Lecture # 18: Potential Flow & Potential Function

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Potential Flow and Potential Function

Velocity Potential: Φ

- **Definition:** ***Velocity potential is defined only for ideal, irrotational flow for either steady or unsteady flows as:***

$$\vec{V} = \nabla \phi$$

- ***Since:***

$$\vec{V} = V_1 \hat{e}_1 + V_2 \hat{e}_2 + V_3 \hat{e}_3$$

$$\nabla \phi = \frac{\partial \phi}{h_1 \partial q_1} \hat{e}_1 + \frac{\partial \phi}{h_2 \partial q_2} \hat{e}_2 + \frac{\partial \phi}{h_3 \partial q_3} \hat{e}_3$$

- ***Therefore:***

$$V_1 = \frac{1}{h_1} \frac{\partial \phi}{\partial q_1}; \qquad V_2 = \frac{1}{h_2} \frac{\partial \phi}{\partial q_2}; \qquad V_3 = \frac{1}{h_3} \frac{\partial \phi}{\partial q_3}$$

❑ Potential Flow and Potential Function

Stream function and potential function (2D)

- Stream function

Cartesian: $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$

Polar: $v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$, $v_\theta = -\frac{\partial \psi}{\partial r}$

- Potential function

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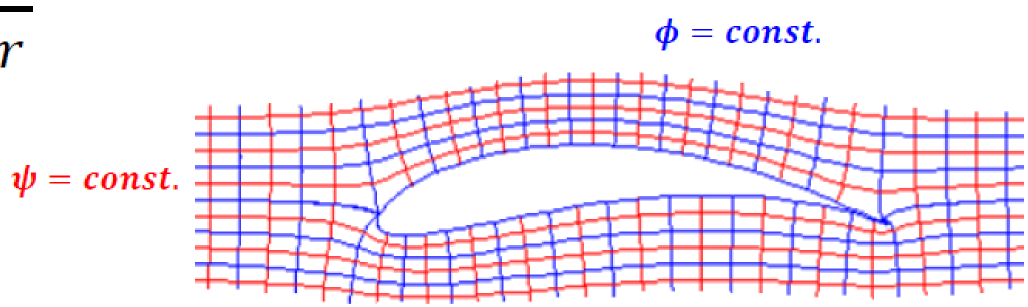

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$\psi = const.$ And $\phi = const.$ are normal to each other

Potential Flow and Potential Function

The stream function for an incompressible flow field is given by

$$\psi(x, y) = 3x^2 y - y^3$$

Sketch the streamlines passing through the origin and find the potential function.

❑ Potential Flow and Potential Function

The stream function for an incompressible flow field is given by

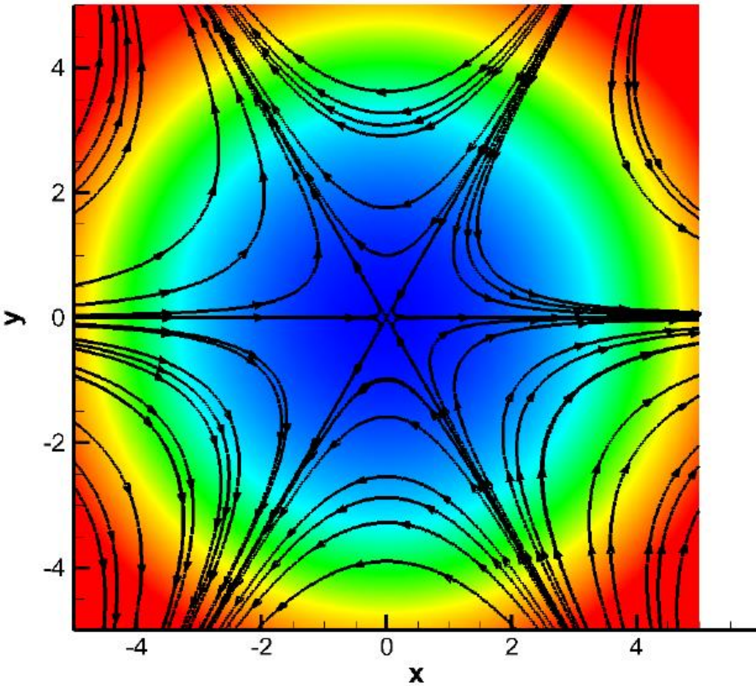
$$\psi(x,y) = 3x^2y - y^3$$

Sketch the streamlines passing through the origin and find the potential function.

$$u = 3x^2 - 3y^2, \qquad v = -6xy$$

Streamlines are curves of $\psi = const$. At origin $x = 0, y = 0 \rightarrow \psi = 0$ then

$3x^2y - y^3 = 0 \rightarrow y = 0$, and $y = \pm\sqrt{3}x$



$$u = \frac{\partial\psi}{\partial y} = \frac{\partial}{\partial y}(3x^2y - y^3) = 3x^2 - 3y^2$$

$$v = -\frac{\partial\psi}{\partial x} = -\frac{\partial}{\partial x}(3x^2y - y^3) = -6xy$$

$$u = \frac{\partial\phi}{\partial x} \rightarrow \phi(x,y) = \int (3x^2 - 3y^2)dx = x^3 - 3y^2x + f(y)$$

$$v = \frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}(x^3 - 3y^2x + f(y)) = -6xy + f'(y) \equiv -6xy$$

$$f'(y) = 0 \rightarrow f(y) = C$$

$$\boxed{\phi(x,y) = x^3 - 3y^2x}$$

❑ Potential Flow and Potential Function

The 2D flow of inviscid incompressible flow in the vicinity of a 'corner' is described by the stream function:

$$\psi(r,\theta) = 2r^2 \sin 2\theta$$

Find the components of velocity field and the potential function

Potential Flow and Potential Function

The 2D flow of inviscid incompressible flow in the vicinity of a 'corner' is described by the stream function:

$$\psi(r,\theta) = 2r^2 \sin 2\theta$$

Find the components of velocity field and the potential function

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial \theta}(2r^2 \sin 2\theta) = \frac{1}{r}(4r^2 \cos 2\theta)$$

$$\boxed{v_r = 4r \cos 2\theta}$$

$$v_\theta = -\frac{\partial \psi}{\partial r} = -\frac{\partial}{\partial r}(2r^2 \sin 2\theta) = -4r \sin 2\theta \rightarrow \boxed{v_\theta = -4r \sin 2\theta}$$

$$v_r = \frac{\partial \phi}{\partial r} \rightarrow \phi(r,\theta) = \int 4r \cos 2\theta \, dr = 2r^2 \cos 2\theta + f(\theta)$$

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial \theta}(2r^2 \cos 2\theta + f(\theta)) = \frac{1}{r}(-4r^2 \sin 2\theta + f'(\theta)) \equiv -4r \sin 2\theta$$

$$\rightarrow f'(\theta) = 0 \rightarrow f(\theta) = C$$

$$\boxed{\phi(r,\theta) = 2r^2 \cos 2\theta + C}$$

❑ Potential Flow and Potential Function

When could ψ and ϕ be defined?

1. A velocity field that does not satisfy continuity : Not physical! $\rightarrow \psi$ can not be defined and there is no reason to define ϕ!

Example: $u = e^x y$, $v = 2xy$

Continuity : $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = e^x y + 2x \neq 0$ everywhere in (x, y) plane

$$u = \frac{\partial \psi}{\partial y} \rightarrow \psi(x, y) = \int e^x y \, dy = e^x \frac{y^2}{2} + f(x)$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\left(e^x \frac{y^2}{2} + f(x)\right) = -e^x \frac{y^2}{2} - f'(x) \equiv 2xy$$

$$\rightarrow f'(x) = -2xy - e^x \frac{y^2}{2}$$

$f'(x)$ can not be a function of y! Therefore, ψ can not be defined.

Same issue will come up calculating ϕ

❑ Potential Flow and Potential Function

Rotational flow example

2. A velocity field that satisfies continuity, but it is rotational: We can define ψ but not ϕ

Example: $u = xy \;,\; v = -\frac{y^2}{2}$

Continuity : $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = y - y = 0$ ✓ satisfied

$$u = \frac{\partial \psi}{\partial y} \rightarrow \psi(x,y) = \int xy\,dy = x\frac{y^2}{2} + f(x)$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\left(x\frac{y^2}{2} + f(x)\right) = -\frac{y^2}{2} - f'(x) \equiv -\frac{y^2}{2}$$

$$\rightarrow f'(x) = 0 \rightarrow f = C = \text{const.}$$

$$\psi(x,y) = x\frac{y^2}{2} + C$$

Potential Flow and Potential Function

Rotational flow example

$$u = \frac{\partial \phi}{\partial x} \rightarrow \phi(x, y) = \int xy\, dx = \frac{1}{2}x^2 y + f(y)$$

$$v = \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{2}x^2 y + f(y)\right) = \frac{1}{2}x^2 + f'(y) \equiv -\frac{y^2}{2}$$

$$\rightarrow f'(y) = -\frac{1}{2}(x^2 + y^2)$$

Again f' can not be a function of x, therefore ϕ can not be defined.

❑ Potential Flow and Potential Function

Irrotational flow example

3. A velocity field that is irrotational ($\omega = 0$) but doesn't satisfy continuity : We can define a ϕ but not ψ. This is still an unphysical field.
4. A velocity field that satisfies continuity and is irrotational: Both ψ and ϕ can be defined

Example: $u = 2x \;\; , \;\; v = -2y$

Continuity : $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 2 - 2 = 0$ ✓ satisfied

$$u = \frac{\partial \psi}{\partial y} \rightarrow \psi(x,y) = \int 2x\, dy = 2xy + f(x)$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\big(2xy + f(x)\big) = -2y - f'(x) \equiv -2y$$

$$\rightarrow f'(x) = 0 \rightarrow f = C = \text{const.}$$

$$\psi(x,y) = 2xy + C$$

Potential Flow and Potential Function

Irrotational flow example

$$u = \frac{\partial \phi}{\partial x} \rightarrow \phi(x, y) = \int 2x \, dx = x^2 + f(y)$$

$$v = \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}(x^2 + f(y)) = f'(y) \equiv -2y$$

$$\rightarrow f(y) = -y^2 + C$$

$$\phi(x, y) = x^2 - y^2 + C$$

Potential Flow and Potential Function

Stream and potential flow example

The velocity component in a steady, incompressible 2D flow field are: $u = 2y$, $v = 4x$

Determine stream function and draw streamlines. Indicate the direction of flow along the streamlines.

$$u = \frac{\partial \psi}{\partial y} = 2y \rightarrow \psi(x, y) = \int 2y \, dy = y^2 + f(x)$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\left(y^2 + f(x)\right) = -f'(x) \equiv 4x$$

$$\rightarrow f'(x) = -4x \rightarrow f(x) = -2x^2 + C$$

$$\psi(x, y) = -2x^2 + y^2 + C$$

C is an arbitrary constant and we can choose it to be zero. Therefore:

$$\psi(x, y) = -2x^2 + y^2$$

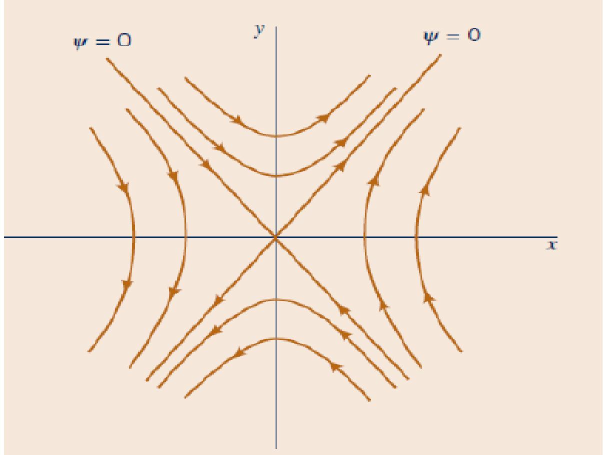
❑ Potential Flow and Potential Function

Example -continued

Streamlines can be determined by setting $\psi = const.$ and plotting the resulting function.

Let's start with $\psi = 0 \rightarrow -2x^2 + y^2 = 0 \rightarrow y = \pm\sqrt{2}x$

Straight lines going through origin



For $\psi = a$

$$-2x^2 + y^2 = a \rightarrow \frac{y^2}{a} - \frac{x^2}{a/2} = 1$$

equation of a hyperbola

To find the streamline direction

If $x > 0 \rightarrow v = 4x > 0$ and if $x < 0 \rightarrow v < 0$

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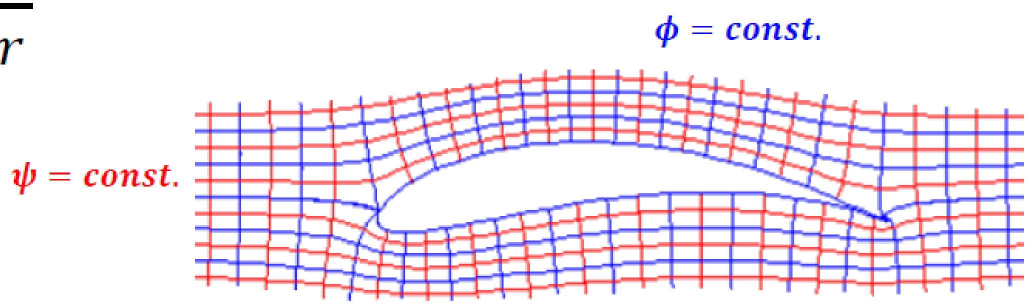

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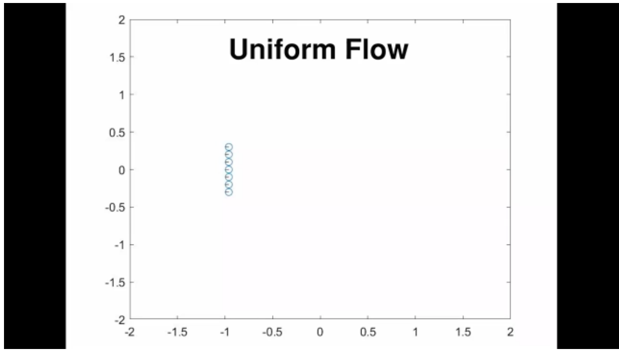
❑ Potential Flow and Potential Function

For 2-D flow

- Any irrotational and incompressible flow has a velocity potential ϕ and stream function ψ that both satisfy Laplace equation.
- Conversely any solution represents the velocity potential ϕ or stream function ψ for an irrotational and incompressible flow.
- A powerful procedure for solving irrotational flow problems is to represent ϕ and ψ by linear combinations of known solutions of Laplace equation.

$$\phi = \sum_{i=1}^{N} C_i \phi_i; \qquad \psi = \sum_{i=1}^{N} C_i \psi_i$$



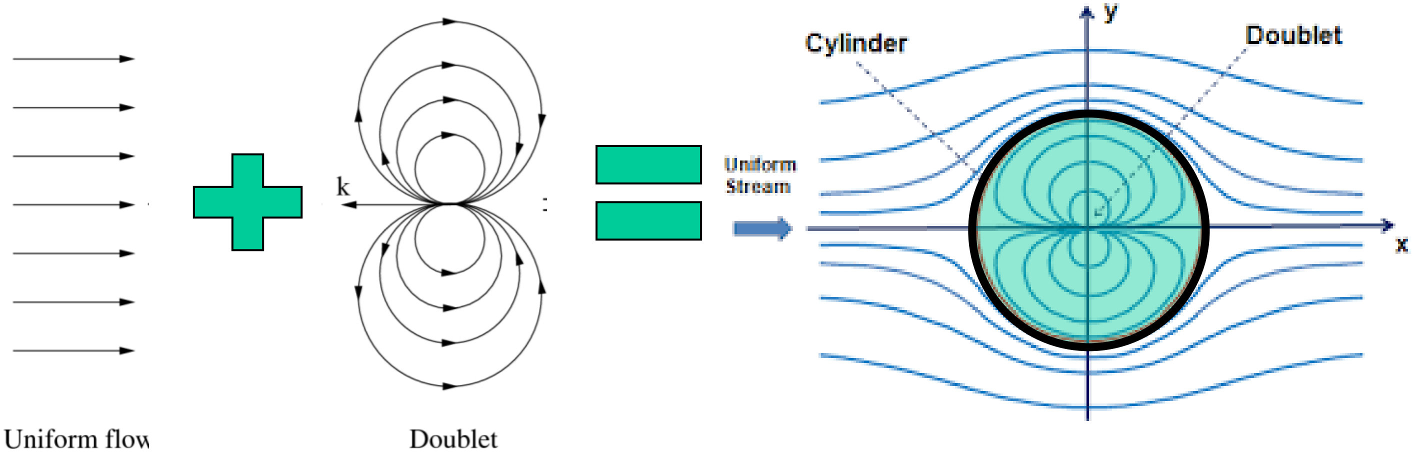
Finding the coefficients C_i so that the boundary conditions are satisfied both far from the body and the body surface.

Say ϕ_1 and ϕ_2 are solutions of $\nabla^2\phi = 0$

i.e., $\nabla^2\phi_1 = 0; \qquad \nabla^2\phi_2 = 0$

Then $\phi = A_1\phi_1 + A_2\phi_2$ is also a solution of the Laplace equation

A complicated flow pattern for an irrotational and incompressible flow can be synthesized by adding together a number of elementary flows which are also irrotational and incompressible.



❑ Potential Flow and Potential Function

The Advantages of using stream function and potential function to solve aerodynamics problems

1. It can replace the nonlinear flow controlling equations (such as N-S equations) by Linear equations (Laplace equation).
2. It reduces the number of unknown numbers, i.e., $\left.\begin{matrix} p \\ u \\ v \\ w \\ \rho \end{matrix}\right\} \Rightarrow \quad \psi \quad or \quad \phi$
3. The solutions of the Laplace equation are adjustable.
4. A complex flow can be decomposed as the summation of many simple flows.