Lecture # 21: Stream & Potential Functions for Basic Flows – Part 3

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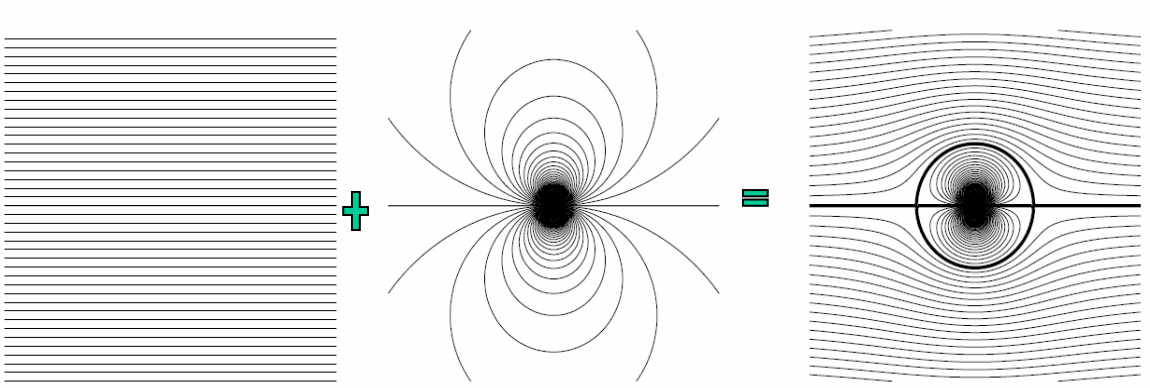
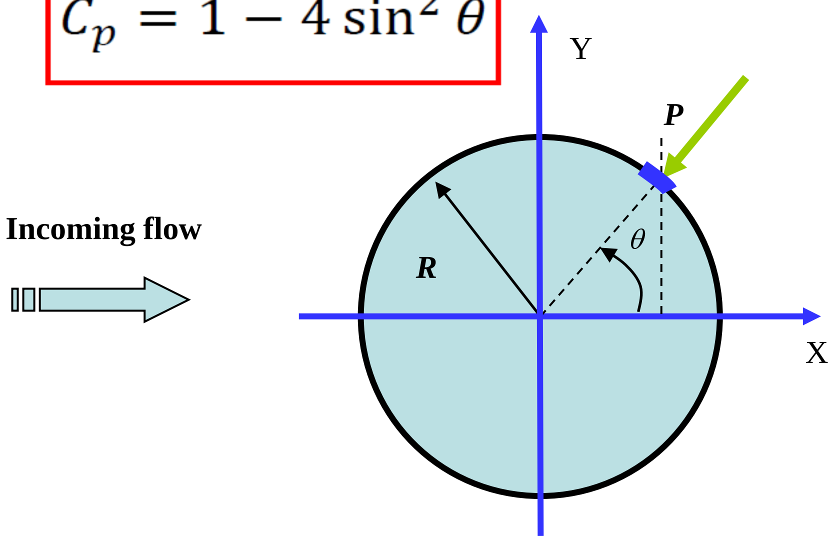
Potential & Stream Functions for Basic Flows

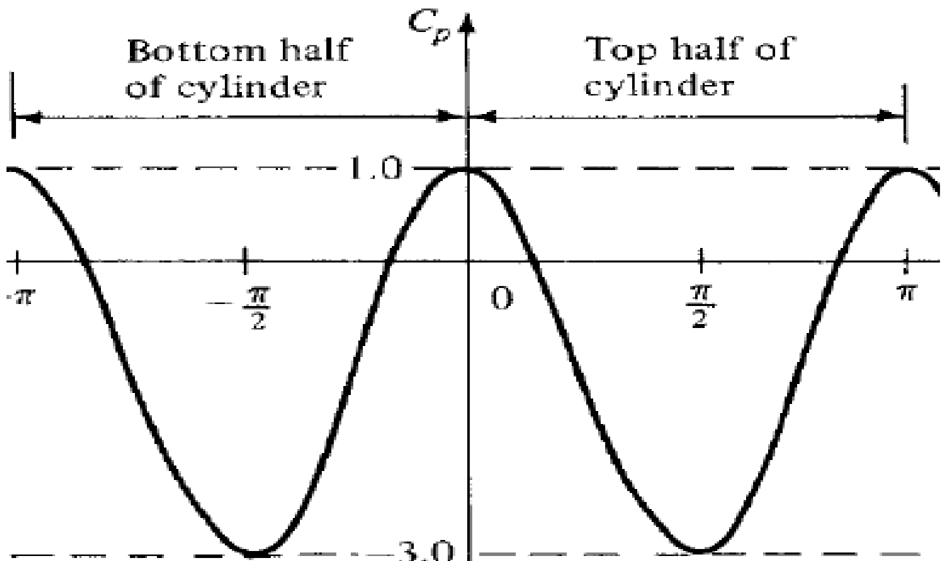
- ***Uniform Flow to the Right + A 2-D Doublet***

$$\left.\begin{aligned}\phi &= V_\infty r\cos\theta\left(1+\frac{R^2}{r^2}\right)\\ \psi &= V_\infty r\sin\theta\left(1-\frac{R^2}{r^2}\right)\\ V_r &= V_\infty\cos\theta\left(1-\frac{R^2}{r^2}\right)\\ V_\theta &= -V_\infty\sin\theta\left(1+\frac{R^2}{r^2}\right)\end{aligned}\right\}(r\geq R)$$



$$C_p = 1 - 4\sin^2\theta$$



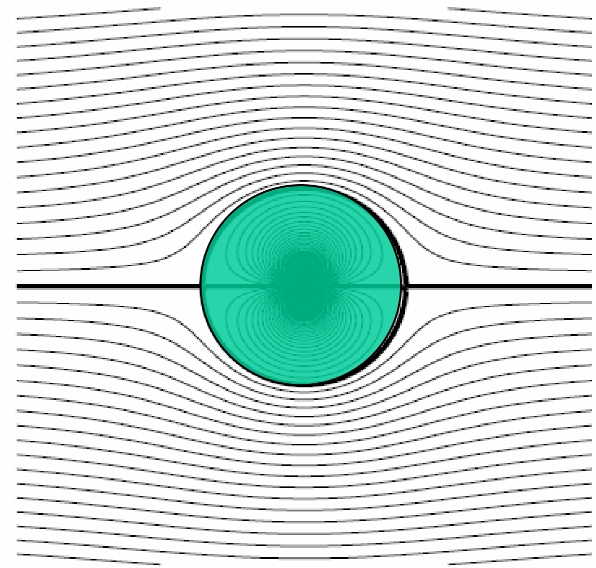


❑ Potential & Stream Functions for Basic Flows

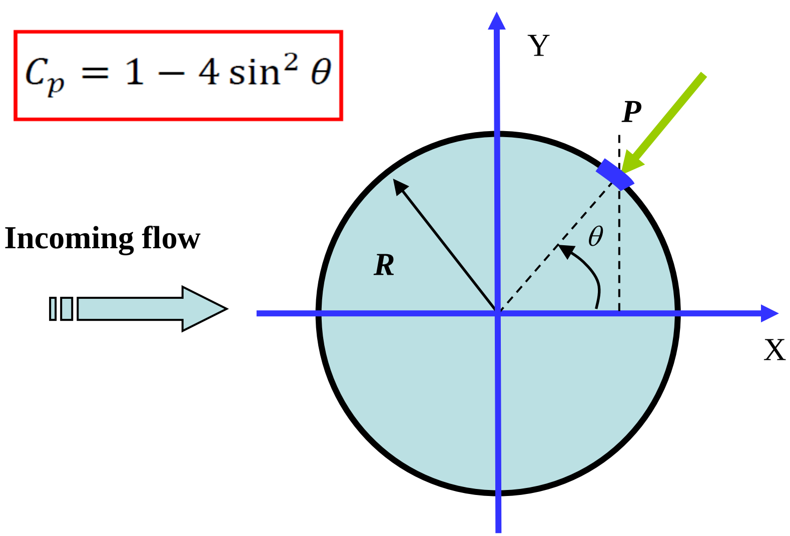
Lift coefficient: $C_L \equiv \dfrac{L}{q_\infty S}$

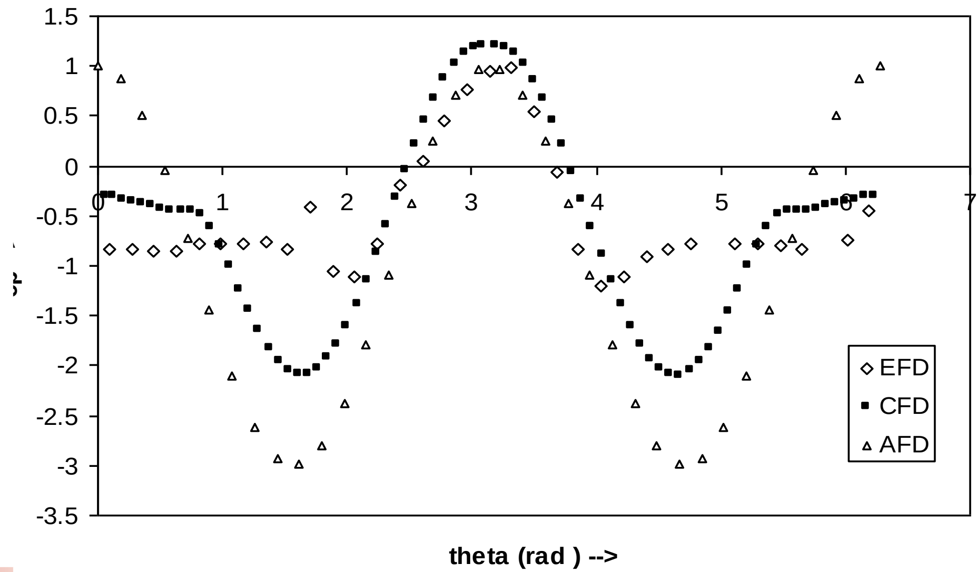
Drag coefficient: $C_D \equiv \dfrac{D}{q_\infty S}$

Moment coefficient: $C_M \equiv \dfrac{M}{q_\infty S \ell}$

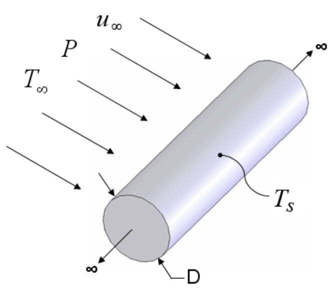


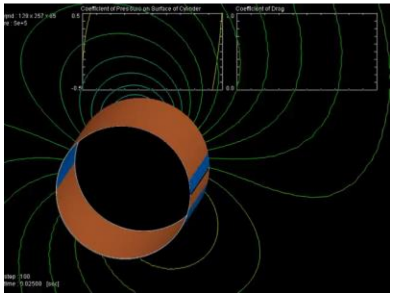
$$C_p = 1 - 4\sin^2\theta$$

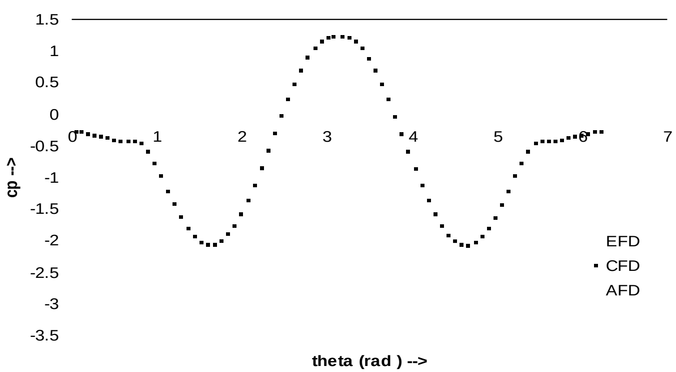


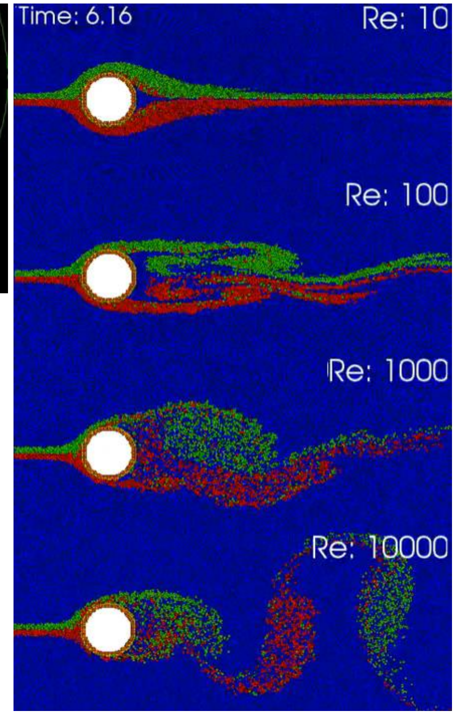


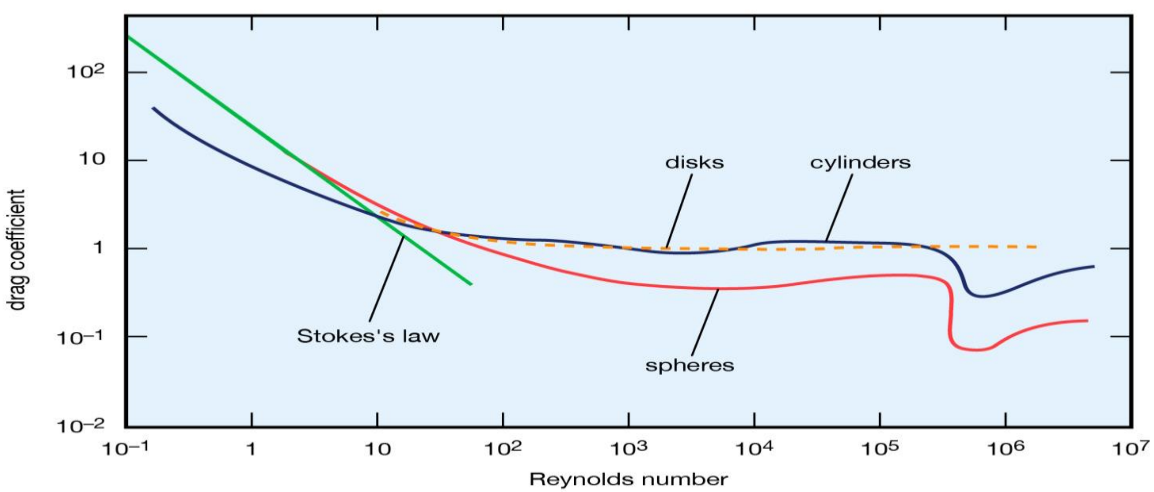
❑ Drag Coefficient of a Circular Cylinder in a Real Rlow

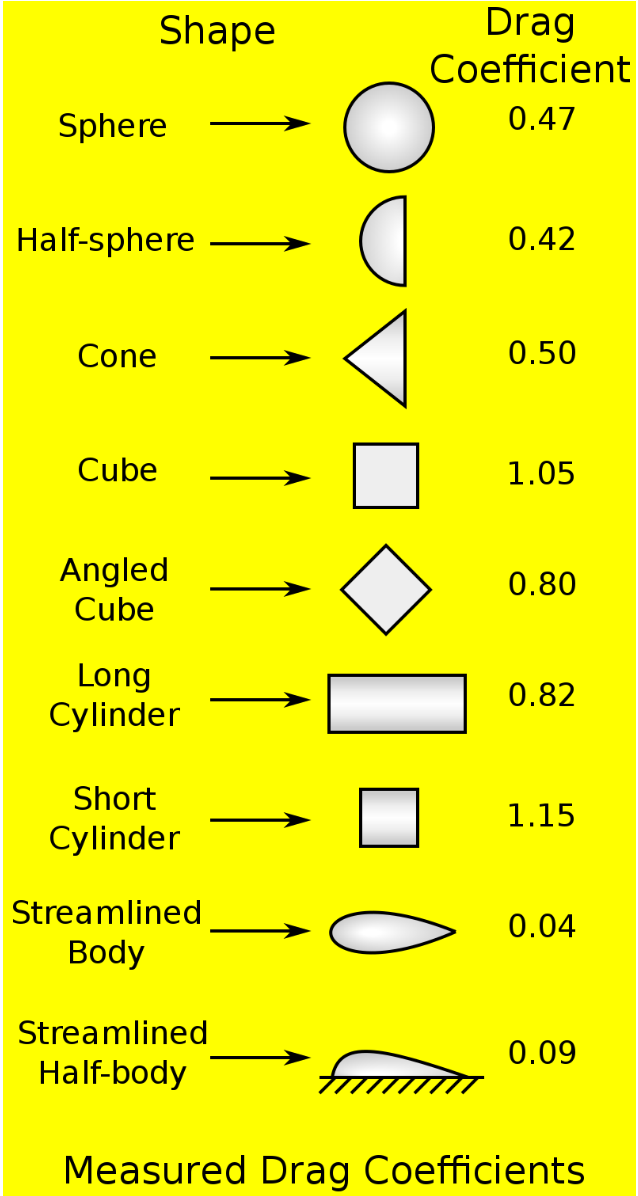






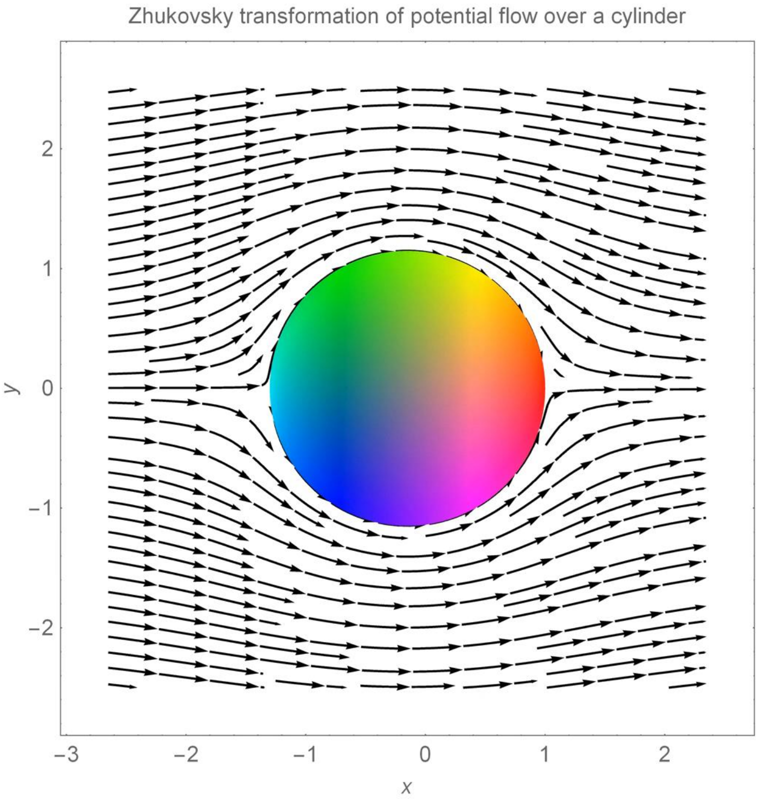


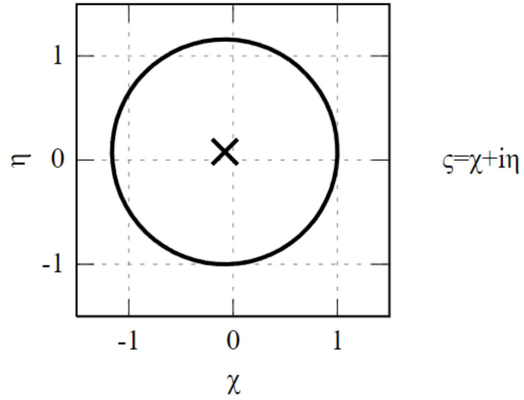


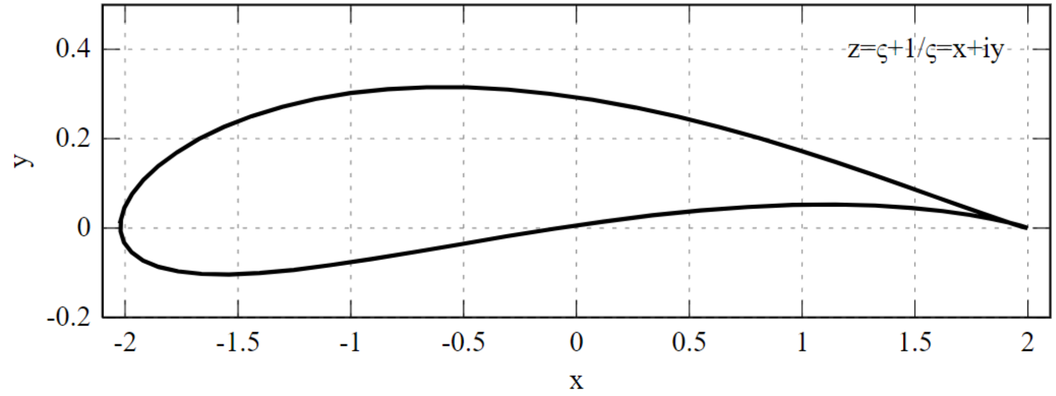


❑ Zhukovsky Transform – Topics to be covered in AerE541

- *By using Zhukovsky transformation, the spinning circle above can be transformed into the Zoukovsky airfoil below.*







❑ Potential & Stream Functions for Basic Flows

- ## *2D Vortex flow*

A 2-D point vortex is a mathematical concept that induces a velocity field given by

$$V_r = 0, \quad V_\theta = \frac{const.}{r} = \frac{C}{r}$$

1. Check if the flow satisfies conservation of mass (Is it a physically possible flow?)

$$\nabla \cdot \vec{V} \stackrel{?}{=} 0$$

$$\nabla \cdot \vec{V} = \frac{1}{r}\left[\frac{\partial(V_r r)}{\partial r} + \frac{\partial V_\theta}{\partial \theta}\right] = 0 \rightarrow \psi \text{ exist.}$$

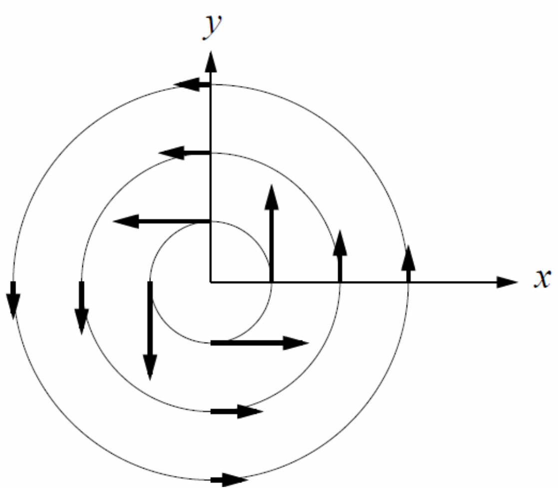
$$V_r = \frac{\partial \psi}{r\partial\theta} = 0 \rightarrow \psi = g(r)$$

$$V_\theta = -\frac{\partial \psi}{\partial r} = \frac{C}{r} \rightarrow \psi = -C\ln r + f(\theta)$$

$$\frac{\partial \psi}{\partial \theta} = f'(\theta) = 0$$

$$f(\theta) = const.$$

$$\therefore \psi = -C\ln r + const.$$



When $r \rightarrow 0$, $V_\theta = \infty$ and $\psi \rightarrow \infty$. To eliminate the infinite velocity it is arbitrary assumed that $\psi = 0$ at $r = R$

$$\therefore \psi = -C\ln R + const. = 0$$

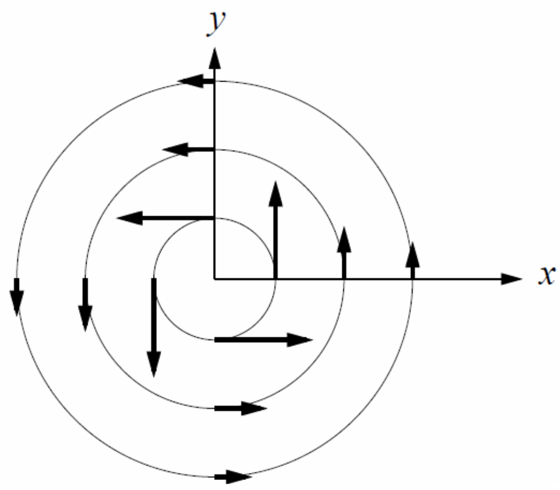
$$const. = C\ln R$$

$$\psi = -C\ln\left(\frac{r}{R}\right) \quad \text{for} \quad (r \geq R)$$

❑ Potential & Stream Functions for Basic Flows

- ## *2D Vortex Flow*

$$V_r = 0, \quad V_\theta = \frac{const.}{r} = \frac{C}{r}$$



Check if the flow is irrotational

$$\nabla \times \vec{V} \stackrel{?}{=} 0$$

$$\nabla \times \vec{V} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix} = \frac{1}{r} \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ V_r & rV_\theta & V_z \end{vmatrix}$$

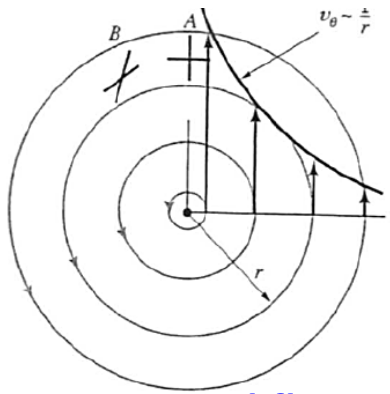
Vorticity or $(\nabla \times \vec{V})$ in the $r - \theta$ plane

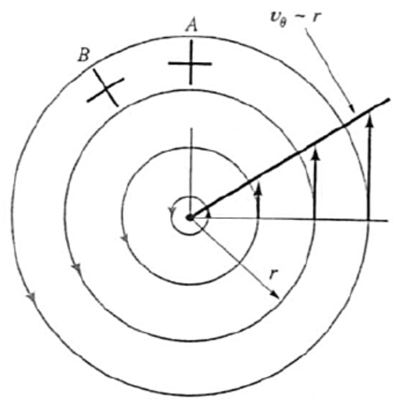
$$\frac{1}{r}\left(\frac{\partial r V_\theta}{\partial r} - \frac{\partial V_r}{\partial \theta}\right) = \left(\frac{\partial C}{\partial r} - \frac{\partial 0}{\partial \theta}\right) = 0 \rightarrow \phi \;\text{ exist.}$$

❑ Potential & Stream Functions for Basic Flows

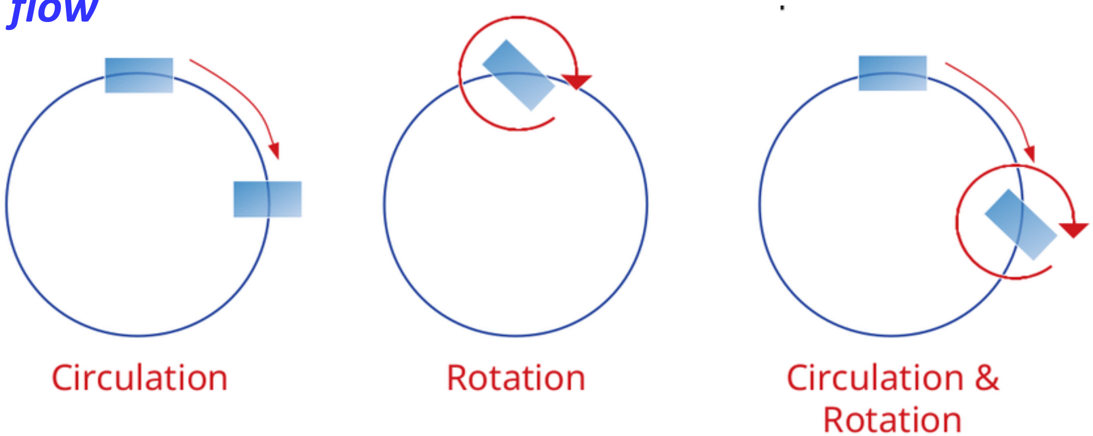
Free and forced vortices

- Free or potential vortex is irrotational : $v_\theta r = const.$
- Forced vortex is rotational: $\frac{v_\theta}{r} = const.$





- ***Irrotational flow***
- ***Rotational flow***



Potential & Stream Functions for Basic Flows

- The velocity components are

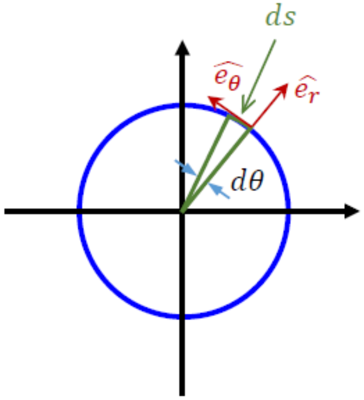
$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0$$

$$v_\theta = -\frac{\partial \psi}{\partial r} = \frac{c_1}{r}$$

- Now let's define circulation Γ around a circle with radius r.

Adding velocity contribution from each term

$$\Gamma = \oint \vec{V} . d\vec{s}$$



Note:

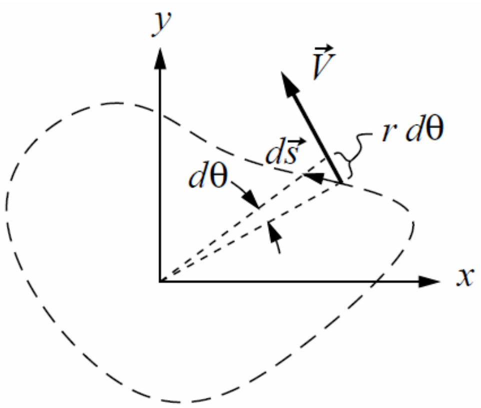
$$\vec{V} = v_\theta \hat{e}_\theta$$
$$d\vec{s} = r d\theta \hat{e}_\theta$$

Therefore:

$$\Gamma = \oint v_\theta \hat{e}_\theta . r d\theta \hat{e}_\theta = \int_0^{2\pi} \frac{c_1}{r} r d\theta \; = c_1 \int_0^{2\pi} d\theta \; = 2\pi c_1 \rightarrow c_1 = \frac{\Gamma}{2\pi}$$

Potential & Stream Functions for Basic Flows

Around closed curve $C1$ that does not include the point vortex



$$\Gamma_{C1} = -\oint_{C1} \vec{V}\cdot d\vec{l} = \iint_{S_1} \underbrace{(\nabla \times \vec{V})}_{0} \cdot d\vec{A} = 0$$

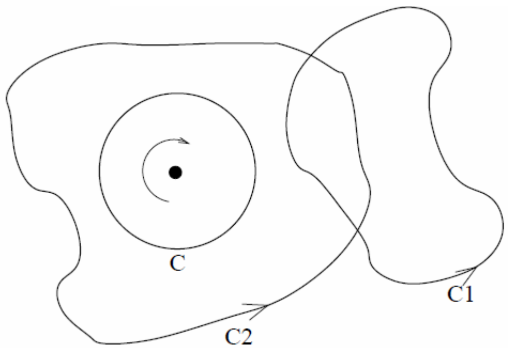
Around $C2$ that includes the point vortex.

$$\Gamma_{C2} = -\left[\oint_{C2} (V_r\hat{e}_r + V_\theta\hat{e}_\theta)\cdot(dr\ \hat{e}_r + r\ d\theta\hat{e}_\theta)\right]$$

$$= -\left[\oint_{C} (V_r\hat{e}_r + V_\theta\hat{e}_\theta)\cdot(dr\ \hat{e}_r + r\ d\theta\hat{e}_\theta)\right] + \left[\oint_{C2-C} (V_r\hat{e}_r + V_\theta\hat{e}_\theta)\cdot(dr\ \hat{e}_r + r\ d\theta\hat{e}_\theta)\right]$$

$$= -\left[\oint_{C} (V_r\hat{e}_r + V_\theta\hat{e}_\theta)\cdot(dr\ \hat{e}_r + r\ d\theta\hat{e}_\theta) + 0\right]$$

$$= -\left[\oint_{C} V_r dr + \oint_{C} V_\theta r\ d\theta\right] = -\left[0 + \int_0^{2\pi}\left(\frac{C}{r}\right) r\ d\theta\right] = -2\pi C$$



$$\Gamma_{C2} = -2\pi C \quad \text{or} \quad -\frac{\Gamma}{2\pi} = C$$

This implies that the circulation evaluated for a curve enclosing the 2-D vortex is a constant and not equal to zero

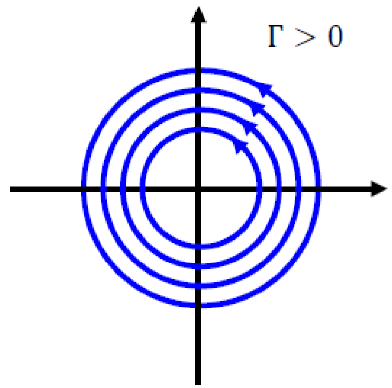
$$\Gamma = \begin{cases} -2\pi C & , \quad \text{(circuit encircles origin)} \\ 0 & , \quad \text{(circuit doesn't encircle origin)} \end{cases}$$

Potential & Stream Functions for Basic Flows

- The final form is:

$$\psi = -\frac{\Gamma}{2\pi}\ln r$$

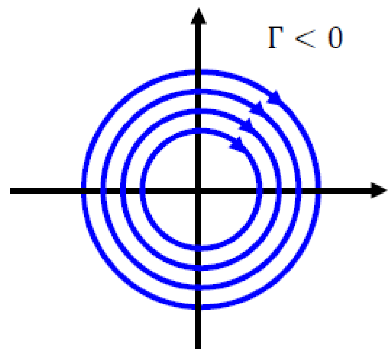
$$v_r = 0 \;, \qquad v_\theta = \frac{\Gamma}{2\pi r}$$



Potential function

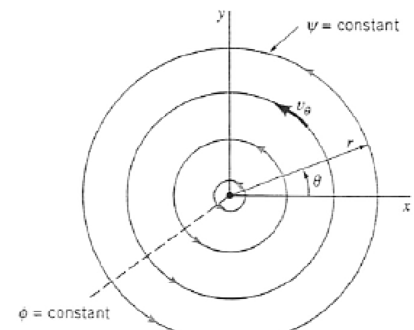
$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{\Gamma}{2\pi r} \rightarrow \phi(r,\theta) = \int \frac{\Gamma}{2\pi} d\theta$$

$$\rightarrow \phi = \frac{\Gamma}{2\pi}\theta + f(r)$$



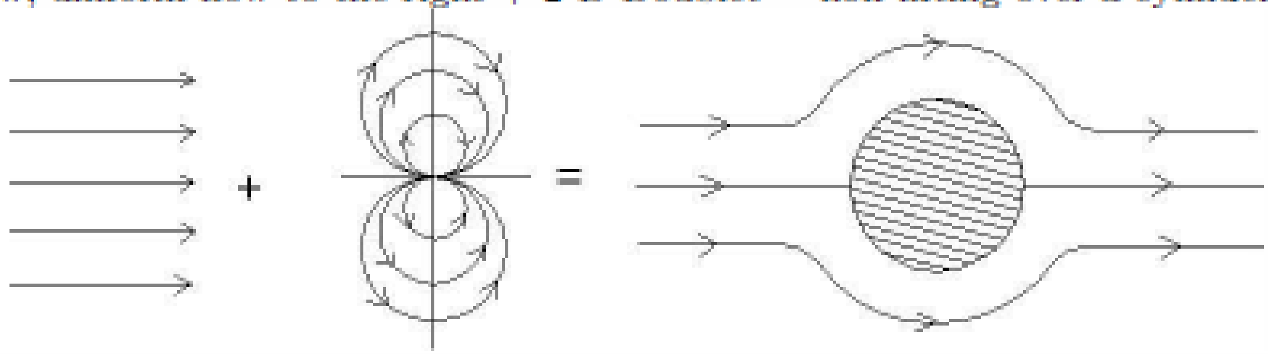
$$v_r = \frac{\partial \phi}{\partial r} = 0 \rightarrow f(r) = c = 0$$

$$\phi = \frac{\Gamma}{2\pi}\theta$$

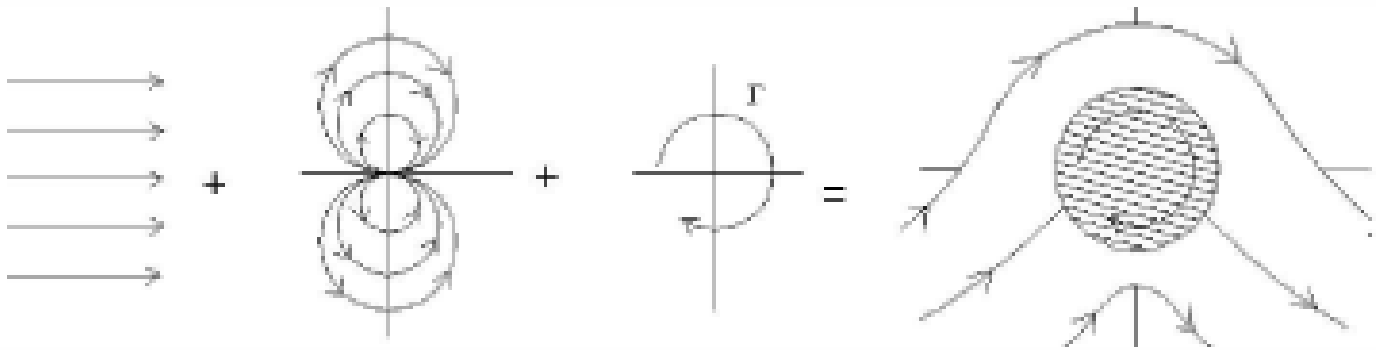


❑ Potential & Stream Functions for Basic Flows

- As we all know, uniform flow to the right + 2-D Doublet = non-lifting over a cylinder



- Uniform flow to the right + 2-D Doublet + 2-D Point Vortex = Lifting flow over a cylinder



The parameters for lifting flow over a cylinder are as follow (spinning cylinder):

Quantity	Non-lifting flow over a cylinder	Vortex of Strength Γ	Combination
ψ	$V_\infty r \sin\theta(1 - \frac{R^2}{r^2})$	$\frac{\Gamma}{2\pi} \ln \frac{r}{R}$	$V_\infty r \sin\theta(1 - \frac{R^2}{r^2}) + \frac{\Gamma}{2\pi} \ln \frac{r}{R}$
ϕ	$V_\infty r \cos\theta(1 + \frac{R^2}{r^2})$	$-\frac{\Gamma}{2\pi}\theta$	$V_\infty r \cos\theta(1 + \frac{R^2}{r^2}) - \frac{\Gamma}{2\pi}\theta$
V_r	$V_\infty \cos\theta(1 - \frac{R^2}{r^2})$	0	$V_\infty \cos\theta(1 - \frac{R^2}{r^2})$
V_θ	$-V_\infty \sin\theta(1 + \frac{R^2}{r^2})$	$-\frac{\Gamma}{2\pi r}$	$-V_\infty \sin\theta(1 + \frac{R^2}{r^2}) - \frac{\Gamma}{2\pi r}$

Potential & Stream Functions for Basic Flows

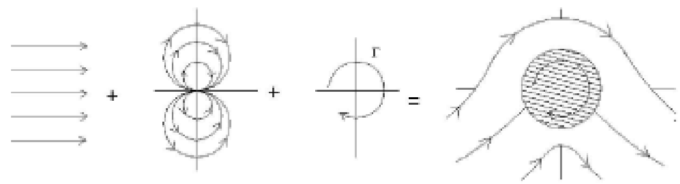
Superposition of free vortex and non-lifting cylinder flow

- Let's add a vortex flow to the non-lifting flow around the cylinder, i.e.

$$\psi = \psi_{\text{uniform}} + \psi_{\text{doublet}} + \psi_{\text{vortex}}$$

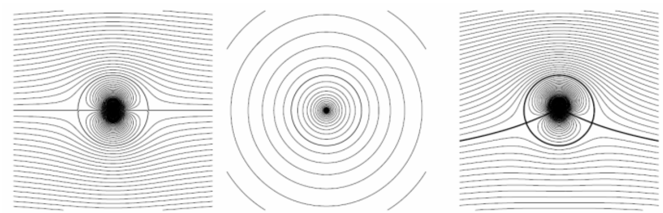
Recall for cylinder flow:

$$\psi = V_\infty r \sin\theta \left(1 - \frac{R^2}{r^2}\right)$$



Therefore:

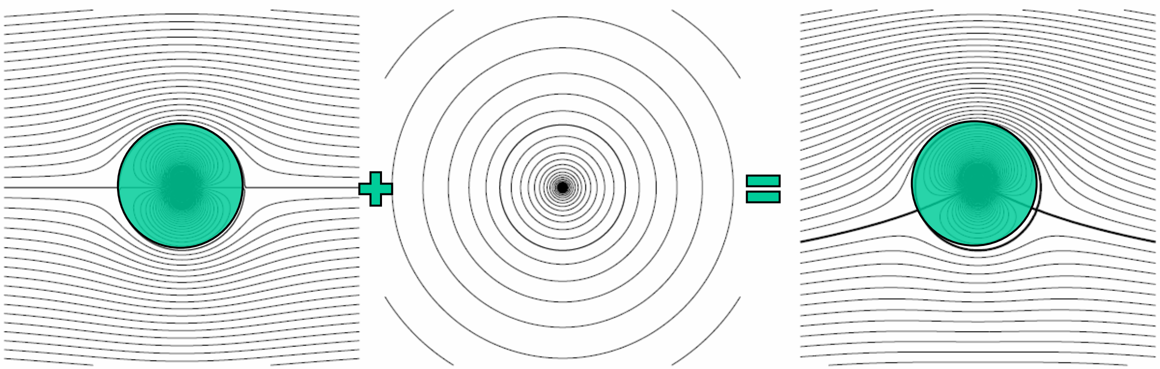
$$\psi = V_\infty r \sin\theta \left(1 - \frac{R^2}{r^2}\right) - \frac{\Gamma}{2\pi} \ln r + c$$

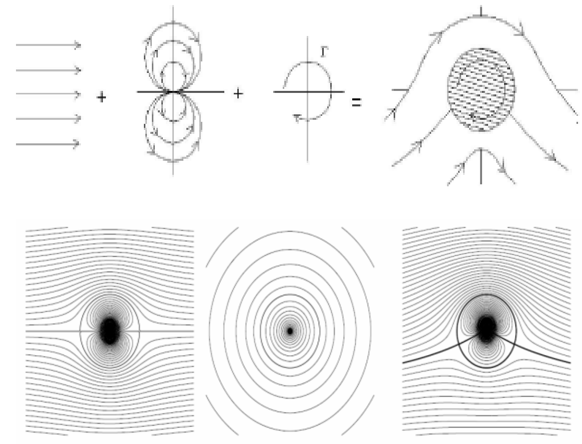


We can always add a constant to stream function. Adding c here to keep $\psi = 0$ on the surface:

$$r = R \rightarrow \psi = 0 = 0 - \frac{\Gamma}{2\pi} \ln R + c \rightarrow c = \frac{\Gamma}{2\pi} \ln R$$

❑ Potential & Stream Functions for Basic Flows





Lifting flow past a cylinder

$$\psi = V_\infty r \sin\theta\left(1 - \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi}\ln\frac{R}{r}$$

Velocity components

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = V_\infty \cos\theta\left(1 - \frac{R^2}{r^2}\right)$$

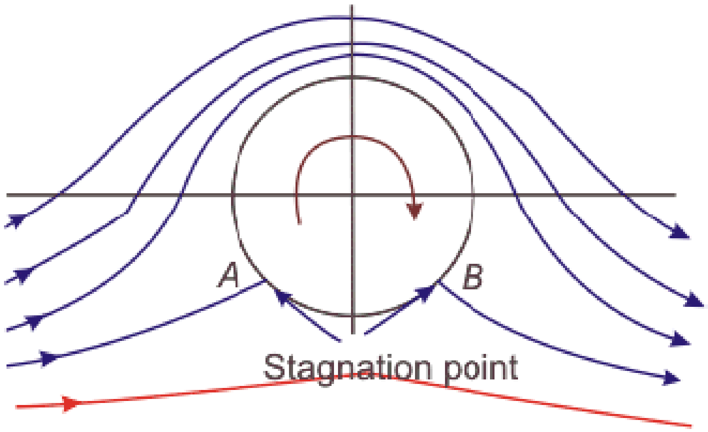
$$v_\theta = -\frac{\partial\psi}{\partial r} = -V_\infty \sin\theta - V_\infty \sin\theta\frac{R^2}{r^2} + \frac{\Gamma}{2\pi r}$$

At $r = R$

$$v_r = 0\,, v_\theta = -2V_\infty \sin\theta + \frac{\Gamma}{2\pi R}$$

❑ Potential & Stream Functions for Basic Flows

Stagnation points are located at $v_r = 0$, $v_\theta = 0$



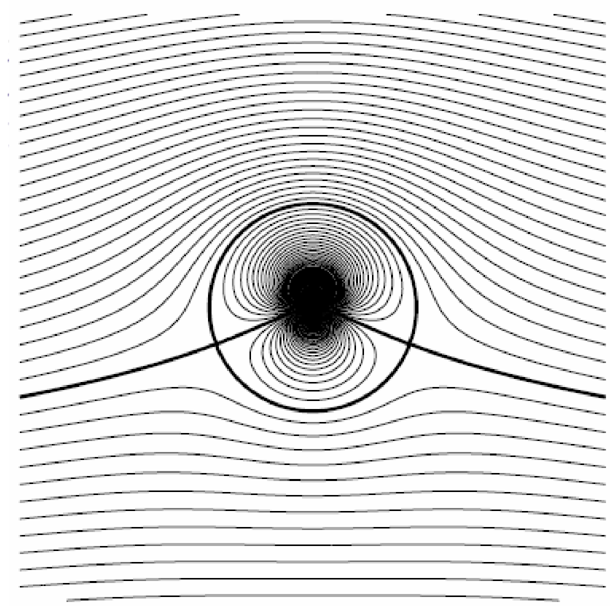
$$v_\theta = -2V_\infty \sin\theta + \frac{\Gamma}{2\pi R} = 0$$

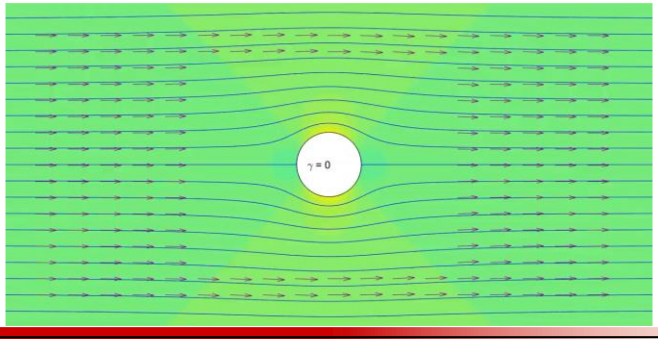
$$\sin\theta = \frac{\Gamma}{4\pi V_\infty R}$$

On cylinder surface $y = R\sin\theta = \frac{\Gamma}{4\pi V_\infty}$

Note $y > 0$ for $\Gamma > 0$ and $y < 0$ for $\Gamma < 0$

(remember we defined $\Gamma > 0$ counter-clockwise but textbook is backward, so let's consider $\Gamma < 0$!

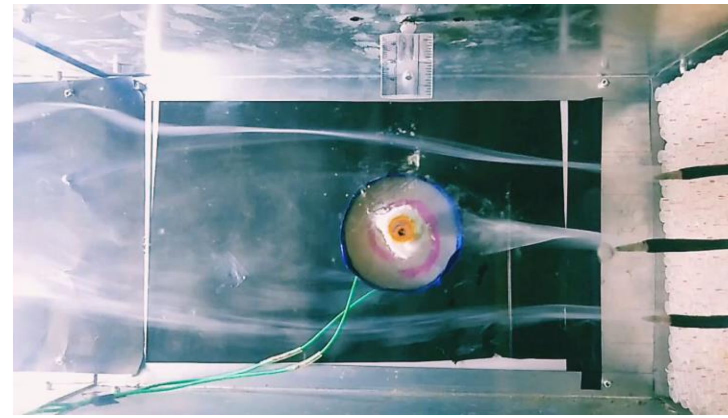


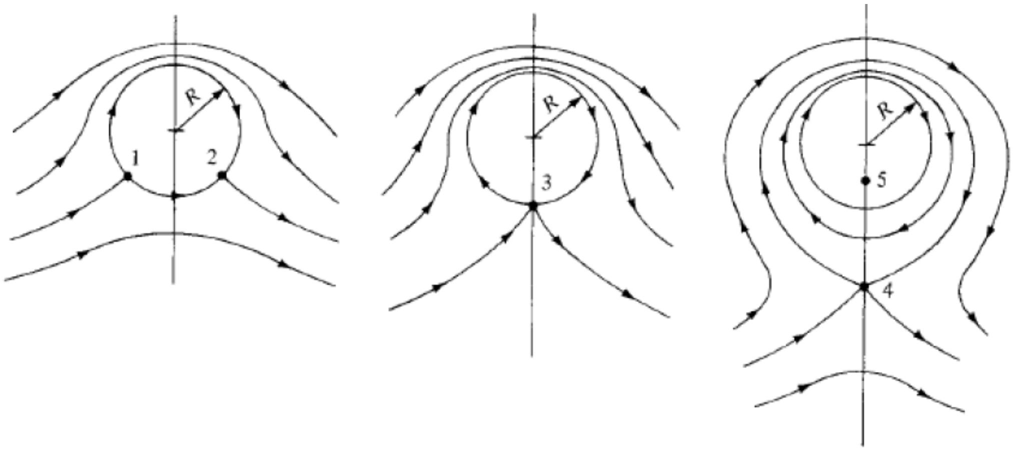


❑ Potential & Stream Functions for Basic Flows

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = V_\infty \cos\theta\left(1 - \frac{R^2}{r^2}\right)$$

$$v_\theta = -\frac{\partial \psi}{\partial r} = -V_\infty \sin\theta - V_\infty \sin\theta\,\frac{R^2}{r^2} + \frac{\Gamma}{2\pi r}$$





$|\Gamma| < 4\pi V_\infty R$ $\quad$ $|\Gamma| = 4\pi V_\infty R$ $\quad$ $|\Gamma| > 4\pi V_\infty R$

For $|\Gamma| > 4\pi V_\infty R$, the equation for $v_\theta = 0$ doesn’t give us a solution. We need to look at the $v_r = 0$

$$v_r = V_\infty \cos\theta\left(1 - \frac{R^2}{r^2}\right) = 0 \rightarrow r = R \text{ or } \cos\theta = 0 \rightarrow \theta = \pm\frac{\pi}{2}$$

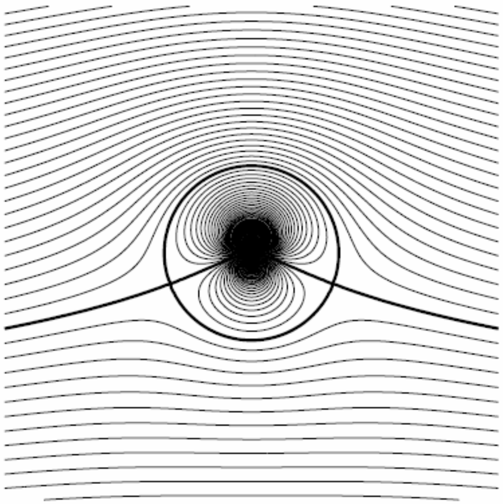
$$r = \frac{\Gamma}{4\pi V_\infty} \pm \sqrt{\left(\frac{\Gamma}{4\pi V_\infty}\right)^2 - R^2}$$

❑ Potential & Stream Functions for Basic Flows

Pressure coefficient

$$V^2 = v_r^2 + v_\theta^2 = 0 + \left(-2V_\infty \sin\theta + \frac{\Gamma}{2\pi R}\right)^2$$

$$C_p = 1 - \left(\frac{V}{V_\infty}\right)^2$$



$$C_p = 1 - \left(-2\sin\theta + \frac{\Gamma}{2\pi R V_\infty}\right)^2$$

$$C_p = 1 - \left[4\sin^2\theta - \frac{2\Gamma\sin\theta}{\pi R V_\infty} + \left(\frac{\Gamma}{2\pi R V_\infty}\right)^2\right]$$

$$p - p_\infty = \frac{1}{2}\rho V_\infty^2 C_p$$

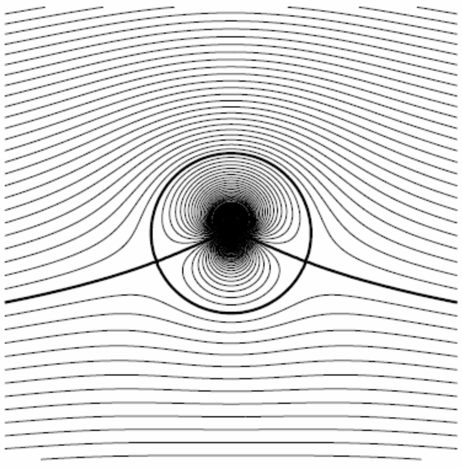
Potential & Stream Functions for Basic Flows

Pressure integral

- Force on a small arc on the cylinder

$$dF = (p - p_\infty) l ds = (p - p_\infty) l R d\theta$$

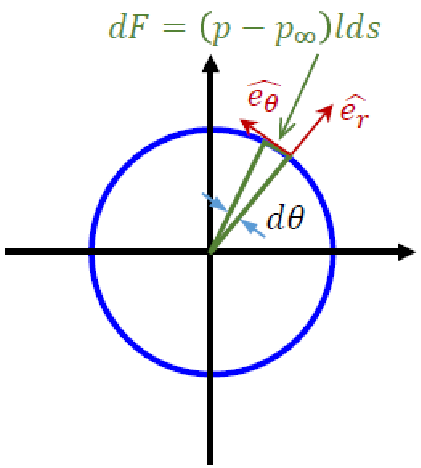
$$F = \int_0^{2\pi} dF$$



Or in terms of its components:

$$D = F_x = F \cos\theta = \int_0^{2\pi} (p - p_\infty) l R \cos\theta \, d\theta$$

$$L = F_y = F \sin\theta = \int_0^{2\pi} (p - p_\infty) l R \sin\theta \, d\theta$$



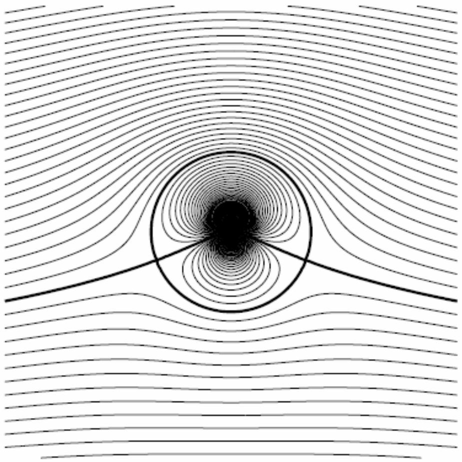
$$C_D = \frac{D}{\frac{1}{2}\rho V_\infty^2 2Rl} = \frac{\int_0^{2\pi} (p - p_\infty) l R \cos\theta \, d\theta}{\frac{1}{2}\rho V_\infty^2 2Rl}$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{(p - p_\infty)}{1/2 \rho V_\infty^2} \cos\theta \, d\theta = \frac{1}{2}\int_0^{2\pi} C_p \cos\theta \, d\theta$$

Note we are using the projection area $2Rl$ as the reference area for C_D

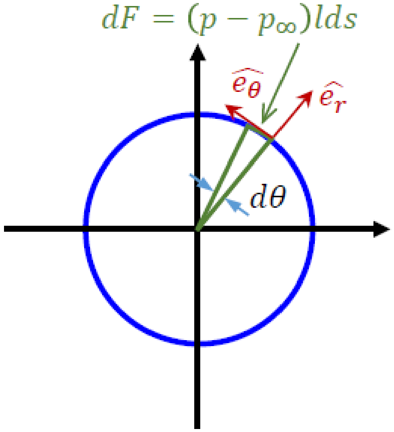
❑ Potential & Stream Functions for Basic Flows

Drag on the cylinder



$$C_D = \frac{1}{2}\int_0^{2\pi} C_p \cos\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(\cos\theta - \left[4\sin^2\theta\cos\theta - \frac{2\Gamma\sin\theta\cos\theta}{\pi R V_\infty} + \left(\frac{\Gamma}{2\pi R V_\infty}\right)^2\cos\theta\right]\right)d\theta$$



Note $\int_0^{2\pi}\cos\theta\, d\theta = 0$, $\int_0^{2\pi}\sin^2\theta\cos\theta\, d\theta = 0$,
$\int_0^{2\pi}\sin\theta\cos\theta\, d\theta = 0$

Therefore

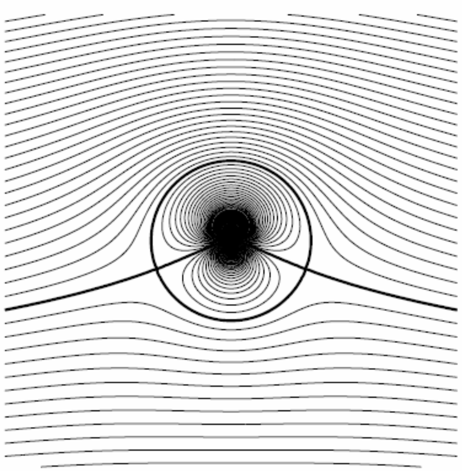
$$\boxed{C_D = 0}$$

d 'Alembert paradox

Note we are using the projection area $2Rl$ as the reference area for C_D

❑ Potential & Stream Functions for Basic Flows

Lift on the cylinder



$$C_L = \frac{1}{2}\int_0^{2\pi} C_p \sin\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(\sin\theta - \left[4\sin^3\theta - \frac{2\Gamma\sin^2\theta}{\pi R V_\infty} + \left(\frac{\Gamma}{2\pi R V_\infty}\right)^2 \sin\theta\right]\right) d\theta$$

Note $\int_0^{2\pi} \sin\theta \, d\theta = 0$, $\int_0^{2\pi} \sin^3\theta \, d\theta = 0$, $\int_0^{2\pi} \sin^2\theta \, d\theta = \pi$

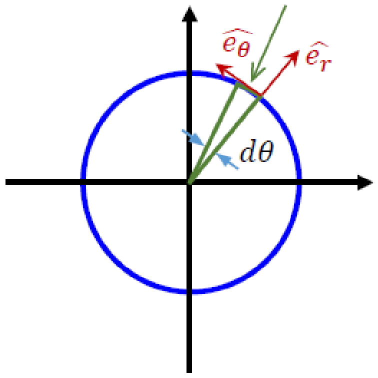
Therefore

$$C_L = \frac{\Gamma}{\pi R V_\infty}(\pi) \rightarrow C_L = \frac{\Gamma}{R V_\infty}$$

$$L = \frac{1}{2}\rho V_\infty^2 (2Rl)\frac{\Gamma}{R V_\infty} = \rho V_\infty \Gamma l$$

$$L' = \rho V_\infty \Gamma$$

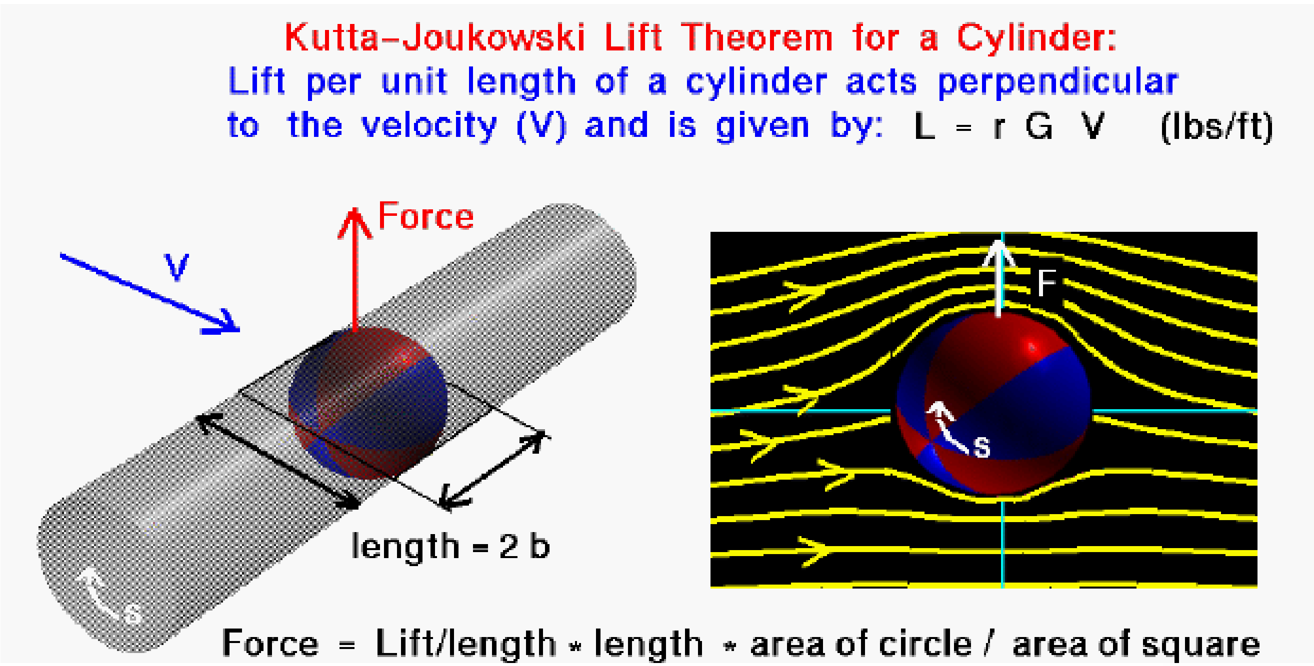
Kutta-Joukowski theorem



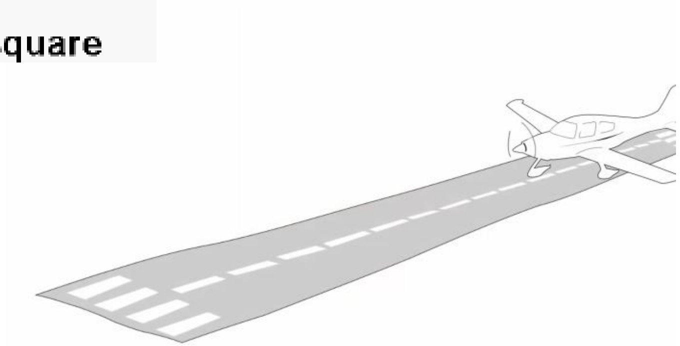
Note we are using the projection area $2Rl$ as the reference area for C_D

❑ The Kutta-Joukowski Lift Theorem

- ***The Kutta-Joukowski Lift Theorem* states the lift per unit length of a spinning cylinder is equal to the density (ρ) of the air times the strength of the rotation (Γ) times the velocity (V) of the air.**



$$L' = \rho V_\infty \Gamma$$



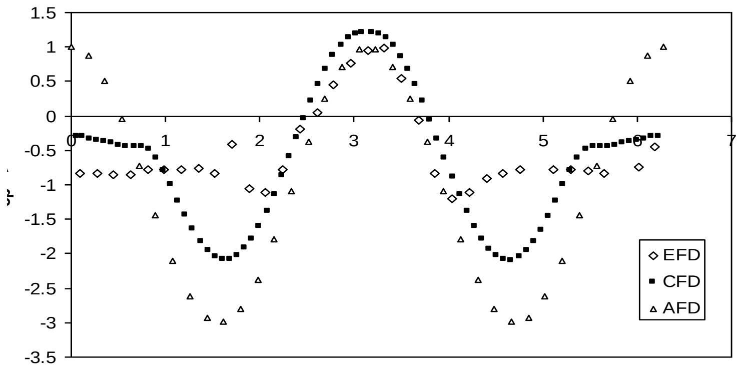
- *Theories in the Production of Lift*

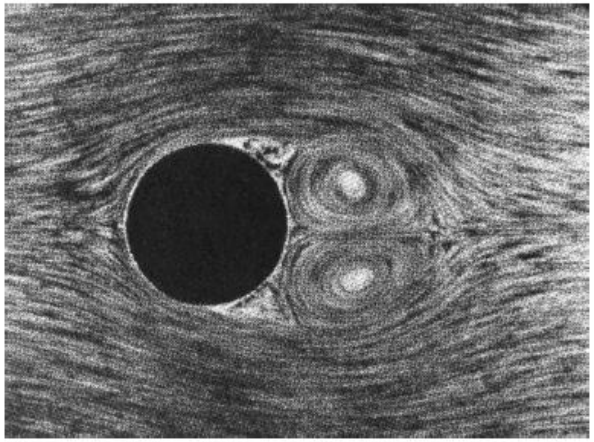
https://www.youtube.com/watch?v=mBRCUU_fQrQ

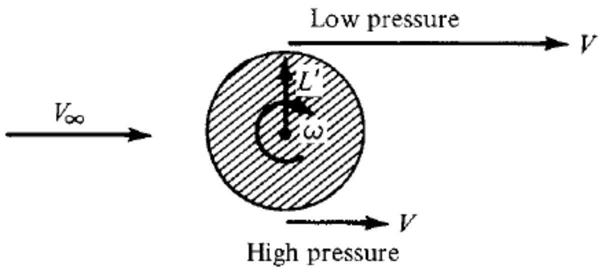
❑ Potential & Stream Functions for Basic Flows

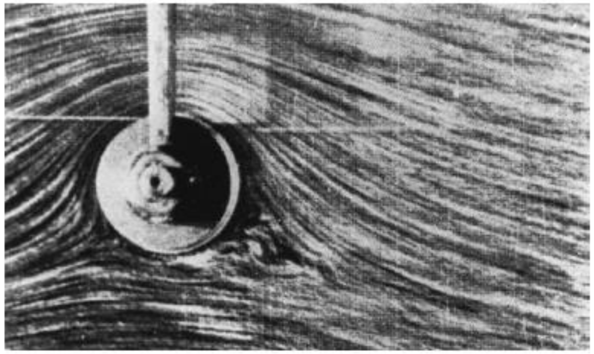
Theoretical vs real flow

- In real flow past cylinder $C_D \neq 0$
  - Drag is primarily caused by the flow separation, a viscous effect. The potential theory neglects the viscous effects hence can not predict the drag force.
- Flow around a spinning cylinder resembles that of theory and generates lift proportional to the circulation









❑ Potential & Stream Functions for Basic Flows

Theoretical vs real flows

- For streamlined shapes (airfoils), potential flow can predict the pressure distribution and the resulting forces with a good accuracy
  - Low angle of attacks, where the flow does not separate
  - Can estimate pressure drag (as long as flow is attached)!
  - Can't get friction drag!