Lecture # 19: Stream & Potential Functions for Basic Flows – Part 1

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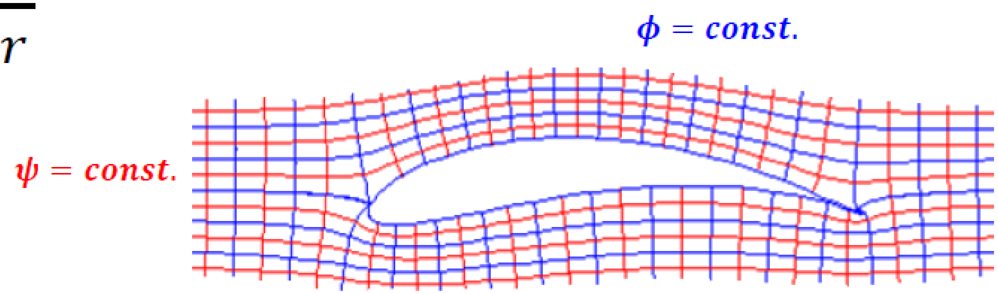
❑ Potential & Stream Functions for Basic Flows

Stream function and potential function (2D)

- Stream function

Cartesian: $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$

Polar: $v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$, $v_\theta = -\frac{\partial \psi}{\partial r}$



The Joukowski airfoil
by John H. Mathews and Russell W. Howell
California State University Fullerton

- Potential function

Cartesian: $u = \frac{\partial \phi}{\partial x}$, $v = \frac{\partial \phi}{\partial y}$

Polar: $v_r = \frac{\partial \phi}{\partial r}$, $v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$

$\psi = const.$ And $\phi = const.$ are normal to each other

Potential & Stream Functions for Basic Flows

Uniform Flow

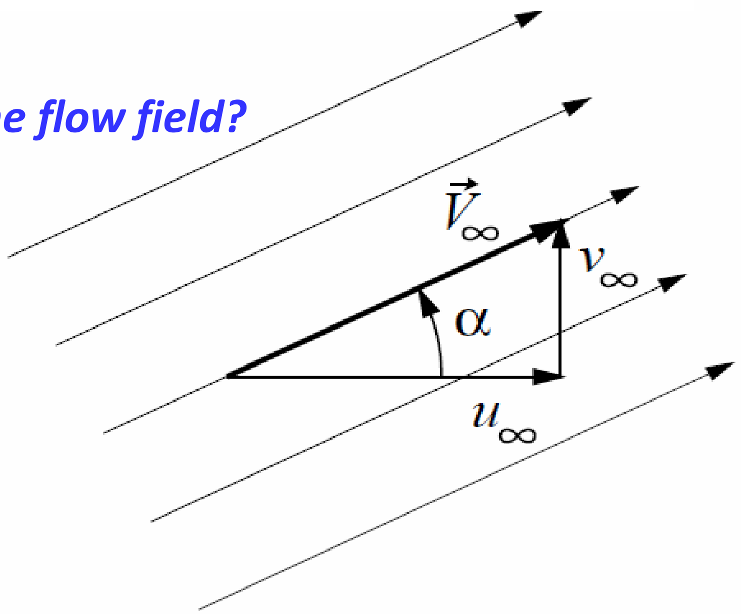
Definition

A *uniform flow* consists of a velocity field where $\vec{V} = u\hat{\imath} + v\hat{\jmath}$ is a constant. In 2-D, this velocity field is specified either by the freestream velocity components u_∞, v_∞, or by the freestream speed V_∞ and flow angle α.

$$\begin{aligned} u &= u_\infty = V_\infty \cos\alpha \\ v &= v_\infty = V_\infty \sin\alpha \end{aligned}$$

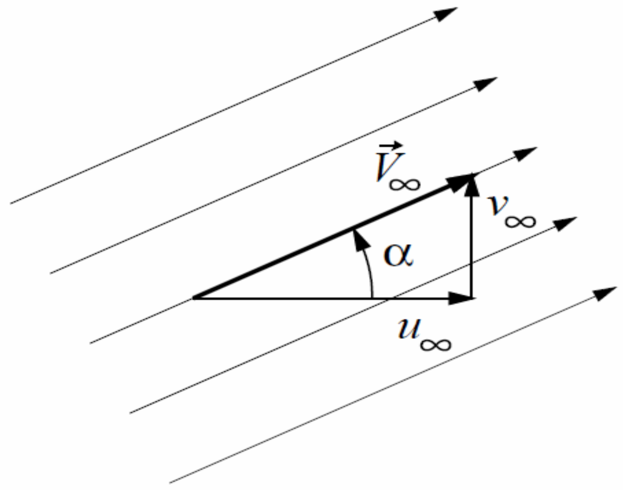
Note also that $V_\infty^2 = u_\infty^2 + v_\infty^2$.

- ***What are the stream and potential functions of the flow field?***



Potential & Stream Functions for Basic Flows

- ***Stream function***



Since u and v are both constants, $\nabla \cdot \vec{V} = 0$
Therefore ψ exists. From conservation of mass,

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$

$$u = V_\infty \cos\alpha = \frac{\partial \psi}{\partial y}$$

$$\psi = V_\infty \cos\alpha \;\; y + f(x)$$

$$\frac{\partial \psi}{\partial x} = -v = 0 + f'(x)$$

$$f'(x) = -V_\infty \sin\alpha \quad or \quad f(x) = -V_\infty \sin\alpha \;\; x + g(y)$$

$$\psi = -V_\infty \sin\alpha \;\; x + V_\infty \cos\alpha \;\; y$$

$$\psi = const. = -V_\infty \sin\alpha \;\; x + V_\infty \cos\alpha \;\; y$$

$$\frac{\psi}{V_\infty} = -\sin\alpha \;\; x + \cos\alpha \;\; y$$

$$\frac{\psi}{V_\infty \cos\alpha} = -\tan\alpha \;\; x + y$$

Equation of streamlines:

$$y = tan\alpha \;\; x + \frac{\psi}{V_\infty \cos\alpha}$$

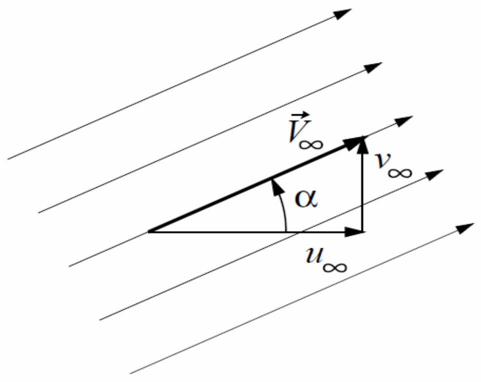
❑ Potential & Stream Functions for Basic Flows

- ***Potential function***

Check if the given flow is a potential flow?

$$\nabla \times \vec{V} \stackrel{?}{=} 0$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \stackrel{?}{=} 0$$



Since V_∞ and α are constant throughtout the flow, $\nabla \times V = 0$
Therefore ϕ exists and $\vec{V} = \nabla\phi$.

$$\vec{V} = \nabla\phi = \frac{\partial \phi}{\partial x}\hat{\imath} + \frac{\partial \phi}{\partial y}\hat{\jmath}$$

$$\frac{\partial \phi}{\partial x} = V_\infty \cos\alpha = u$$

$$\phi = V_\infty \cos\alpha \;\; x + f(y)$$

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) = v$$

$$f'(y) = V_\infty \sin\alpha \quad or \quad f(y) = V_\infty \sin\alpha \;\; y + f(x)$$

$$\phi = V_\infty \cos\alpha \;\; x + V_\infty \sin\alpha \;\; y \quad (\text{uniform flow at an angle } \; \alpha)$$

$$\phi = const. = V_\infty \cos\alpha \;\; x + V_\infty \sin\alpha \;\; y$$

$$\frac{\phi}{V_\infty \sin\alpha} = \frac{x}{\tan\alpha} + y$$

Equation of Equipotential lines:

$$y = -\frac{1}{\tan\alpha}x + \frac{\phi}{V_\infty \sin\alpha}$$

ϕ constant lines are orthogonal to ψ constant lines.

❑ Potential & Stream Functions for Basic Flows

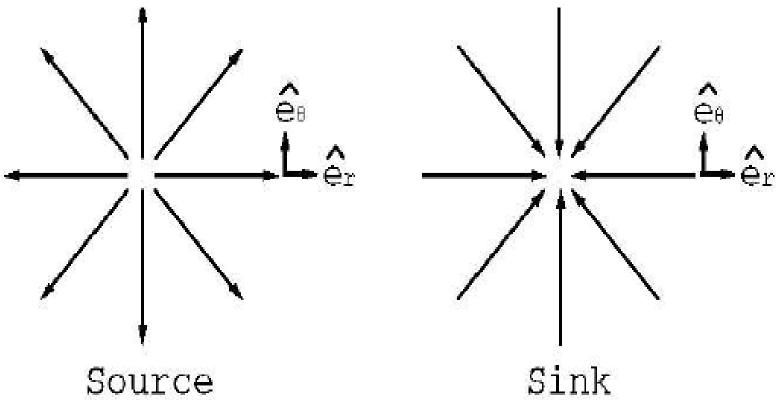
- ## *2D Source or Sink flow*

Definition: A source is a point from which fluid issues along radial lines. Streamlines are straight lines emanating from a central point. Velocity varies inversely with distance from the origin.

From the definition of the source the velocity vector can be written as:

$$\vec{V} = v_r \hat{e}_r$$

where $v_r \propto \frac{1}{r}$ or $v_r = \frac{c}{r}$, and $v_\theta = 0$ where C is a constant.
Check if the assumed flow is physically possible.



$$\vec{V} = \frac{c}{r}\hat{e}_r + 0\hat{e}_\theta$$

$$\nabla \cdot \vec{V} \stackrel{?}{=} 0$$

$$\nabla \cdot \vec{V} = \frac{1}{r}\left[\frac{\partial(r v_r)}{\partial r} + \frac{\partial(v_\theta)}{\partial \theta}\right] = \frac{1}{r}\left[\frac{\partial(c)}{\partial r} + \frac{\partial(0)}{\partial \theta}\right] \equiv 0$$

Flow is physically possible and ψ exists.

❑ Potential & Stream Functions for Basic Flows

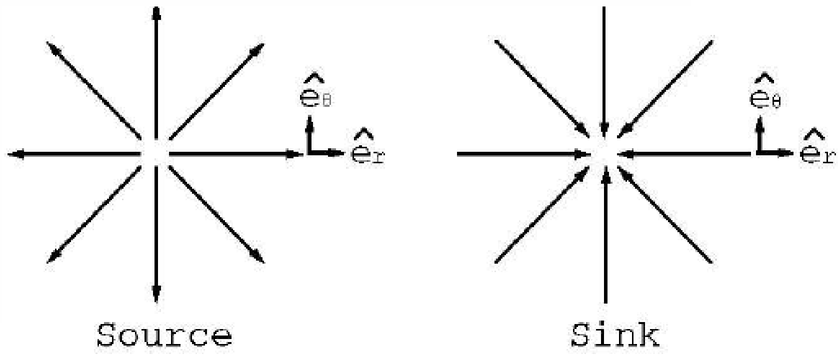
- ## *2D Source or Sink flow*

Define K as the source strength. It is physically the rate of volume flow from the source per unit depth into the page (2-D).

$$K = 2\pi c \quad or \quad c = \frac{K}{2\pi}$$

then the velocity becomes:

$$v_r = \frac{K}{2\pi r}$$



Is the flow irrotational?

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial r}(-c) - \frac{1}{r}\frac{\partial}{\partial \theta}(0) = 0$$

- ***What are the stream and potential functions of the flow field?***

❑ Potential & Stream Functions for Basic Flows

- ***Stream function***

Since $\nabla \cdot \vec{V} = 0$ is satisfied, the flow is physically possible and from the definition of ψ in polar coordinates, ψ can be found.

$$v_r = \frac{\partial \psi}{r \partial \theta} \quad and \quad v_\theta = -\frac{\partial \psi}{\partial r}$$

$$\frac{\partial \psi}{r \partial \theta} = \frac{K}{2\pi r}$$

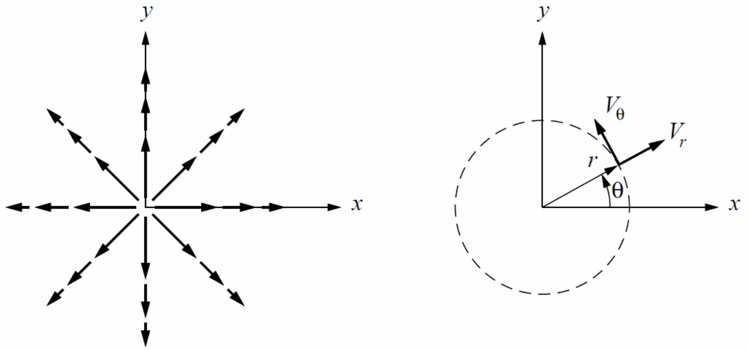
$$\frac{\partial \psi}{\partial \theta} = \frac{K}{2\pi}$$

$$\psi = \frac{K}{2\pi}\theta + f(r)$$

$$\frac{\partial \psi}{\partial r} = -v_\theta = 0 + f'(r) = 0$$

$$f(r) = const.$$

$$\psi = \frac{K}{2\pi}\theta$$



Since the source strength, K is a constant, ψ constant lines are radial lines.

$$\psi = \frac{K}{2\pi}\theta = const.$$

Potential & Stream Functions for Basic Flows

- ***Potential function***

Is $\nabla \times \vec{V} \stackrel{?}{=} 0.$

$$\nabla \times \vec{V} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ v_r & rv_\theta & v_z \end{vmatrix} = \frac{1}{r}\left[\frac{\partial(rv_\theta)}{\partial r} - \frac{\partial(v_r)}{\partial \theta}\right]\hat{e}_z \equiv 0$$

therefore ϕ exists.

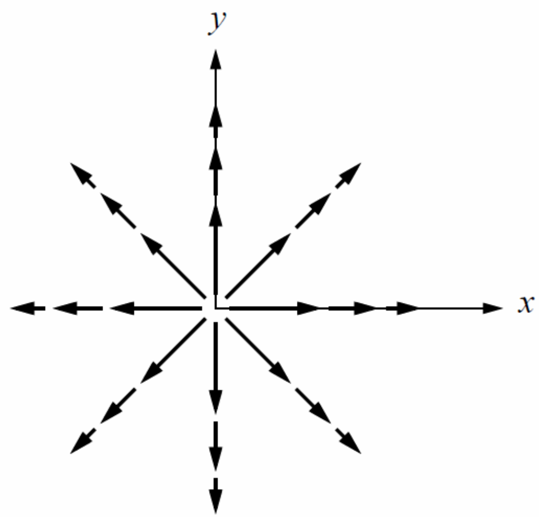
$$v_r = \frac{\partial \phi}{\partial r} = \frac{K}{2\pi r} \quad and \quad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0$$

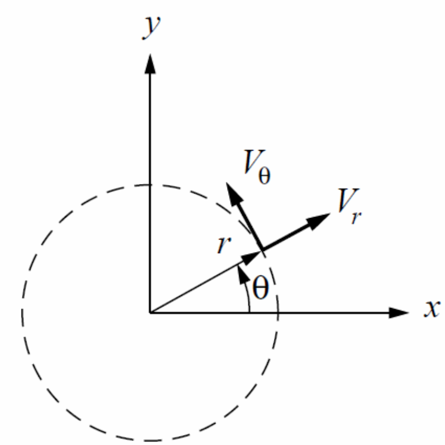
$$\phi = \frac{K}{2\pi}\ln r + g(\theta)$$

$$\frac{\partial \phi}{\partial \theta} = 0 + g'(\theta) = 0$$

$$g(\theta) = const.$$

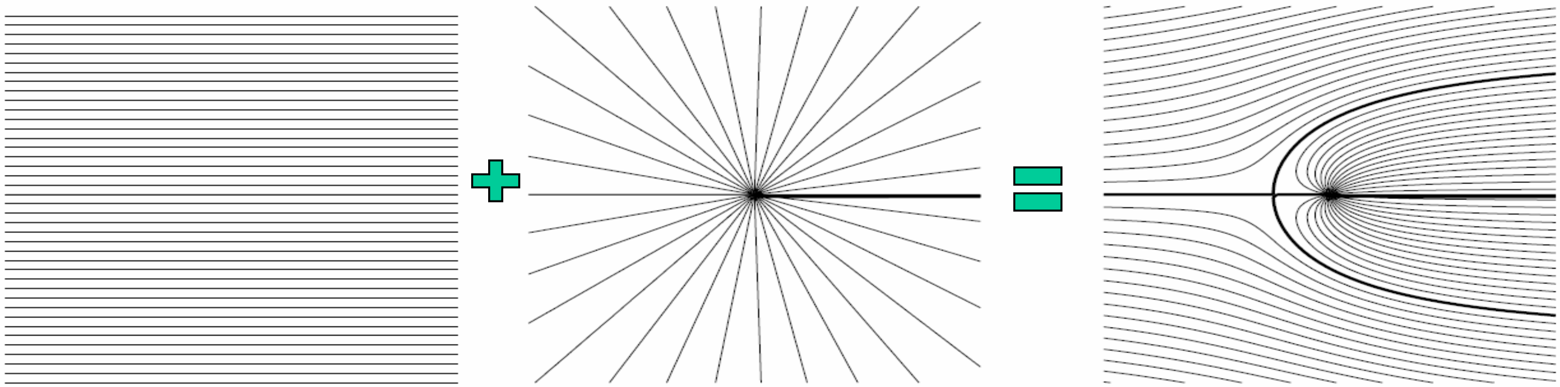
$$\therefore \phi = \frac{K}{2\pi}\ln r$$





❑ Potential & Stream Functions for Basic Flows

- ***Uniform Flow with Source***



Quantity	Uniform flow	Source/Sink	Combination
$\vec{V}$	$V_\infty \hat{\imath}$	$\pm \frac{K}{2\pi r} \hat{e}_r$	$V_\infty \hat{\imath} \pm \frac{K}{2\pi r} \hat{e}_r$
ϕ	$V_\infty x$	$\pm \frac{K}{2\pi} \ln r$	$V_\infty x \pm \frac{K}{2\pi} \ln r$
ψ	$V_\infty y$	$\pm \frac{K}{2\pi} \theta$	$V_\infty y \pm \frac{K}{2\pi} \theta$

Stagnation Point:

At the stagnation point, $\vec{V} \equiv 0$

$$v_r = V_\infty \cos\theta + \frac{K}{2\pi r} = 0$$

$$v_\theta = -V_\infty \sin\theta = 0$$

Solve for θ and r at the stagnation point to get $(r_{stag}, \theta_{stag})$. Proceed to find ψ_{stag} to get the shape of the body. From $v_\theta = 0$:

$$\sin\theta = 0$$

$$\theta = 0 \;\; or \;\; \pm\pi$$

Potential & Stream Functions for Basic Flows

- *Case 1: $\theta_s = 0$. Solve for r_s from $v_r = 0$*

$$v_r = V_\infty \cos\theta + \frac{K}{2\pi r} = 0$$

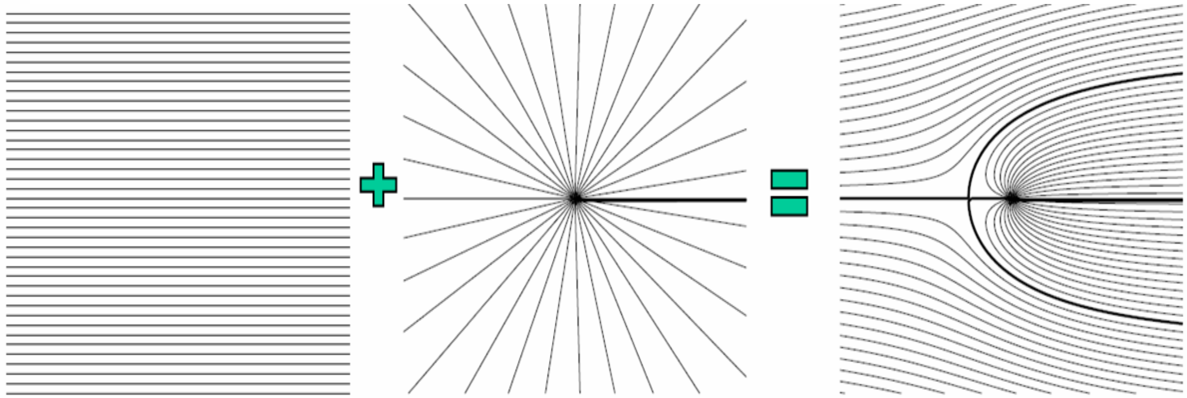
$$\text{if} \quad \theta = \theta_s = 0$$

$$\cos\theta_s = 1$$

$$v_r = V_\infty + \frac{K}{2\pi r_s} = 0$$

$$\text{or} \quad r_s = -\frac{K}{2\pi V_\infty}$$

Impossible solution since $r_s > 0$



- *Case 2: $\theta_s = \pm\pi$. Solve for r_s from $v_r = 0$*

$$v_r = -V_\infty + \frac{K}{2\pi r_s} = 0$$

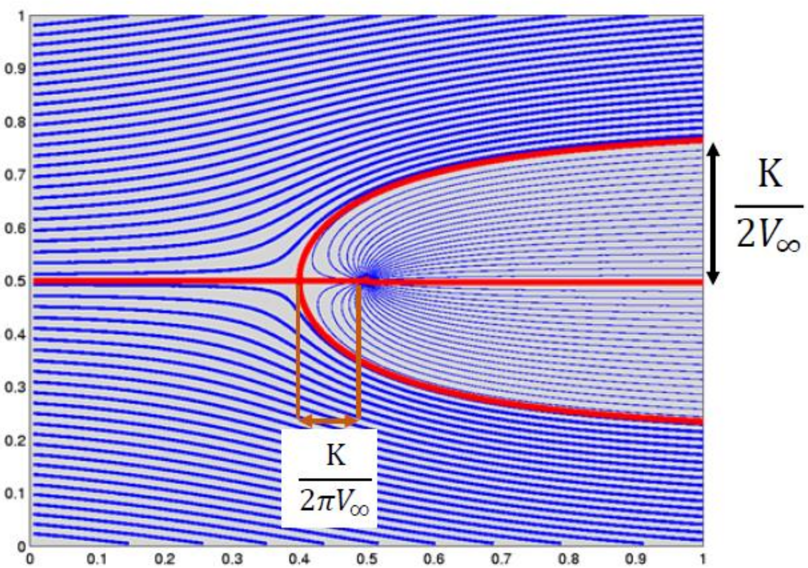
$$r_s = \frac{K}{2\pi V_\infty}$$

$\theta_s = +\pi$ for the upper half of the body

$\theta_s = -\pi$ for the lower half of the body

Coordinates of the stagnation point:

$$(r_s, \theta_s) = \left(\frac{K}{2\pi V_\infty}, \pm\pi\right)$$



❑ Potential & Stream Functions for Basic Flows

Body Shape (Stagnation Streamline):

A general expression for the streamfunction for the combined flow is:

$$\psi = V_\infty r \sin\theta + \frac{K}{2\pi}\theta$$

Find ψ_s (= the body shape) by substituting (r_s, θ_s).

$$\psi_s = V_\infty r \sin(\pm\pi) + \frac{K}{2\pi}(\pm\pi) = \pm\frac{K}{2} = const.$$

In Cartesian coordinate, the general expression for the body streamfunction becomes:

$$\pm\frac{K}{2} = V_\infty y + \frac{K}{2\pi}\tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{K}{2\pi}\tan^{-1}\left(\frac{y}{x}\right) = \left(\pm\frac{K}{2} - V_\infty y\right)$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \left(\pm\pi - \frac{2\pi V_\infty y}{K}\right)$$

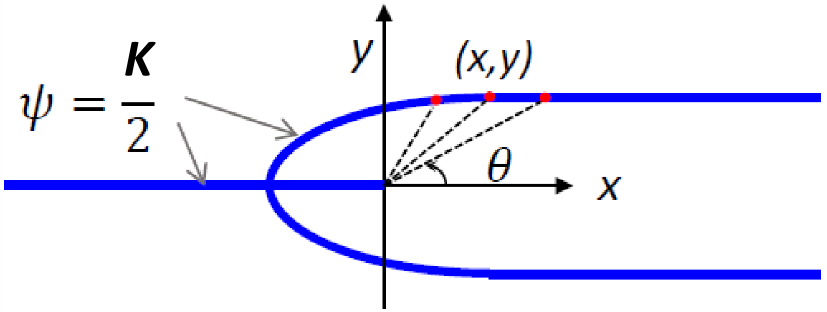
$$\frac{y}{x} = \tan\left(\pm\pi - \frac{2\pi V_\infty y}{K}\right)$$

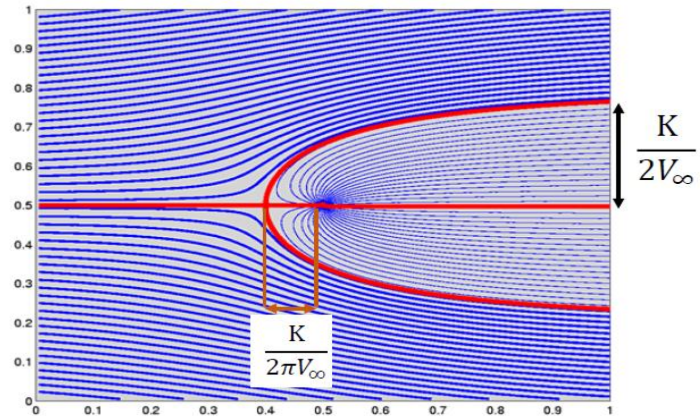
$$x = \frac{y}{\tan\left(\pm\pi - \frac{2\pi V_\infty y}{K}\right)}$$

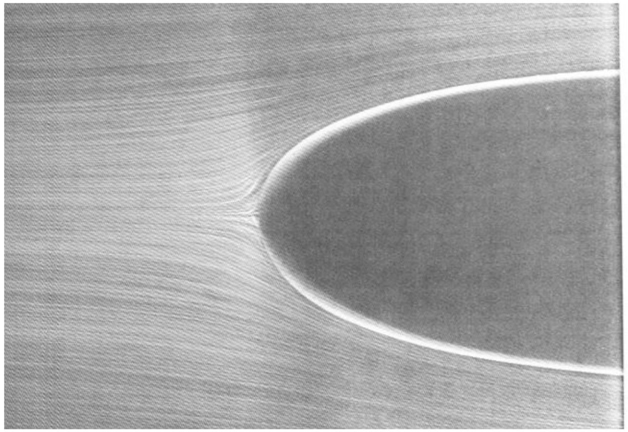
To find maximum y value, consider $\psi = K/2$ (upper half of the body).

$$\frac{K}{2} = V_\infty y + \frac{K}{2\pi}\tan^{-1}\left(\frac{y}{x}\right)$$

$$y_{max} = y_{@x=\infty} = \frac{K}{2V_\infty}$$







❑ Potential & Stream Functions for Basic Flows

6.3.2 Combined Flow of a Source at $(-b, 0)$ and a Sink at $(b, 0)$

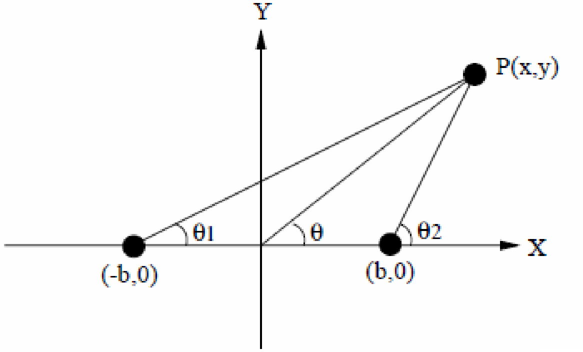
$$\psi = \frac{K}{2\pi}\theta_1 - \frac{K}{2\pi}\theta_2$$

where θ_1 and θ_2 are measured from the center of the source and sink respectively.

$$\theta_1 = \tan^{-1}\left(\frac{y}{x+b}\right), \quad \theta_2 = \tan^{-1}\left(\frac{y}{x-b}\right)$$

$$\theta_2 - \theta_1 = \tan^{-1}\left(\frac{y}{x-b}\right) - \tan^{-1}\left(\frac{y}{x+b}\right)$$

$$\theta_2 - \theta_1 = \tan^{-1}\left(\frac{2by}{x^2+y^2-b^2}\right)$$



$$\psi = \psi_1 + \psi_2 = \frac{K}{2\pi}(\theta_1 - \theta_2)$$

$$\theta_1 - \theta_2 = -\tan^{-1}\left(\frac{2by}{x^2+y^2-b^2}\right)$$

$$\psi_{source+sink} = -\frac{K}{2\pi}\tan^{-1}\left(\frac{2by}{x^2+y^2-b^2}\right)$$

$$\frac{2\pi\psi}{K} = -\tan^{-1}\left(\frac{2by}{x^2+y^2-b^2}\right)$$

$$\tan\left(\frac{2\pi\psi}{K}\right) = -\frac{2by}{x^2+y^2-b^2}$$

$$x^2 + y^2 + 2by\cot\left(\frac{2\pi\psi}{K}\right) = b^2$$

$$(x-0)^2 + \left(y + b\cot\left[\frac{2\pi\psi}{K}\right]\right)^2 = b^2\left(1 + \cot^2\left[\frac{2\pi\psi}{K}\right]\right)$$

$$(x-0)^2 + \left(y + b\cot\left[\frac{2\pi\psi}{K}\right]\right)^2 = b^2\csc^2\left[\frac{2\pi\psi}{K}\right]$$

Equation of a circle with center at $\left(0, \pm b\cot\frac{2\pi\psi}{K}\right)$ and radius of $\left(b\csc\frac{2\pi\psi}{K}\right)$. When $y = 0$, $x = \pm b$. All streamlines go through $\pm b$.

❑ Potential & Stream Functions for Basic Flows

- *Uniform Flow to the Right + Source (- b; 0)+ Sink (b; 0) (Rankine oval)*

  - The combined stream function

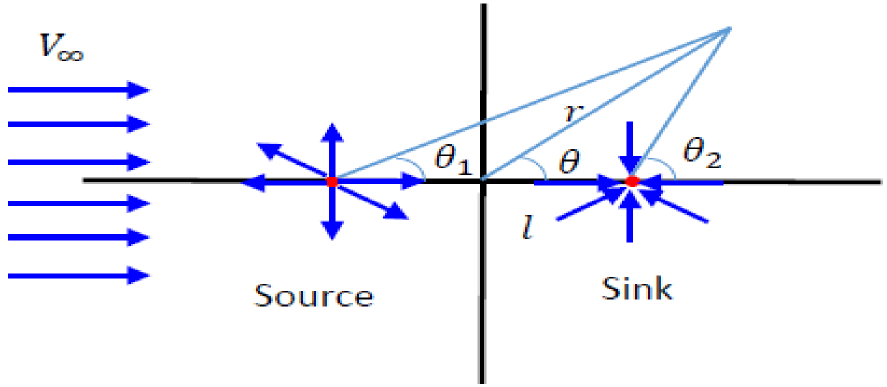
$$\psi = V_\infty r \sin\theta + \frac{K}{2\pi}\theta_1 - \frac{K}{2\pi}\theta_2$$

  - To find stagnation points, note that those will lie on the horizontal line going through source/sink center.

Adding velocity contribution from each term

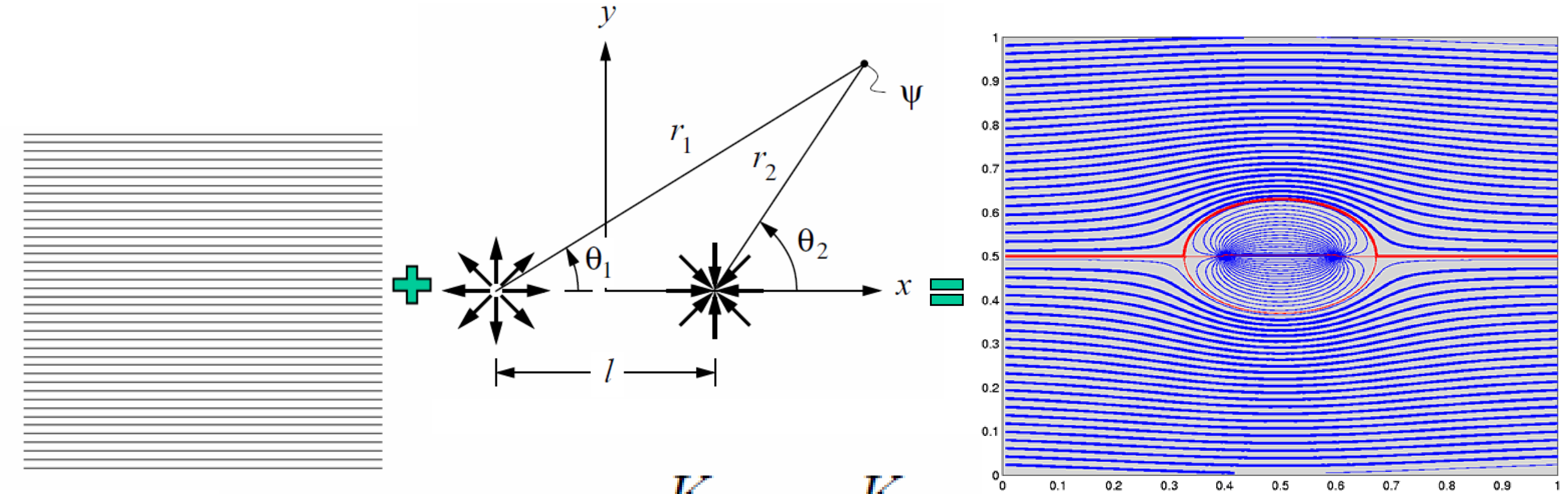
$$V_\infty + \frac{K}{2\pi(r+l)} - \frac{K}{2\pi(r-l)} = 0$$

$$V_\infty - \frac{K}{\pi(r^2 - l^2)} = 0 \rightarrow r = \pm\sqrt{l^2 + \frac{Kl}{\pi V_\infty}}$$

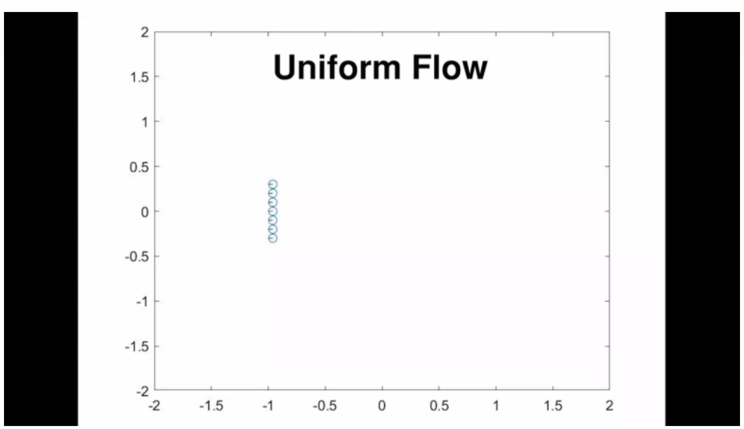


Potential & Stream Functions for Basic Flows

- *Uniform Flow to The Right + Source (- b; 0)+ Sink (b; 0) (Rankine oval)*



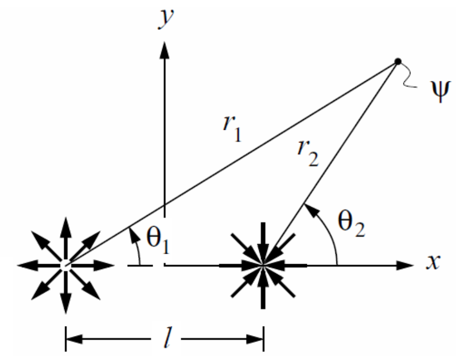
$$\psi = V_\infty r \sin\theta + \frac{K}{2\pi}\theta_1 - \frac{K}{2\pi}\theta_2$$



❑ Potential & Stream Functions for Basic Flows

• *2D Doublet flow*

Definition: A doublet is obtained when a source and sink of equal strength approach each other so that the product of their strength and the distance apart remains a constant.



- Consider a pair of source/sink with equal strength of K separated by a distance l. The stream function is

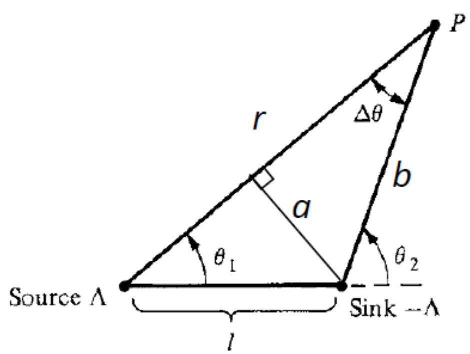
$$\psi = \frac{K}{2\pi}(\theta_1 - \theta_2) = -\frac{K}{2\pi}\Delta\theta$$

- Now let l go toward zero
- If K remains constant, the net flow will be zero. Sink cancels the source!
  - Instead consider a case where K increases at the same rate as l decreases such that

$$Kl = \mu = const.$$

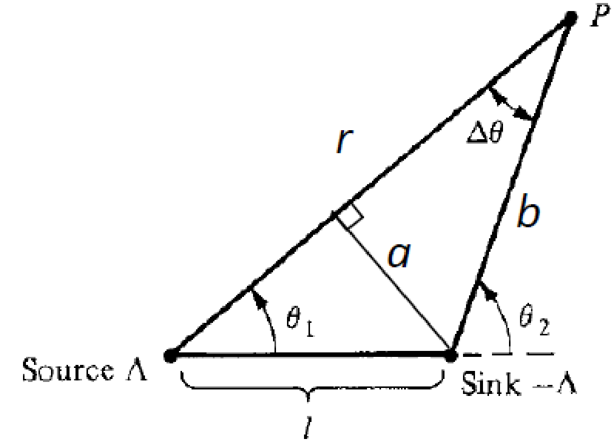
$$\psi = \lim_{\substack{l\to 0 \\ \kappa=\Lambda l=const}} \left(-\frac{K}{2\pi}d\theta\right)$$

$$d\theta = \frac{l\sin\theta}{r - l\cos\theta}$$



❑ Potential & Stream Functions for Basic Flows

Doublet: third elementary flow



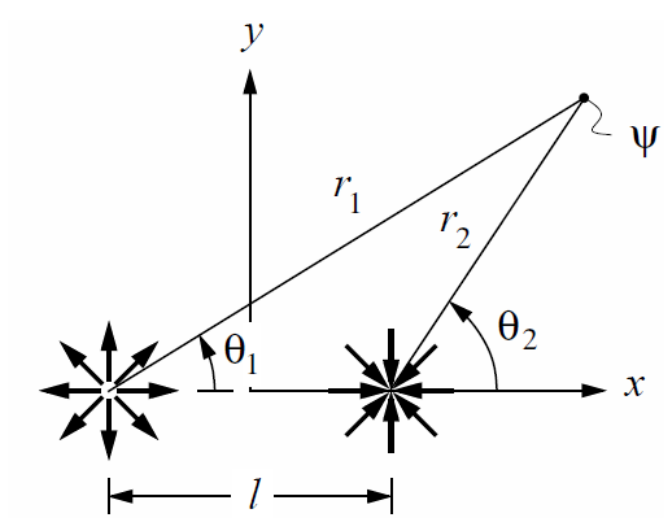
$$\psi = \lim_{\substack{l \to 0 \\ \kappa = const}} \left(-\frac{K}{2\pi} \frac{l \sin\theta}{r - l\cos\theta} \right)$$

$$= \lim_{\substack{l \to 0 \\ \kappa = const}} \left(-\frac{\mu}{2\pi} \frac{\sin\theta}{r - l\cos\theta} \right)$$

$$\boxed{\psi = -\frac{\mu}{2\pi} \frac{\sin\theta}{r}}$$

We can also show :

$$\boxed{\phi = \frac{\mu}{2\pi} \frac{\cos\theta}{r}}$$



❑ Potential & Stream Functions for Basic Flows

- ***Streamlines of a 2D doublet***

$$\psi = \lim_{\substack{\ell \to 0 \\ \kappa = \text{const.}}} -\frac{\mu}{2\pi\,\ell}\Delta\theta = -\frac{\mu}{2\pi}\frac{\sin\theta}{r}$$

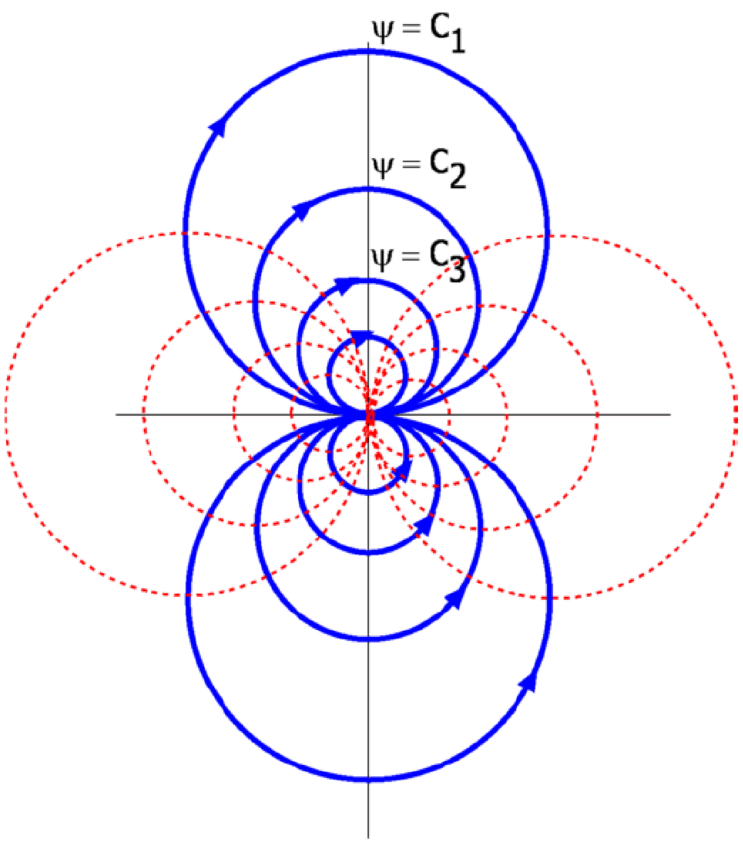
$$\psi = -\frac{\mu}{2\pi}\frac{\sin\theta}{r} = c = const.$$

$$\Rightarrow r = -\frac{\mu}{2\pi c}\sin\theta \Rightarrow r^2 = -\frac{\mu}{2\pi c} r\sin\theta$$

$$\Rightarrow x^2 + y^2 = -\frac{\mu}{2\pi c} y$$

$$\Rightarrow x^2 + y^2 + \frac{\mu}{2\pi c} y + \left(\frac{\mu}{4\pi c}\right)^2 = \left(\frac{\mu}{4\pi c}\right)^2$$

$$\Rightarrow (x-0)^2 + \left(y + \frac{\mu}{4\pi c}\right)^2 = \left(\frac{\mu}{4\pi c}\right)^2$$



Streamlines are circles centerd on the y-axis a distance $-\frac{\mu}{4\pi\psi}$ from the x-axis with a radius of $\left|\frac{\mu}{4\pi\psi}\right|$. All circles pass through the origin.