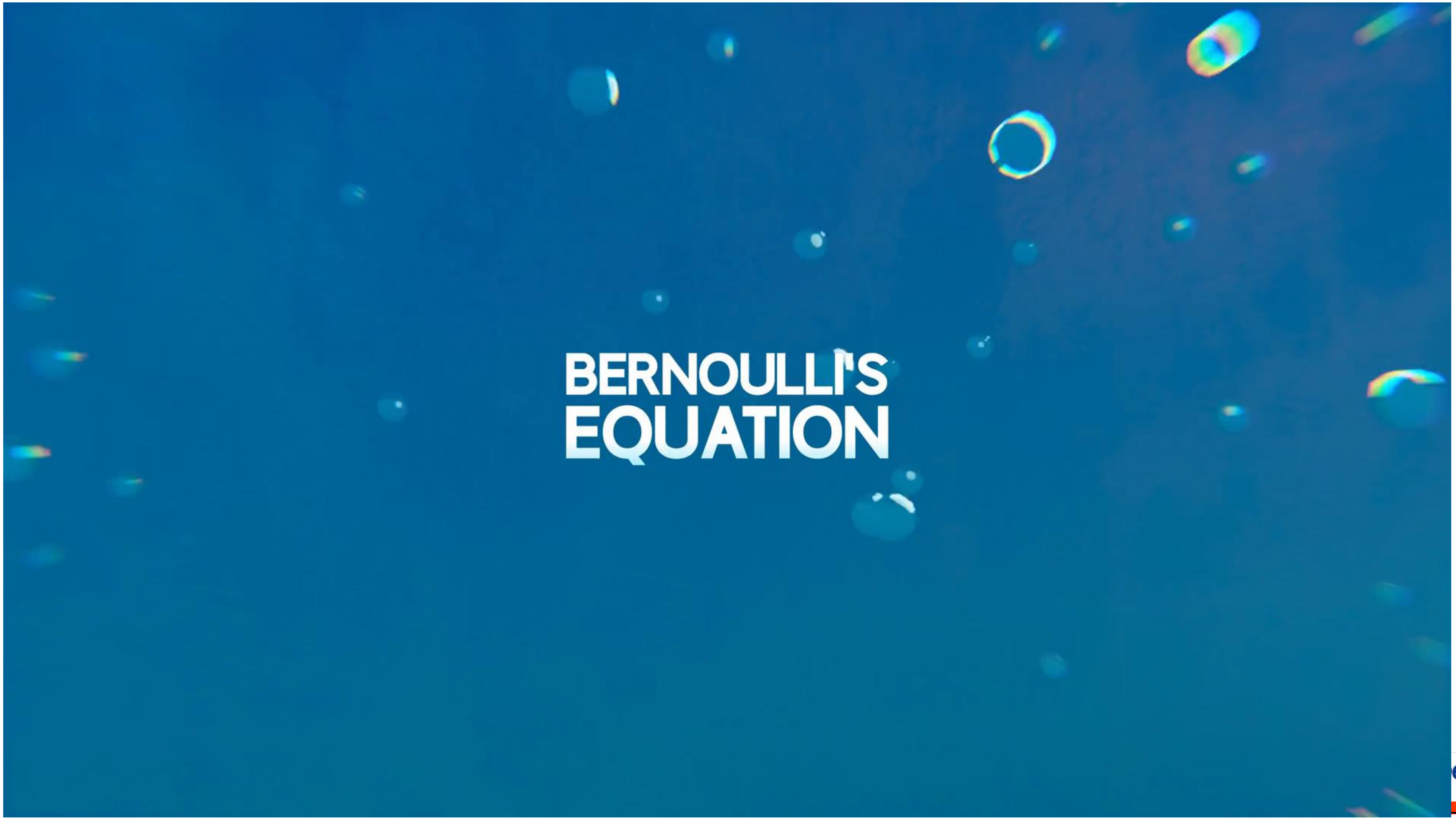
$$p_1 = 1.048\ atm$$

Only a ~4% increase in pressure needed to increase velocity by 100%!

❑ Bernoulli equation for irrotational flow

- Understanding Bernoulli's Equation

https://www.youtube.com/watch?v=DW4rItB20h4



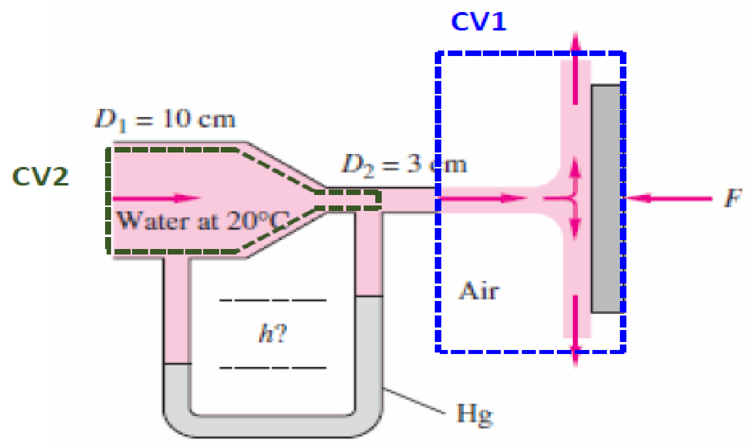
❑ BERNOULLI EQUATION FOR IRROTATIONAL FLOW

Examples involving Bernoulli's equation

- Water flows through a circular nozzle, exits into the air as a jet, and strikes a plate, as shown in Figure. The force required to hold the plate steady is *F*=70 N. Assuming steady, frictionless, one-dimensional flow, estimate

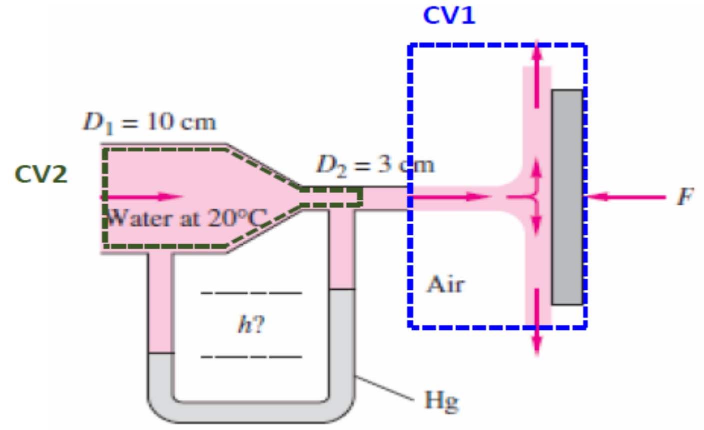
(a) The velocities at sections (1) and (2) and

(b) the mercury manometer reading *h* ($\rho_{Hg} = 13550\ kg/m^3$).



❑ Bernoulli equation for irrotational flow

Example-solution



- X-momentum for CV1

$$\frac{\partial}{\partial t}\iiint_V \rho u dV + \iint_S \rho u(\vec{V}.\hat{n})dS = \iiint_V \rho f_x \, dV + \iint_S p_x \, dS + \sum F_x$$

=0 (steady); $f_x = 0$; $p = p_{atm}$ everywhere; $= -F$

$$-\rho_w u_2^2 A_2 = -F \rightarrow F = \frac{\pi}{4}\rho_w D_2^2 u_2^2$$

$$u_2 = \sqrt{\frac{4F}{\pi \rho_w D_2^2}} = \sqrt{\frac{4 \times 70}{\pi \times 1000 \times 0.03^2}} = 9.95 \, m/s$$

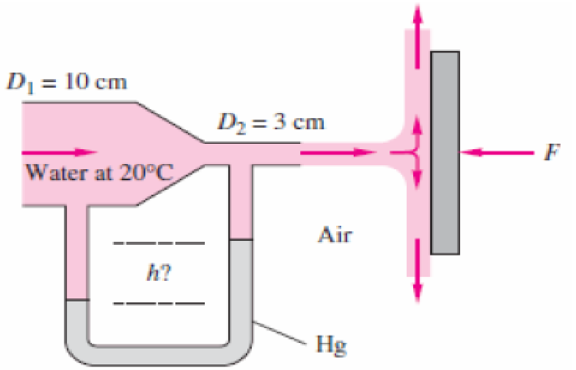
- Conservation of mass for CV2

$$\rho_w u_1 A_1 = \rho_w u_2 A_2 \rightarrow u_1 = u_2 \frac{A_2}{A_1} = u_2 \left(\frac{D_2}{D_1}\right)^2$$

$$u_1 = 9.95 \left(\frac{3}{10}\right)^2 \rightarrow u_1 = 0.9 \, m/s$$

❑ Bernoulli equation for irrotational flow

Example-solution



- Bernoulli's equation on a streamline from 1 to 2

$$p_1 + \frac{1}{2}\rho_w u_1^2 = p_2 + \frac{1}{2}\rho_w u_2^2$$

$$\rightarrow p_1 - p_2 = \frac{1}{2}\rho_w (u_2^2 - u_1^2)$$

For the manometer

$$p_1 - \rho_{Hg} g h = p_2 \rightarrow p_2 - p_1 = \rho_{Hg} g h$$

Therefore

$$\frac{1}{2}\rho_w (u_2^2 - u_1^2) = \rho_{Hg} g h$$

$$h = \frac{1}{2g}\frac{\rho_w}{\rho_{Hg}}(u_2^2 - u_1^2) = \frac{1}{2 \times 9.81}\frac{1000}{13550}(9.95^2 - 0.9^2)$$

$$h = 0.37\ m$$