Lecture # 16: Streamlines & Stream function Part-2

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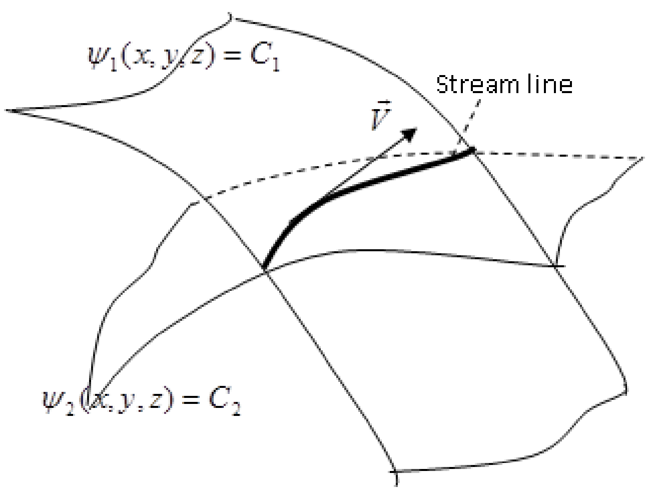
❑ Streamlines & Stream function

As shown in the figure, along the streamline, the velocity $\vec{V}$ lies in both the surface of $\psi_1(x,y,z)=C_1$ and $\psi_2(x,y,z)=C_2$. Since direction of the gradients of $\psi_1(x,y,z)$ and $\psi_2(x,y,z)$ (i.e., $\nabla\psi_1$ and $\nabla\psi_2$) being parallel to the direction that normal to the their respective surfaces, thus, $\vec{V}$ is normal to $\nabla\psi_1$ and $\nabla\psi_2$.

$\vec{V}\bullet\nabla\psi_1=0$ and $\vec{V}\bullet\nabla\psi_2=0$

This shows that $\vec{V}$ is normal to the plane formed by the vectors $\nabla\psi_1$ and $\nabla\psi_2$. In other words, $\vec{V}$ should be parallel to the cross product of $\nabla\psi_1$ and $\nabla\psi_2$, i.e.,

$$\mu\vec{V}=\nabla\psi_1\times\nabla\psi_2=\begin{vmatrix}\hat{e}_1 & \hat{e}_2 & \hat{e}_3\\ \dfrac{\partial\psi_1}{h_1\partial q_1} & \dfrac{\partial\psi_1}{h_2\partial q_2} & \dfrac{\partial\psi_1}{h_3\partial q_3}\\ \dfrac{\partial\psi_2}{h_1\partial q_2} & \dfrac{\partial\psi_2}{h_2\partial q_2} & \dfrac{\partial\psi_2}{h_3\partial q_3}\end{vmatrix}$$



- ***Where μ is a scalar function of position***

$$\nabla\bullet(\mu\vec{V})=\nabla\bullet(\nabla\psi_1\times\nabla\psi_2)=0$$

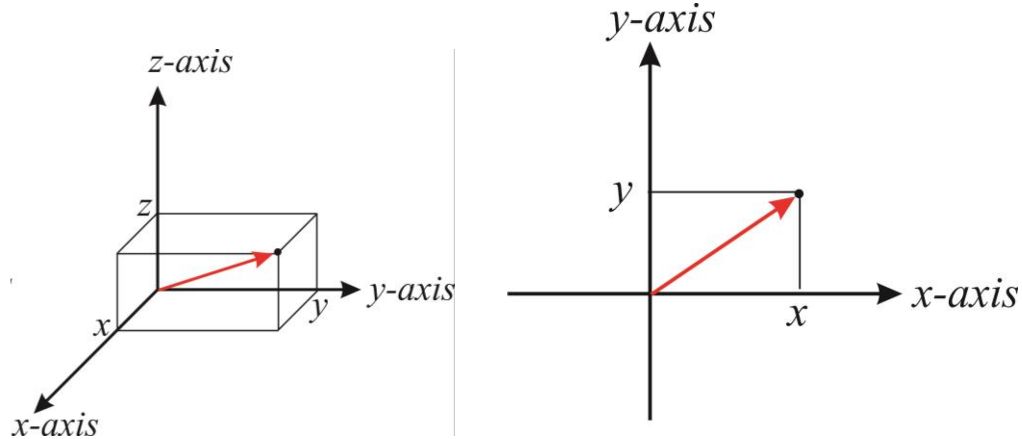
- ***For incompressible flows, we usually chose μ =1.***

❑ Stream Function for 2-D Flows

- *When a flow is called 2-D flow, it means the flow quantities are independent of distance along a certain fixed direction. It we designate Z axis as the direction, we shall have.*

$$\frac{\partial()}{\partial Z} = \frac{\partial}{\partial Z} \text{ of any quantities } \equiv 0$$

In Cartesian coordinate system



In Cartesian system, we will have:

$$d\vec{S} = (dx, dy, dz)$$

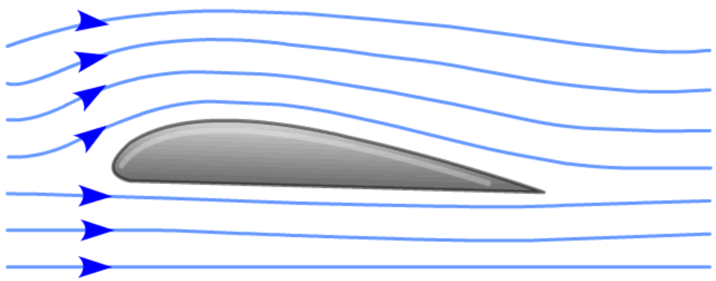
$$\vec{V} = (u, v, 0)$$

$$u = u(x, y)$$

$$v = v(x, y)$$

Along a streamline, it can be written as

$$\frac{dx}{u(x,y)} = \frac{dy}{v(x,y)} = \frac{dz}{0}$$

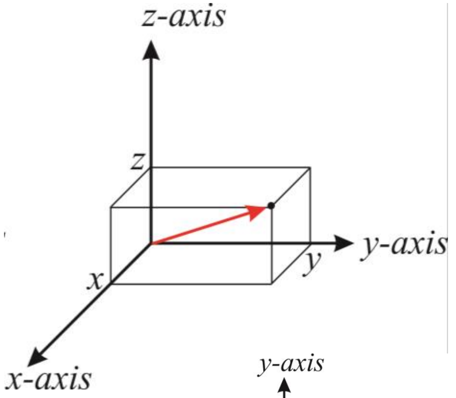


The shape of wing forces air to move faster over the top surface.

It immediately follows that $dz = 0$ or $Z = const$

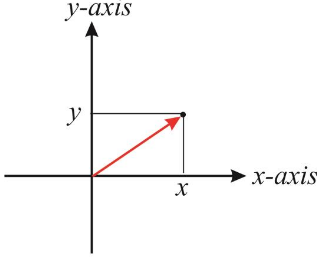
❑ Stream Function for 2-D Flows

For the steam function ψ_1 and ψ_2, we can choose stream function ψ_2 is simply z (i.e., $\psi_2 = z$), the ψ_1 is the only one unknown function which is denoted as ψ. ψ is a function of x and y only. i.e., $\psi_1 = \psi(x, y)$ and $\psi_2 = z$.



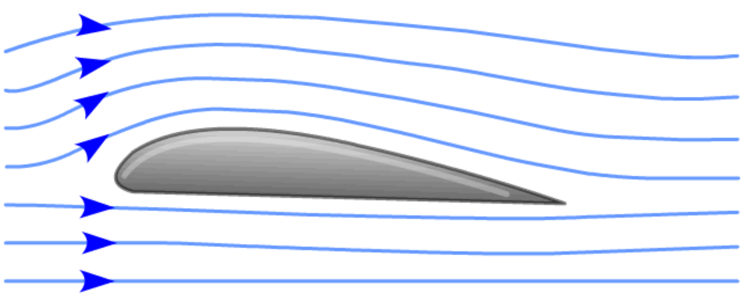
Since $\mu\vec{V} = \nabla\psi_1 \times \nabla\psi_2$, then we have

$$\mu\vec{V} = \mu u\,\hat{i} + \mu v\,\hat{j} + 0\hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial\psi}{\partial x} & \dfrac{\partial\psi}{\partial y} & 0 \\ 0 & 0 & 1 \end{vmatrix} \Rightarrow \begin{cases} \mu u = \dfrac{\partial\psi}{\partial y} \\ \mu v = -\dfrac{\partial\psi}{\partial x} \end{cases}$$



For incompressible flows, $\mu = 1$, $\Rightarrow$
$$\begin{cases} u = \dfrac{\partial\psi}{\partial y} \\ v = -\dfrac{\partial\psi}{\partial x} \end{cases}$$

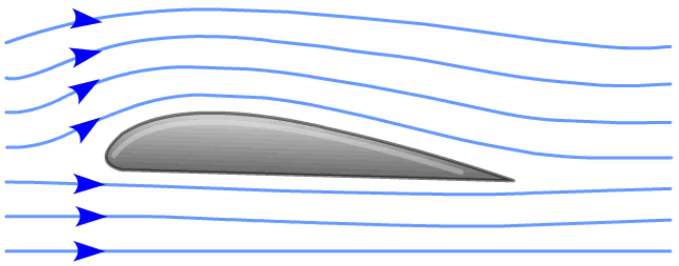
For compressible flows, $\mu = \rho$, $\Rightarrow$
$$\begin{cases} \rho u = \dfrac{\partial\psi}{\partial y} \\ \rho v = -\dfrac{\partial\psi}{\partial x} \end{cases}$$



➔ **The shape of wing forces air to move faster over the top surface.**

Stream Function for 2-D Flows

Uniform flow (free stream flow)



The shape of wing forces air to move faster over the top surface.

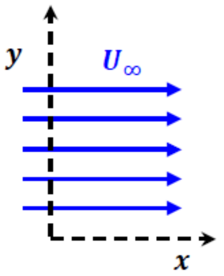
$$u = U_\infty , \qquad v = 0$$

$$u = \frac{\partial \psi}{\partial y} = U_\infty \rightarrow \psi(x, y) = U_\infty y + f(x)$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}[U_\infty y + f(x)] = f'(x) = 0$$

$$\rightarrow f(x) = const. = c$$

$$\psi = U_\infty y + c$$

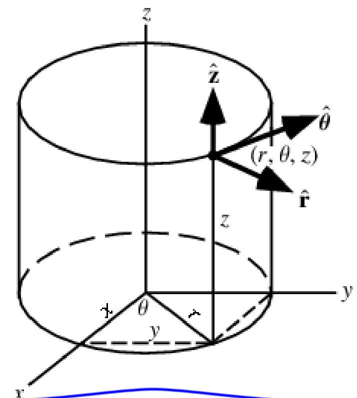


Stream function is undetermined by a constant. Often this constant is taken as zero.

Streamlines of uniform flow are lines of constant ψ that are lines of constant y.

❑ Stream Function for 2-D Flows

In Cylindrical coordinate system



$$d\vec{S} = (dr, rd\theta, dz)$$

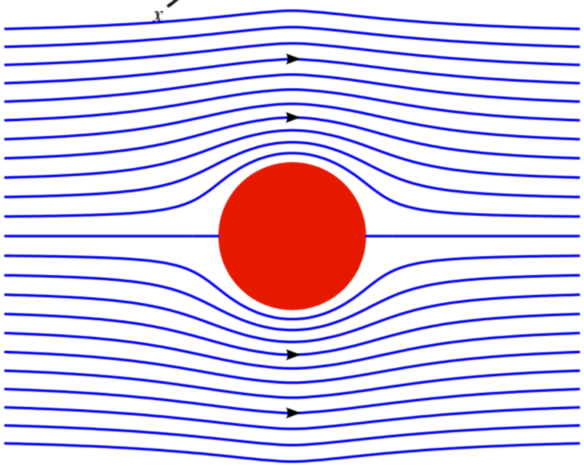
$$\vec{V} = (V_r, V_\theta, 0)$$

$$V_r = V_r(r, \theta)$$

$$V_\theta = V_\theta(r, \theta)$$

Along a streamline, it can be written as

$$\frac{dr}{V_r(r,\theta)} = \frac{d\theta}{V_\theta(r,\theta)} = \frac{dz}{0}$$



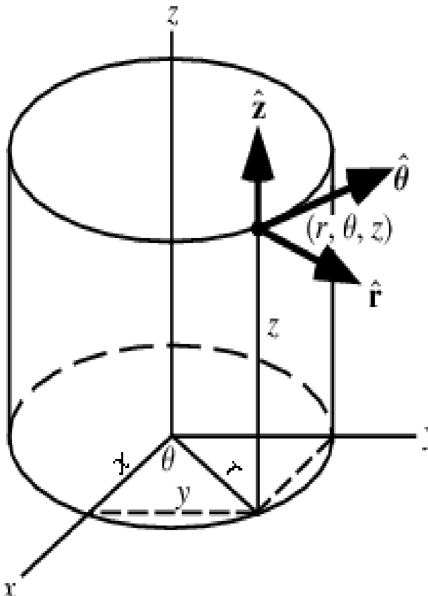
It immediately follows that $dz = 0$ or $Z = const$

For the steam function ψ_1 and ψ_2, we can choose stream function ψ_2 is simply z (i.e., $\psi_2 = z$), the ψ_1 is the only one unknown function which is denoted as ψ. ψ is a function of r and θ only.
i.e., $\psi_1 = \psi(r, \theta)$ and $\psi_2 = z$.

❑ Stream Function for 2-D Flows

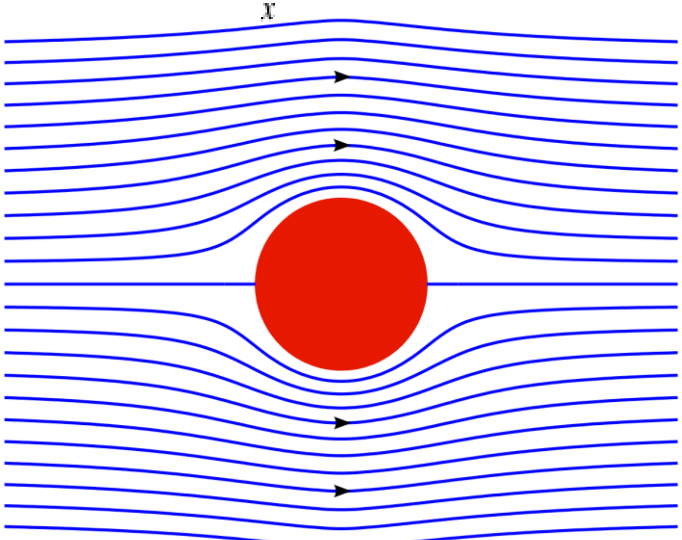
Since $\mu\vec{V} = \nabla\psi_1 \times \nabla\psi_2$, then we have

$$\mu\vec{V} = \mu V_r\,\hat{e}_r + \mu V_\theta\,\hat{e}_\theta + 0\hat{e}_Z = \begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{e}_z \\ \dfrac{\partial\psi}{\partial r} & \dfrac{\partial\psi}{r\partial\theta} & 0 \\ 0 & 0 & 1 \end{vmatrix} \Rightarrow \begin{cases} \mu V_r = \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta} \\ \mu V_\theta = -\dfrac{\partial\psi}{\partial r} \end{cases}$$



For incompressible flows, $\mu = 1$, $\Rightarrow$ $\begin{cases} V_r = \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta} \\ V_\theta = -\dfrac{\partial\psi}{\partial r} \end{cases}$

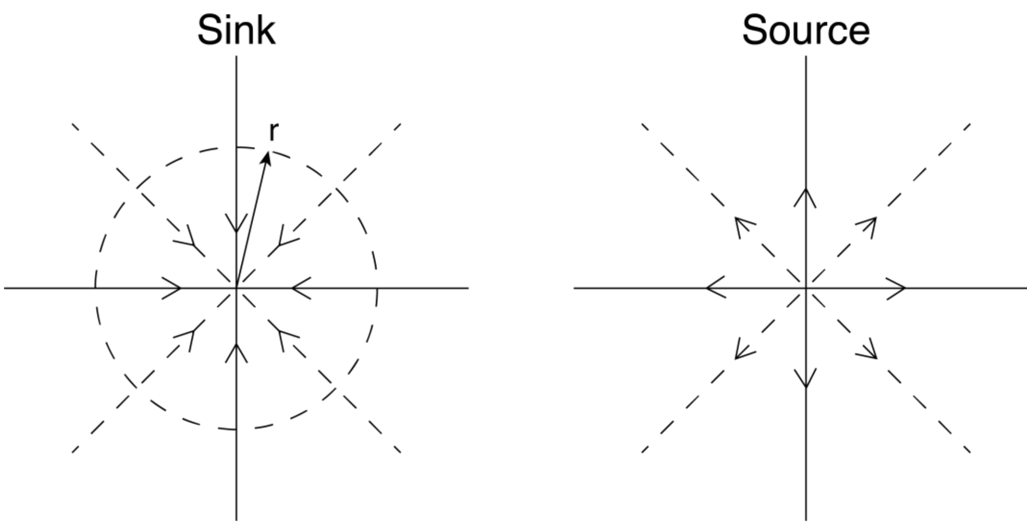
For compressible flows, $\mu = \rho$, $\Rightarrow$ $\begin{cases} \rho V_r = \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta} \\ \rho V_\theta = -\dfrac{\partial\psi}{\partial r} \end{cases}$



❑ Stream Function for 2-D Flows

- *Please determine the stream function for 2-D incompressible flow with the velocity field given as :*

$$u = \frac{cx}{x^2 + y^2}, v = \frac{cy}{x^2 + y^2}$$



Stream Function for 2-D Flows

$$u = \frac{cx}{x^2 + y^2}, v = \frac{cy}{x^2 + y^2}$$

In polar coordinates

$$v_r = u\cos\theta + v\sin\theta = \frac{cr\cos\theta}{r^2}\cos\theta + \frac{cr\sin\theta}{r^2}\sin\theta = \frac{c}{r}$$

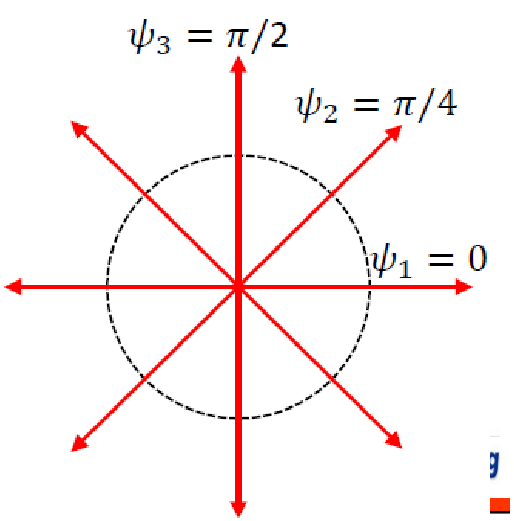
$$v_\theta = -u\sin\theta + v\cos\theta = -\frac{cr\cos\theta}{r^2}\sin\theta + \frac{cr\sin\theta}{r^2}\cos\theta$$
$$= 0$$

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{c}{r} \to \frac{\partial\psi}{\partial\theta} = c \to \psi(r,\theta) = c\theta + f(r)$$

$$v_\theta = -\frac{\partial\psi}{\partial r} = -\frac{\partial}{\partial r}[c\theta + f(r)] = 0$$

$$f'(r) = 0 \to f(r) = const = c_1$$

$$\psi = c\theta + c_1$$



Streamlines are lines of $\theta = const.$

❑ Stream Function for 2-D Flows

Stream function and continuity

Continuity equation for 2D steady incompressible flow

$$\vec{\nabla}.\vec{V} = 0 \rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

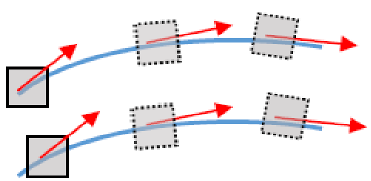
$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0 \quad ✓$$

Stream function satisfies the continuity equation

Consider two fluid elements next to each other on two streamlines

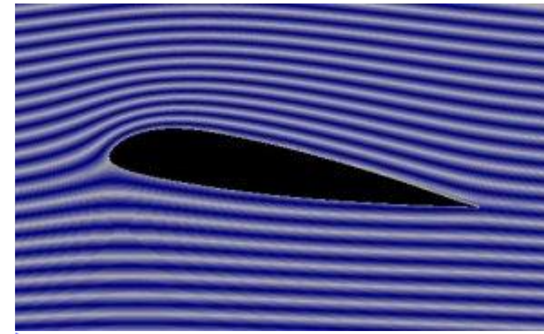
Fluid elements travel tangent to streamlines

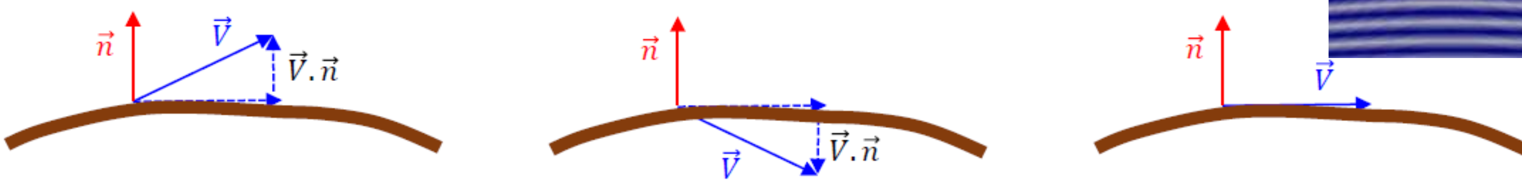
Fluid element does not travel across the streamlines!



❑ Stream Function for 2-D Flows

Stream function and solid boundaries





$\vec{V}.\vec{n} > 0$
Fluid would flow out of the surface!

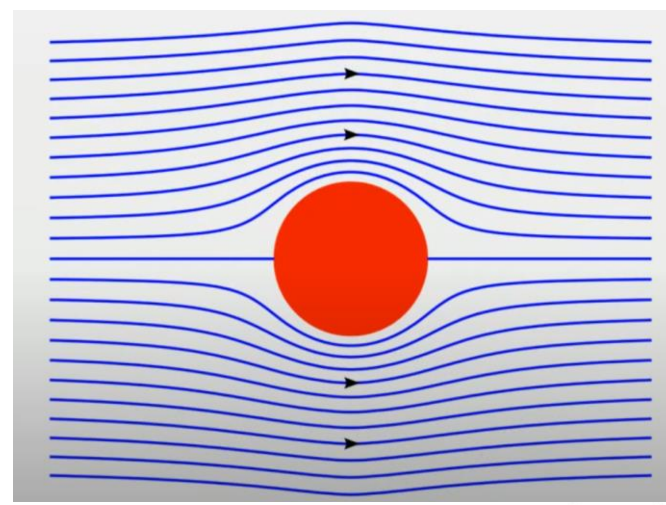
$\vec{V}.\vec{n} < 0$
Fluid would flow into the surface!

$\vec{V}.\vec{n} = 0$
Fluid flows tangent o the surface

- This can’t happen for an impermeable surface, therefore : $\vec{V}.\vec{n} = 0$ at the surface

- For an inviscid flow a tangential component of velocity can be present at the surface

- This means solid surface is a streamline and vice versa.
- Streamlines and solid surfaces are interchangeable!
- Fluid does not flow across the streamlines, only tangent to them!



❑ Stream Function for 2-D Flows

Streamlines and mass flow rate

- Consider two streamlines in a flow field
  - All the fluid that starts between these streamlines stays between them!

- In a steady flow $\dot{m} = const.$ at all locations.
- We can evaluate this mass flow rate in far upstream where $u = V_\infty$ and $v = 0$

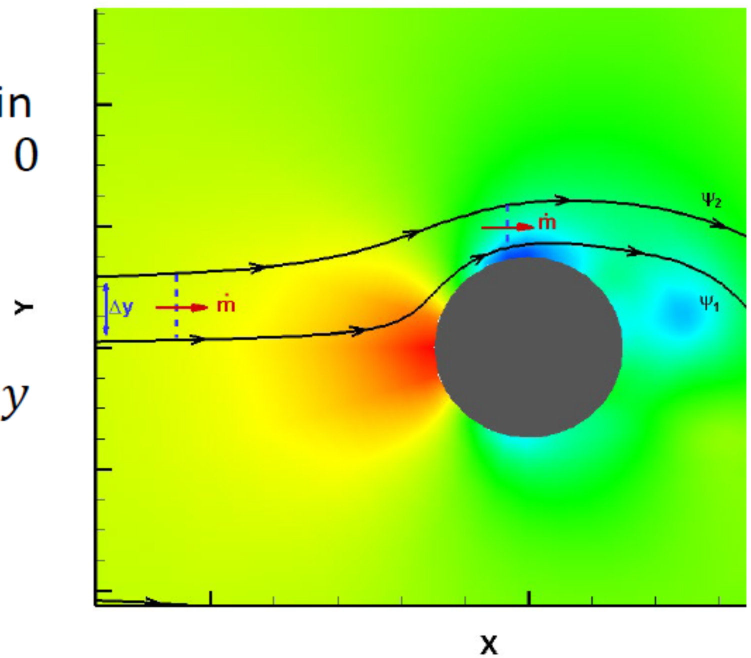
$$\dot{m} = \rho u A = \rho V_\infty \Delta y$$

- For a uniform flow

$$\psi = V_\infty y + c$$

$$\Delta\psi = \psi_2 - \psi_1 = V_\infty(y_2 - y_1) = V_\infty \Delta y$$

$$\rightarrow \Delta y = \frac{\Delta\psi}{V_\infty} \text{ and } V_\infty = \frac{\dot{m}}{\rho \Delta y}$$



Volume flow rate, $Q\ [\frac{m^3}{s}]$ →

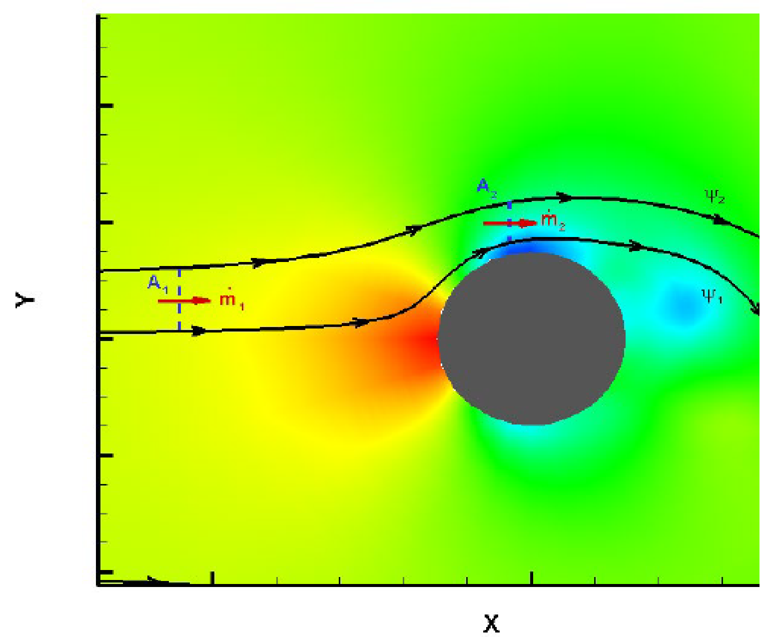
$$\frac{\dot{m}}{\rho} = \Delta\psi$$

❑ Stream Function for 2-D Flows

Streamline and mass flow rate

$$\dot{m}_1 = \dot{m}_2$$
$$\rho V_1 A_1 = \rho V_2 A_2$$
$$V_1 A_1 = V_2 A_2$$

$$A_2 < A_1 \rightarrow V_2 > V_1$$



- In an accelerating flow streamlines come together, and they move apart if flow is decelerating.
- If ψ is known, the flow rate can be calculated:

For example, if $\psi_1 = 0.0$ and ψ_2=0.5

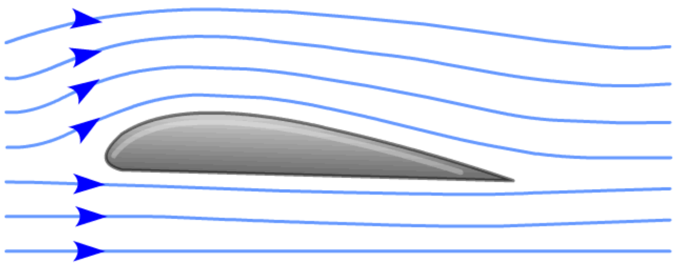
Then

$$\dot{m}_1 = \dot{m}_2 = \rho(\psi_2 - \psi_1) = \rho(0.5 - 0.0) = 0.5\rho$$

❑ Stream Function for 2-D Flows

Viscous flow example

This applies to viscous flows as well



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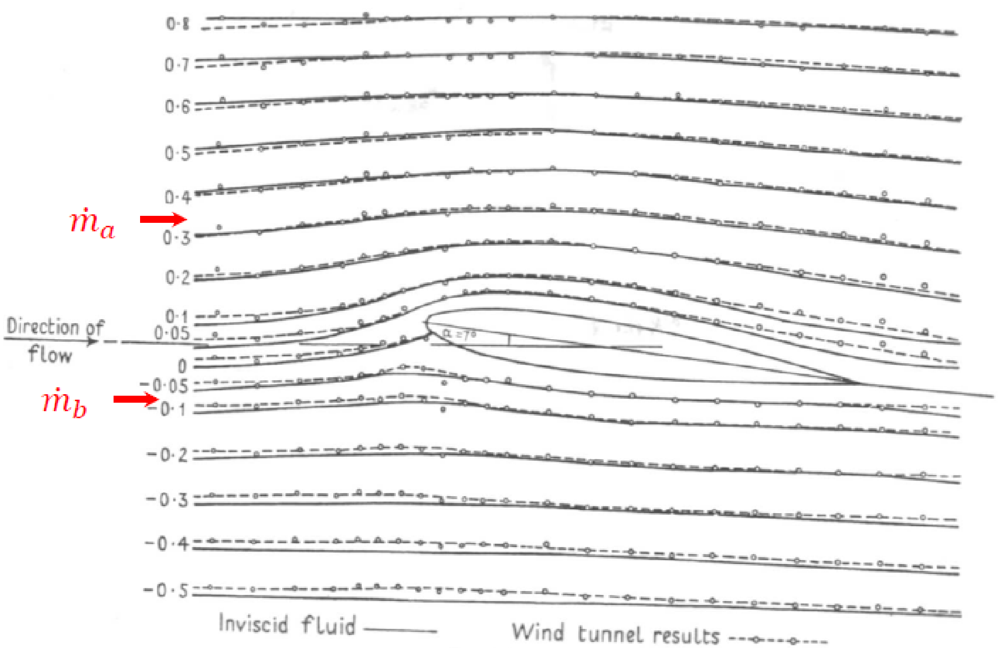

Fig. 180.

$$\dot{m}_a = \rho(0.4 - 0.3) = 0.1\rho$$

$$\dot{m}_b = \rho(-0.05 - (-0.1)) = 0.05\rho$$

❑ Stream Function for Irrotational 2-D Ideal Fluid Flow

Since flow is 2-D and irrotational ideal flow, then

In Cartesian system, $\vec{\Omega} = 0 \;\Rightarrow\; \Omega_z = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} = 0$

Since $\quad u = \dfrac{\partial \psi}{\partial y}; \qquad v = -\dfrac{\partial \psi}{\partial x}$

Thus, $$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial\left(-\frac{\partial \psi}{\partial x}\right)}{\partial x} - \frac{\partial\left(\frac{\partial \psi}{\partial y}\right)}{\partial y} = -\left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right] = 0$$

i.e., $$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \qquad \text{or} \qquad \nabla^2 \psi = 0$$ Laplace Equation

Laplace equation has solutions which are called as harmonic functions.