Lecture # 17: Potential Flow & Potential Function

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❑ Integral of Euler Equation in Irrotational Flows

Potential Flow:

- ***Definition:*** ***A non-heat conducting, homogeneous, inviscid, incompressible (i.e., ideal fluid), and irrotational flow is defined as potential flow.***

- ***For potential flows, the governing equations of the Fluid flow are:***

  1). Continuity equation:

$$\nabla \bullet \vec{V} = 0$$

  2). Euler equation

$$\frac{\partial \vec{V}}{\partial t} + \nabla\left(\frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho} - U\right) - \vec{V} \times (\nabla \times \vec{V}) = 0$$

$$\Rightarrow \qquad \frac{\partial \vec{V}}{\partial t} + \nabla\left(\frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho} - U\right) = 0$$

$$if\ U = -g\,Z \Rightarrow \qquad \frac{\partial \vec{V}}{\partial t} + \nabla\left(\frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho} + gZ\right) = 0$$

Potential Flow and Potential Function

Velocity Potential: Φ

- **Definition:** ***Velocity potential is defined only for ideal, irrotational flow for either steady or unsteady flows as:***

$$\vec{V} = \nabla \phi$$

- ***Since:***

$$\vec{V} = V_1 \hat{e}_1 + V_2 \hat{e}_2 + V_3 \hat{e}_3$$

$$\nabla \phi = \frac{\partial \phi}{h_1 \partial q_1} \hat{e}_1 + \frac{\partial \phi}{h_2 \partial q_2} \hat{e}_2 + \frac{\partial \phi}{h_3 \partial q_3} \hat{e}_3$$

- ***Therefore:***

$$V_1 = \frac{1}{h_1} \frac{\partial \phi}{\partial q_1}; \qquad V_2 = \frac{1}{h_2} \frac{\partial \phi}{\partial q_2}; \qquad V_3 = \frac{1}{h_3} \frac{\partial \phi}{\partial q_3}$$

❑ Potential Flow and Potential Function

***Velocity Potential:** Φ*

$$Since: \quad \vec{V} = \nabla\Phi \qquad and \qquad \nabla \bullet \vec{V} = 0$$

$$\Rightarrow \nabla \bullet \vec{V} = \nabla \bullet \nabla\Phi = \nabla^2\Phi = 0$$

- ***Therefore, the potential function satisfies the Laplace Equation.***

$$\nabla^2\Phi = 0$$

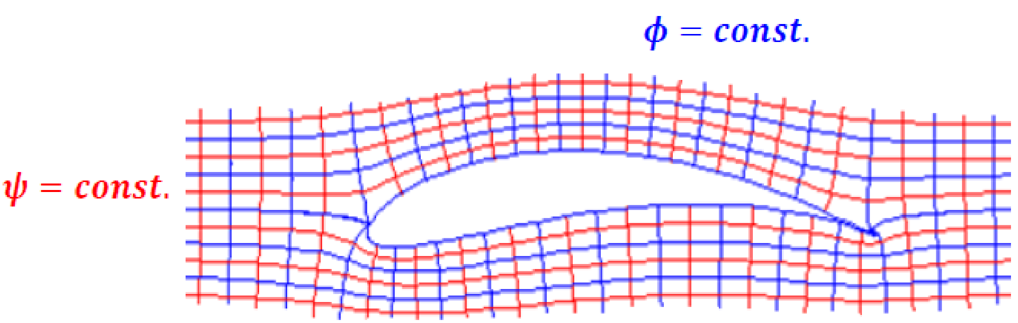
❑ Potential Flow and Potential Function

Potential function

- Stream function $\psi(x, y)$ is defined so that $\vec{V}$ is parallel to level contours at every point
- Potential function $\phi(x, y)$ is defined so that $\vec{V}$ is normal to level surfaces $\phi = const.$ at every point

Can define: $\vec{V} = \vec{\nabla}\phi$

i.e.: $$u = \frac{\partial \phi}{\partial x}\ ,\ v = \frac{\partial \phi}{\partial y}$$



The Joukowski airfoil
by John H. Mathews and Russell W. Howell
California State University Fullerton

$\psi = const.$ And $\phi = const.$ are normal to each other

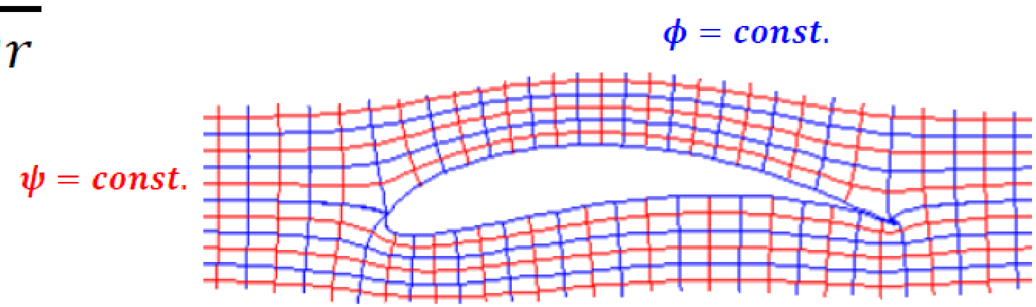
❑ Potential Flow and Potential Function

Stream function and potential function (2D)

- Stream function

Cartesian: $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$

Polar: $v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$, $v_\theta = -\frac{\partial \psi}{\partial r}$



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❑ Potential Flow and Potential Function

Potential function and irrotational flow

Recall vorticity

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\omega = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} = 0$$

Therefore, potential function can only be defined for irrotational flows

❑ Potential Flow and Potential Function

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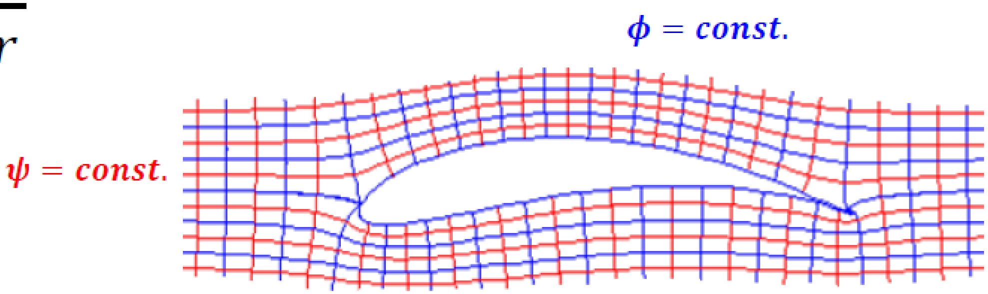

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Potential Flow and Potential Function

The stream function for an incompressible flow field is given by

$$\psi(x, y) = 3x^2y - y^3$$

Sketch the streamlines passing through the origin and find the potential function.

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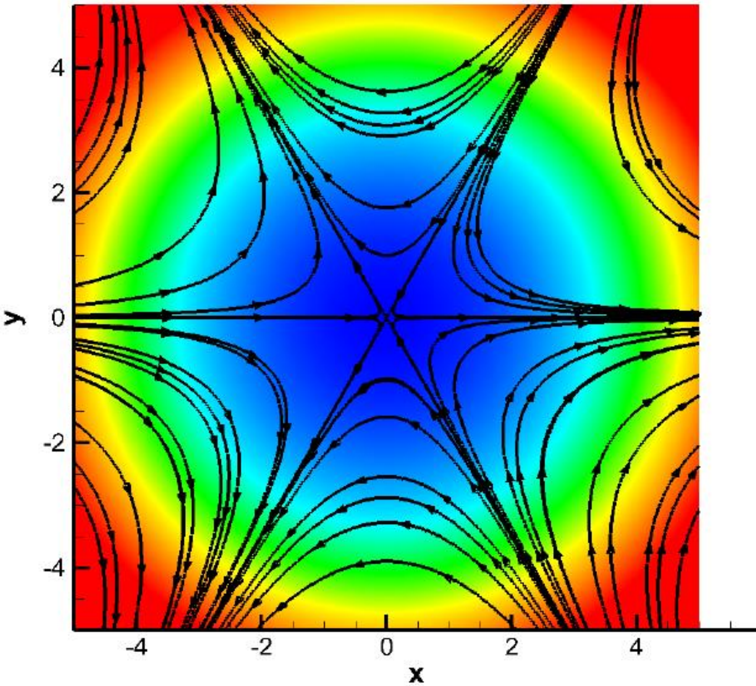
Sketch the streamlines passing through the origin and find the potential function.

$$u = 3x^2 - 3y^2, \qquad v = -6xy$$

Streamlines are curves of $\psi = const$. At origin $x = 0, y = 0 \rightarrow \psi = 0$ then

$3x^2y - y^3 = 0 \rightarrow y = 0$, and $y = \pm\sqrt{3}x$



$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}(3x^2y - y^3) = 3x^2 - 3y^2$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}(3x^2y - y^3) = -6xy$$

$$u = \frac{\partial \phi}{\partial x} \rightarrow \phi(x, y) = \int (3x^2 - 3y^2)dx = x^3 - 3y^2x + f(y)$$

$$v = \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}(x^3 - 3y^2x + f(y)) = -6xy + f'(y) \equiv -6xy$$

$$f'(y) = 0 \rightarrow f(y) = C$$

$$\boxed{\phi(x, y) = x^3 - 3y^2x}$$