Lecture # 15: Streamlines & Stream function

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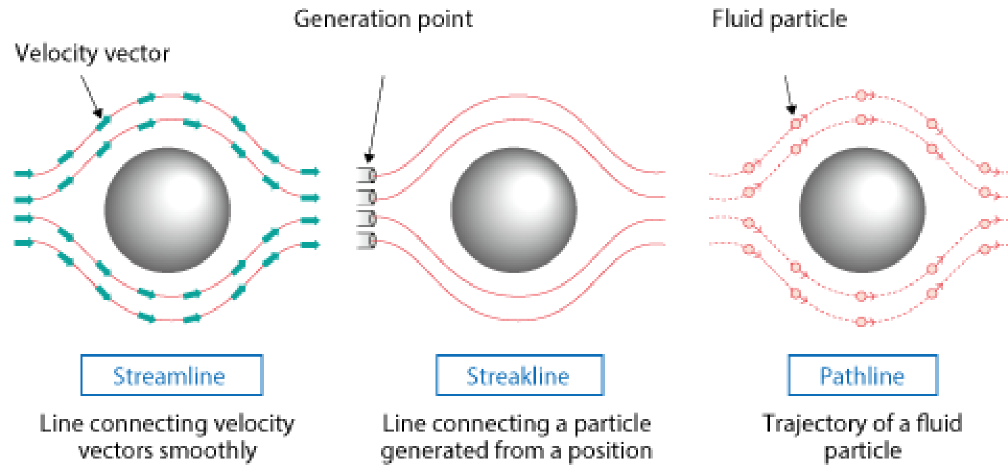
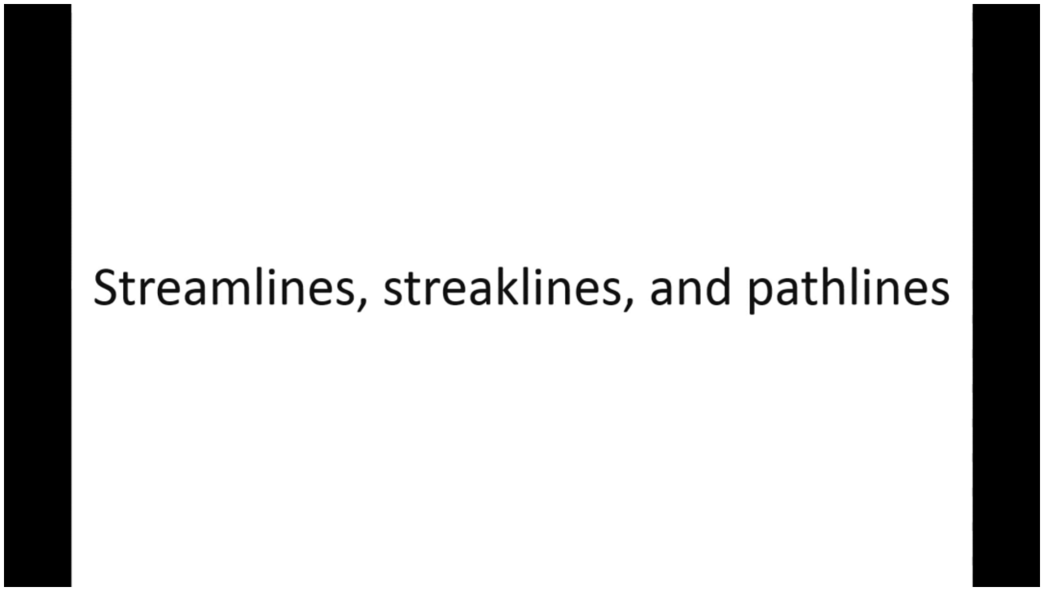
• Streamlines; Streaklines, and Pathline

- ***Streamlines** are a family of curves that are instantaneously tangent to the velocity vector of the flow. These show the direction in which a massless fluid element will travel at any point in time (Eularian approach).*
- ***Streaklines** are the loci of points of all the fluid particles that have passed continuously through a particular spatial point in the past. Dye steadily injected into the fluid at a fixed point extends along a streakline (Langragian approach).*
- ***Pathlines** are the trajectories that individual fluid particles follow. These can be thought of as "recording" the path of a fluid element in the flow over a certain period. The direction the path takes will be determined by the streamlines of the fluid at each moment in time (Langragian approach).*

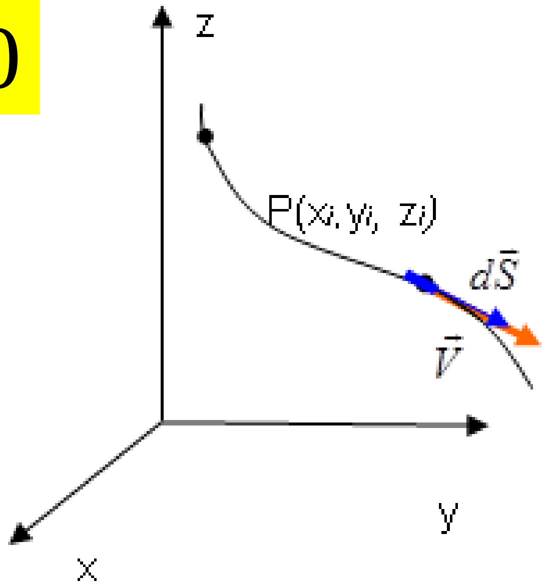




❑ Streamlines & Stream function

- ***By definition, the equation of a streamline is given as:***

$$\vec{V} \times d\vec{S} = 0$$



$$\vec{V} \times d\vec{S} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ V_1 & V_2 & V_3 \\ h_1 dq_1 & h_2 dq_2 & h_3 dq_3 \end{vmatrix}$$

$$= \hat{e}_1 (V_2 h_3 dq_3 - V_3 h_2 dq_2) + \hat{e}_2 (V_3 h_1 dq_1 - V_1 h_3 dq_3)$$

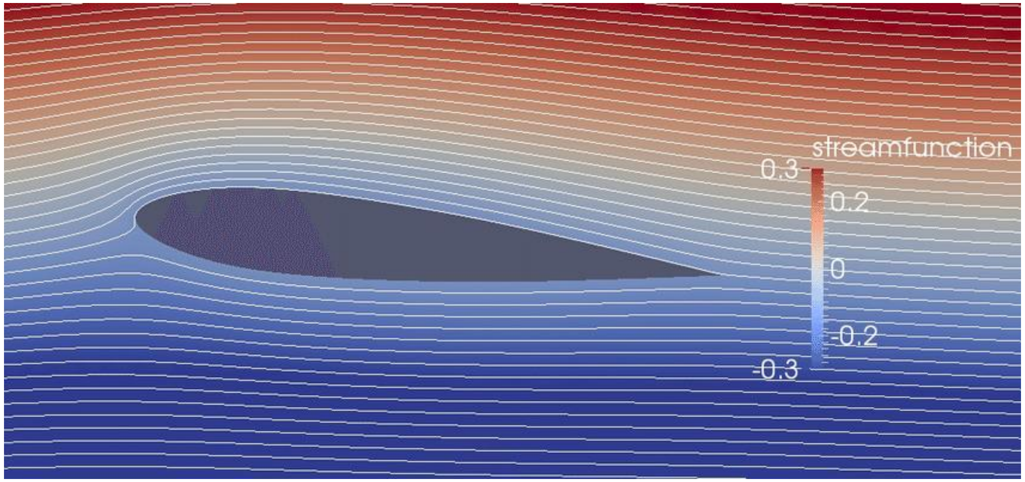
$$+ \hat{e}_3 (V_1 h_2 dq_2 - V_2 h_1 dq_1) = 0$$

- ***In differential form:***

$$V_2 h_3 dq_3 - V_3 h_2 dq_2 = 0 \Rightarrow \frac{h_3 dq_3}{V_3} = \frac{h_2 dq_2}{V_2}$$

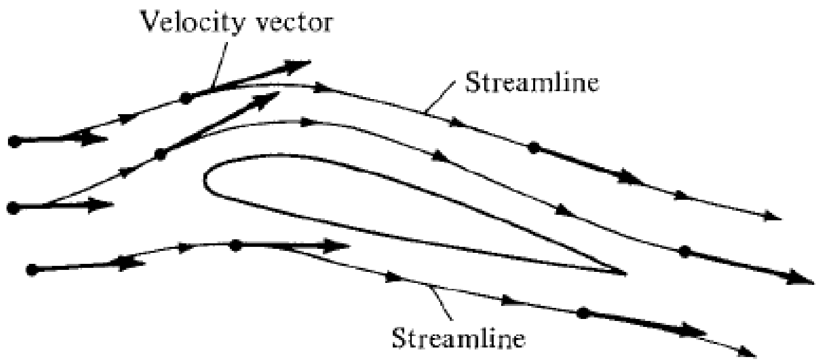
$$V_3 h_1 dq_1 - V_1 h_3 dq_3 = 0 \Rightarrow \frac{h_3 dq_3}{V_3} = \frac{h_1 dq_1}{V_1}$$

$$V_1 h_2 dq_2 - V_2 h_1 dq_1 = 0 \Rightarrow \frac{h_2 dq_2}{V_2} = \frac{h_1 dq_1}{V_1}$$



❑ Streamlines & Stream function

Streamline Definition



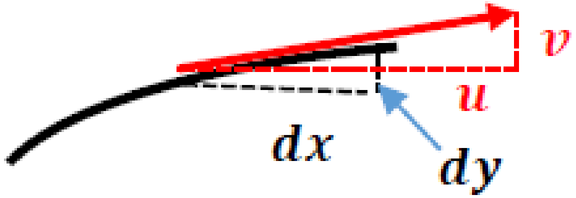
- $\vec{V} \times d\vec{s} = 0$

- In 2D : $\vec{V} = u\hat{\imath} + v\hat{\jmath}\ ,\ \ d\vec{s} = dx\hat{\imath} + dy\hat{\jmath}$

$$\vec{V} \times d\vec{s} = 0 \rightarrow (udy - vdx)\hat{k} = 0$$
$$udy = vdx$$

$$\frac{v}{u} = \frac{dy}{dx}$$

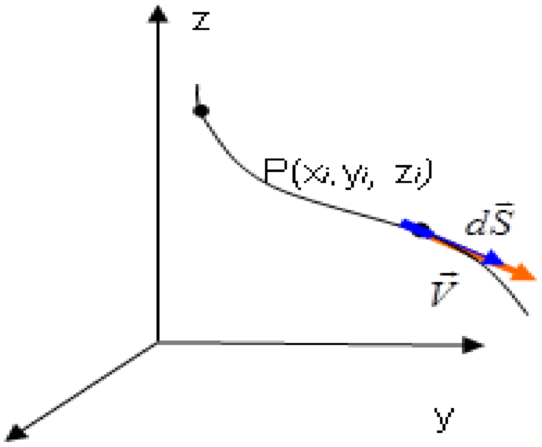
- For $\vec{V}$ to be tangent to the path



$$u = \frac{dx}{dt}\ ,\ v = \frac{dy}{dt} \rightarrow \frac{v}{u} = \frac{dy}{dx}$$

Streamlines & Stream function

- ***According to the definition of streamlines, no fluid can cross a streamline***

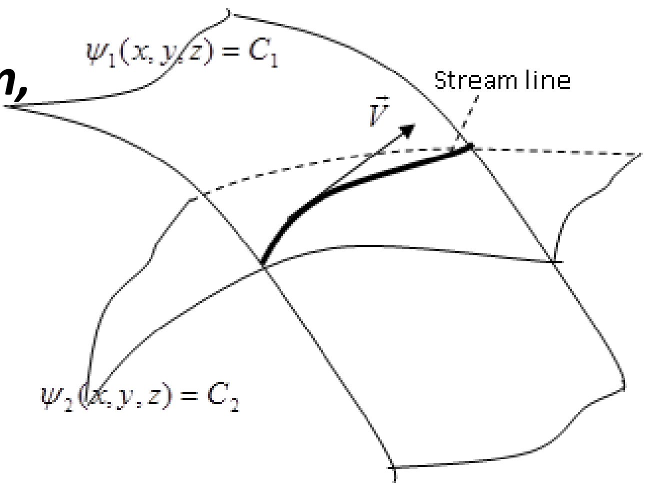


$$V_2 h_3 dq_3 - V_3 h_2 dq_2 = 0 \Rightarrow \frac{h_3 dq_3}{V_3} = \frac{h_2 dq_2}{V_2}$$

$$V_3 h_1 dq_1 - V_1 h_3 dq_3 = 0 \Rightarrow \frac{h_3 dq_3}{V_3} = \frac{h_1 dq_1}{V_1}$$

$$V_1 h_2 dq_2 - V_2 h_1 dq_1 = 0 \Rightarrow \frac{h_2 dq_2}{V_2} = \frac{h_1 dq_1}{V_1}$$

  - ***only two independent equations***



- ***For general purpose, in Cartesian coordinate system, the two independent equations can be written as***

$$a_1(x,y,z)dx + a_2(x,y,z)dy + a_3(x,y,z)dz = 0$$

$$b_1(x,y,z)dx + b_2(x,y,z)dy + b_3(x,y,z)dz = 0$$

- ***The solutions of the above equations can be expressed as***

$$\psi_1(x,y,z) = C_1$$ ***Where C_1 is a constant***

$$\psi_2(x,y,z) = C_2$$ ***Where C_2 is a constant***

- ***A streamline in space can be expressed as the curve of intersection of two surfaces.***

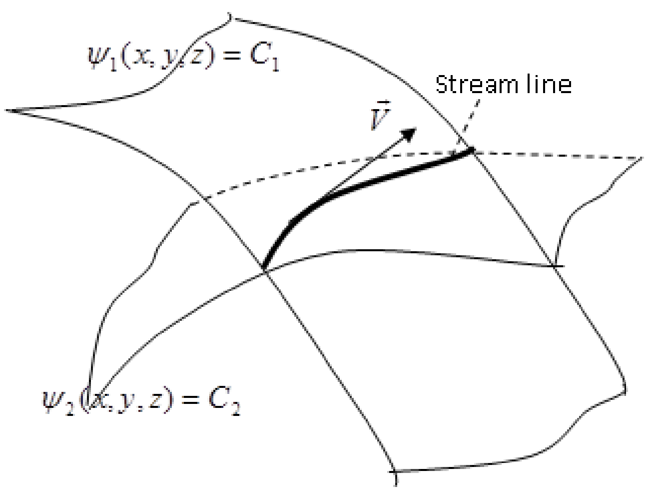
❑ Streamlines & Stream function

As shown in the figure, along the streamline, the velocity $\vec{V}$ lies in both the surface of $\psi_1(x,y,z)=C_1$ and $\psi_2(x,y,z)=C_2$. Since direction of the gradients of $\psi_1(x,y,z)$ and $\psi_2(x,y,z)$ (i.e., $\nabla\psi_1$ and $\nabla\psi_2$) being parallel to the direction that normal to the their respective surfaces, thus, $\vec{V}$ is normal to $\nabla\psi_1$ and $\nabla\psi_2$.

$\vec{V}\bullet\nabla\psi_1=0$ and $\vec{V}\bullet\nabla\psi_2=0$

This shows that $\vec{V}$ is normal to the plane formed by the vectors $\nabla\psi_1$ and $\nabla\psi_2$. In other words, $\vec{V}$ should be parallel to the cross product of $\nabla\psi_1$ and $\nabla\psi_2$, i.e.,

$$\mu\vec{V}=\nabla\psi_1\times\nabla\psi_2=\begin{vmatrix}\hat{e}_1 & \hat{e}_2 & \hat{e}_3\\ \dfrac{\partial\psi_1}{h_1\partial q_1} & \dfrac{\partial\psi_1}{h_2\partial q_2} & \dfrac{\partial\psi_1}{h_3\partial q_3}\\ \dfrac{\partial\psi_2}{h_1\partial q_2} & \dfrac{\partial\psi_2}{h_2\partial q_2} & \dfrac{\partial\psi_2}{h_3\partial q_3}\end{vmatrix}$$



- ***Where μ is a scalar function of position***

$$\nabla\bullet(\mu\vec{V})=\nabla\bullet(\nabla\psi_1\times\nabla\psi_2)=0$$

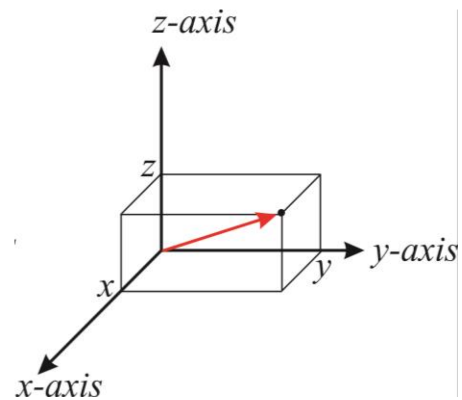
- ***For incompressible flows, we usually chose μ =1.***

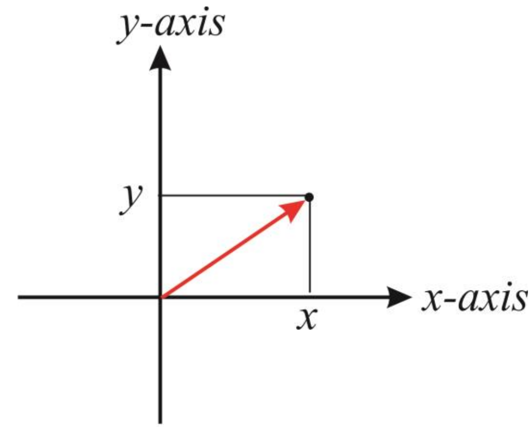
❑ Stream Function for 2-D Flows

- *When a flow is called 2-D flow, it means the flow quantities are independent of distance along a certain fixed direction. It we designate Z axis as the direction, we shall have.*

$$\frac{\partial()}{\partial Z} = \frac{\partial}{\partial Z} \text{ of any quantities } \equiv 0$$

In Cartesian coordinate system





In Cartesian system, we will have:

$$d\vec{S} = (dx, dy, dz)$$

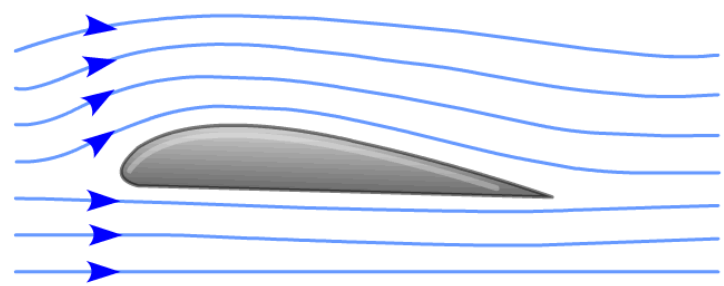
$$\vec{V} = (u, v, 0)$$

$$u = u(x, y)$$

$$v = v(x, y)$$

Along a streamline, it can be written as

$$\frac{dx}{u(x,y)} = \frac{dy}{v(x,y)} = \frac{dz}{0}$$

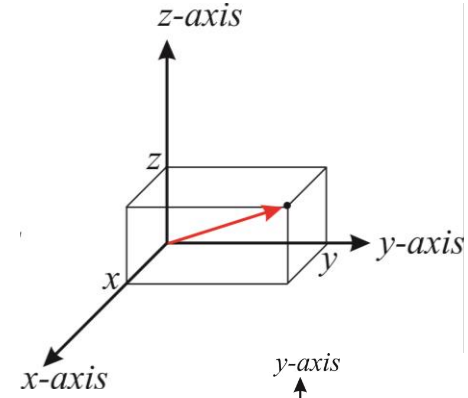


The shape of wing forces air to move faster over the top surface.

It immediately follows that $dz = 0$ or $Z = const$

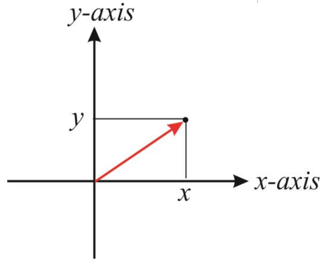
❑ Stream Function for 2-D Flows

For the steam function ψ_1 and ψ_2, we can choose stream function ψ_2 is simply z (i.e., $\psi_2 = z$), the ψ_1 is the only one unknown function which is denoted as ψ. ψ is a function of x and y only. i.e., $\psi_1 = \psi(x, y)$ and $\psi_2 = z$.



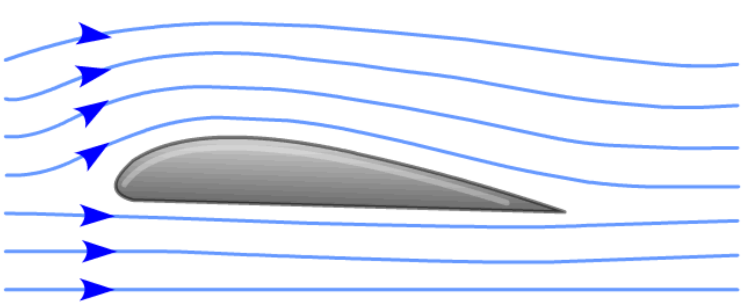
Since $\mu\vec{V} = \nabla\psi_1 \times \nabla\psi_2$, then we have

$$\mu\vec{V} = \mu u\,\hat{i} + \mu v\,\hat{j} + 0\hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial y} & 0 \\ 0 & 0 & 1 \end{vmatrix} \Rightarrow \begin{cases} \mu u = \dfrac{\partial\psi}{\partial y} \\ \mu v = -\dfrac{\partial\psi}{\partial x} \end{cases}$$



For incompressible flows, $\mu = 1$, $\Rightarrow$ $\begin{cases} u = \dfrac{\partial\psi}{\partial y} \\ v = -\dfrac{\partial\psi}{\partial x} \end{cases}$

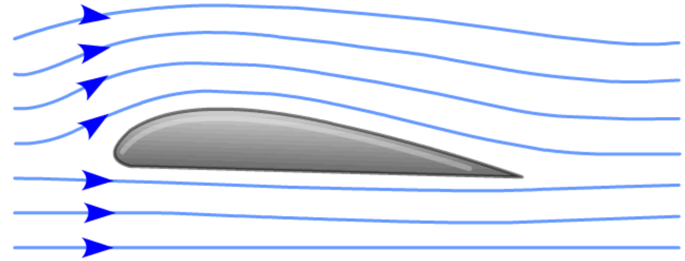
For compressible flows, $\mu = \rho$, $\Rightarrow$ $\begin{cases} \rho u = \dfrac{\partial\psi}{\partial y} \\ \rho v = -\dfrac{\partial\psi}{\partial x} \end{cases}$



➔ The shape of wing forces air to move faster over the top surface.

Stream Function for 2-D Flows

Uniform flow (free stream flow)



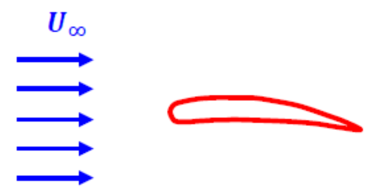
The shape of wing forces air to move faster over the top surface.

$$u = U_\infty \,, \qquad v = 0$$

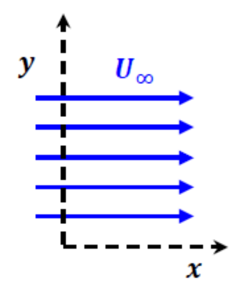
$$u = \frac{\partial \psi}{\partial y} = U_\infty \rightarrow \psi(x, y) = U_\infty y + f(x)$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}[U_\infty y + f(x)] = f'(x) = 0$$

$$\rightarrow f(x) = const. = c$$



$$\psi = U_\infty y + c$$



Stream function is undetermined by a constant. Often this constant is taken as zero.

Streamlines of uniform flow are lines of constant ψ that are lines of constant y.

❑ Stream Function for 2-D Flows

- *Please determine the stream function for 2-D incompressible flow with the velocity field given as :*

$$\vec{V} = 2x\ \hat{i} - 2y\ \hat{j}$$

❑ Stream Function for 2-D Flows

- ***Please determine the stream function for 2-D incompressible flow with the velocity field given as :***

$$\vec{V} = 2x\ \hat{i} - 2y\ \hat{j}$$

Given a velocity field: $\vec{V} = 2x\ \hat{i} - 2y\ \hat{j}$ $\quad u = 2x \quad v = -2y$

Begin with u: $\dfrac{\partial \psi}{\partial y} \equiv u = 2x$

$$\partial \psi = u\ \partial y = 2x\ \partial y \qquad \psi = 2xy + f(x)$$

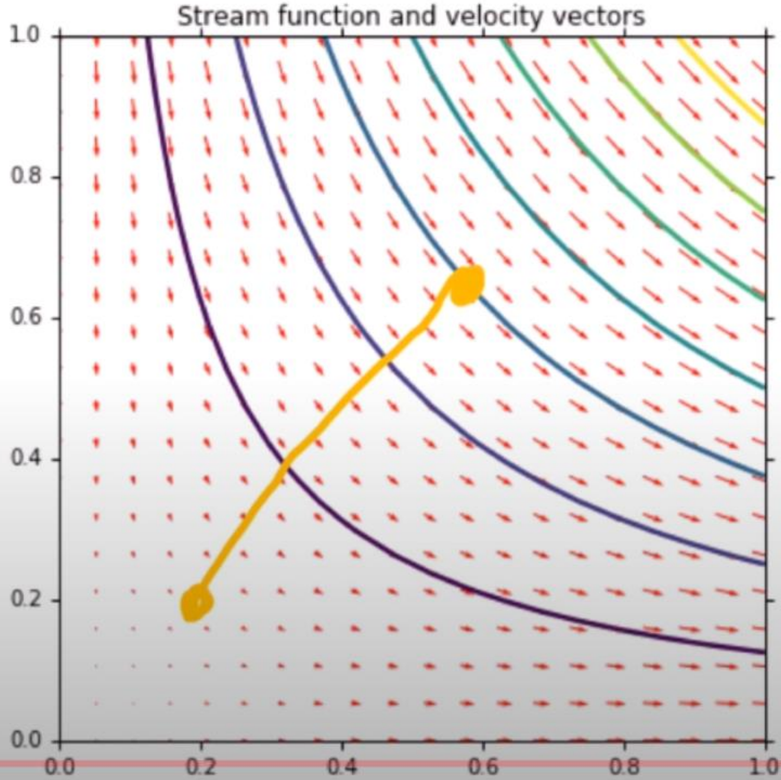
Continue with v: $\dfrac{\partial \psi}{\partial x} \equiv -v = 2y$

$$\partial \psi = -v\ \partial x = 2y\ \partial x \qquad \psi = 2xy + f(y) + C$$

Comparing the two expressions and choosing C=0:

❑ Stream Function for 2-D Flows

Stream function



Contours of the stream function are streamlines

The difference between the value of the stream function at any two locations gives the volume flow passing through the line connecting the two points.