Lecture # 22: Stream & Potential Functions for Basic Flows – Part 4

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❑ Summary of various Basic Flows

- ***Basic flow #01 - Uniform flow***

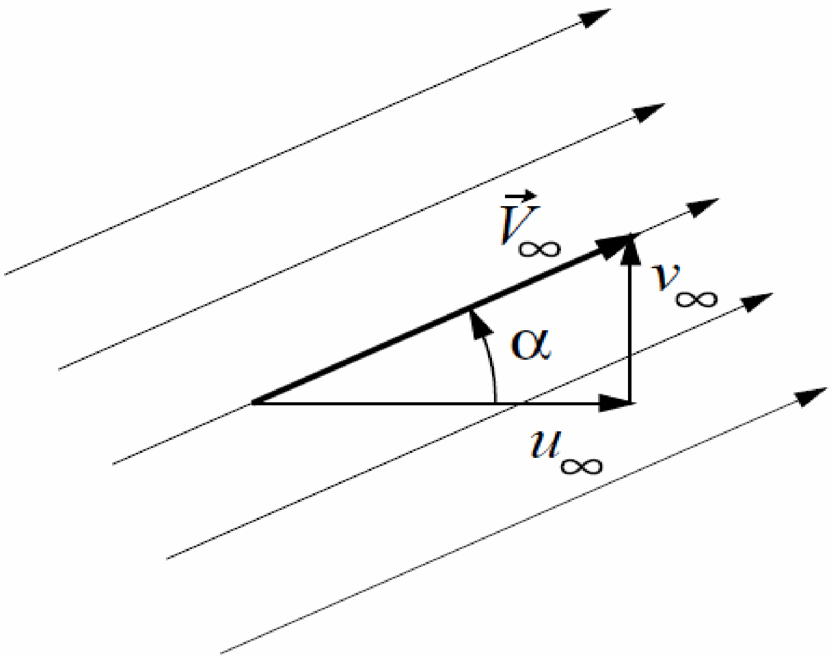
(uniform flow at an angle α)

$$\psi = -V_\infty \sin\alpha \; x + V_\infty \cos\alpha \; y$$

$$\phi = V_\infty \cos\alpha \; x + V_\infty \sin\alpha \; y$$

$$u \;=\; u_\infty \;=\; V_\infty \cos\alpha$$

$$v \;=\; v_\infty \;=\; V_\infty \sin\alpha$$



❑ Summary of various Basic Flows

- ***Basic flow # 2: 2D Source or Sink Flow***

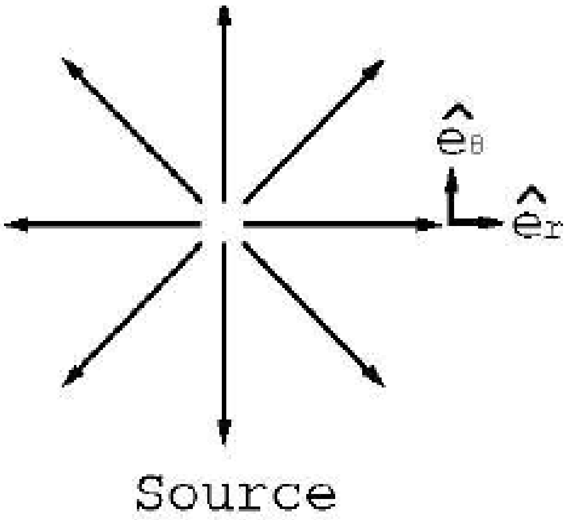
$$\psi = \frac{K}{2\pi}\theta$$

$$\phi = \frac{K}{2\pi}\ln r$$

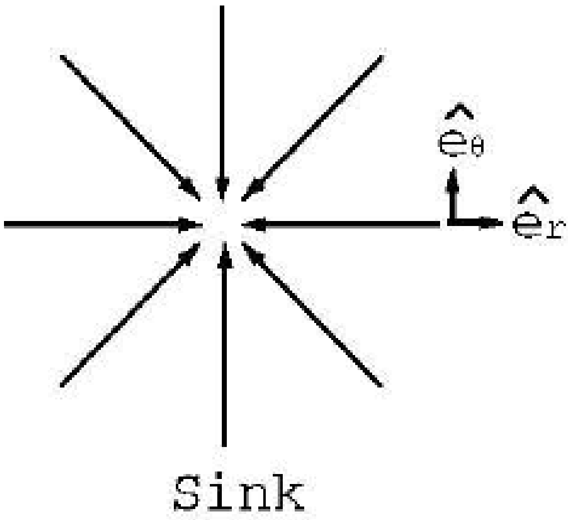
$$V_r = \frac{K}{2\pi r}$$

$$V_\theta = 0$$

$K > 0$



$K < 0$



❑ Summary of various Basic Flows

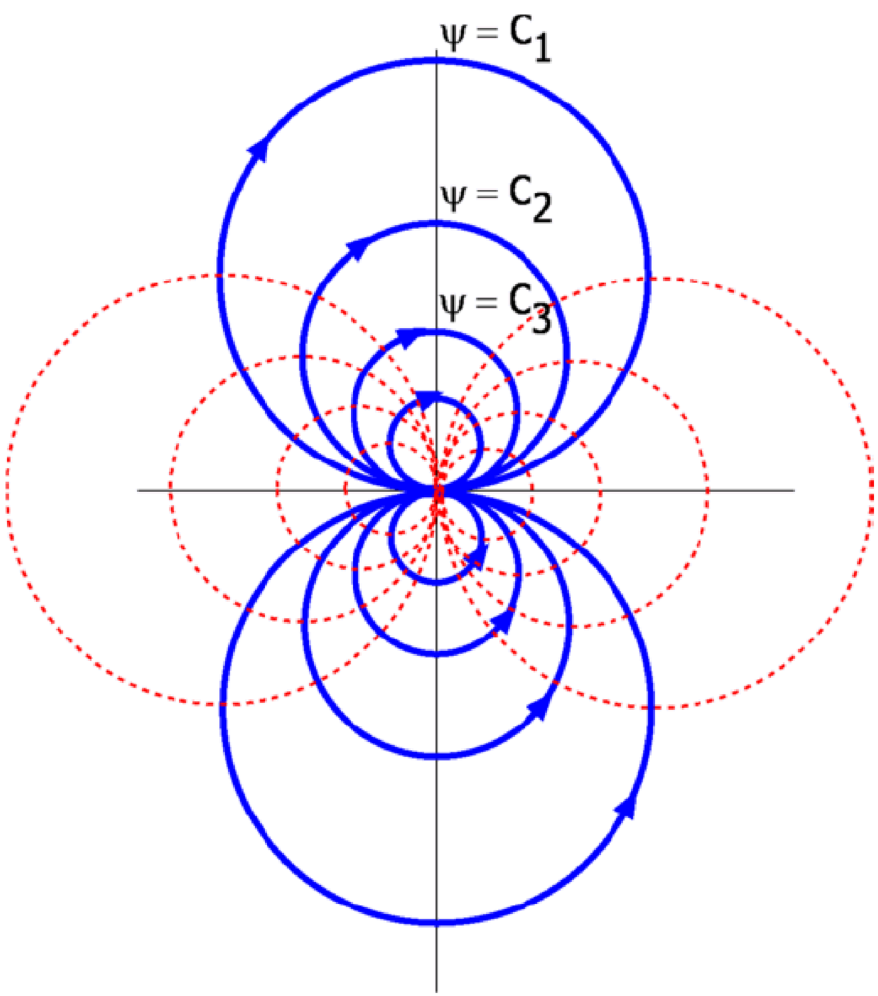
- *Basic flow # 03: 2D doublet flow*

$$\psi = -\frac{\mu}{2\pi}\frac{\sin\theta}{r}$$

$$\phi = \frac{\mu}{2\pi}\frac{\cos\theta}{r}$$

$$V_r = \frac{\partial\phi}{\phi r} = -\frac{\mu}{2\pi}\frac{\cos\theta}{r^2}$$

$$V_\theta = \frac{1}{r}\frac{\partial\phi}{\phi\theta} = -\frac{\mu}{2\pi}\frac{\sin\theta}{r^2}$$



❑ Summary of various Basic Flows

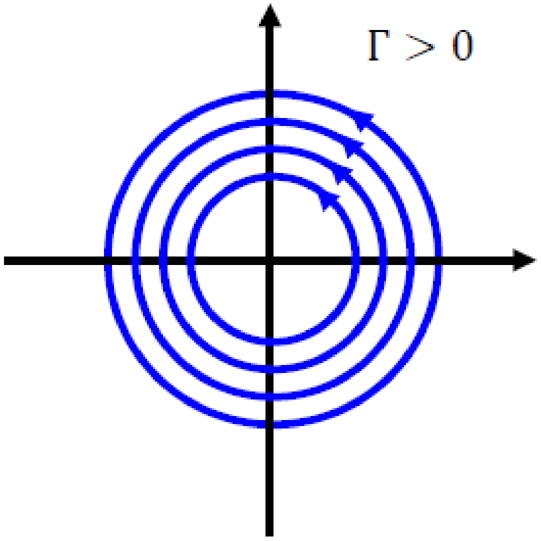
- ***Basic flow # 04: 2D vortex***

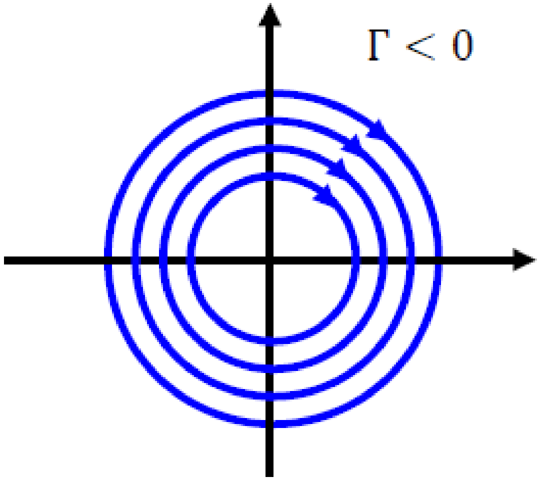
$$\psi = \frac{\Gamma}{2\pi} \ln r$$

$$\phi = -\frac{\Gamma}{2\pi} \theta$$

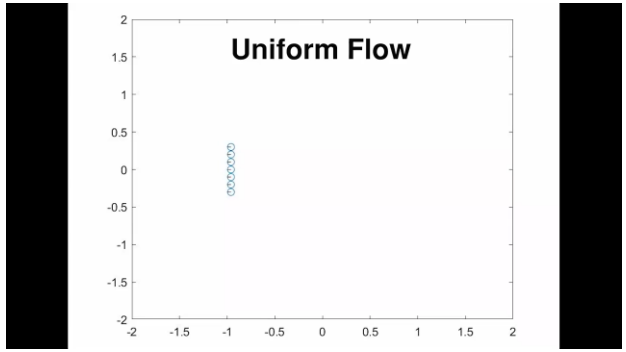
$$V_r = 0$$

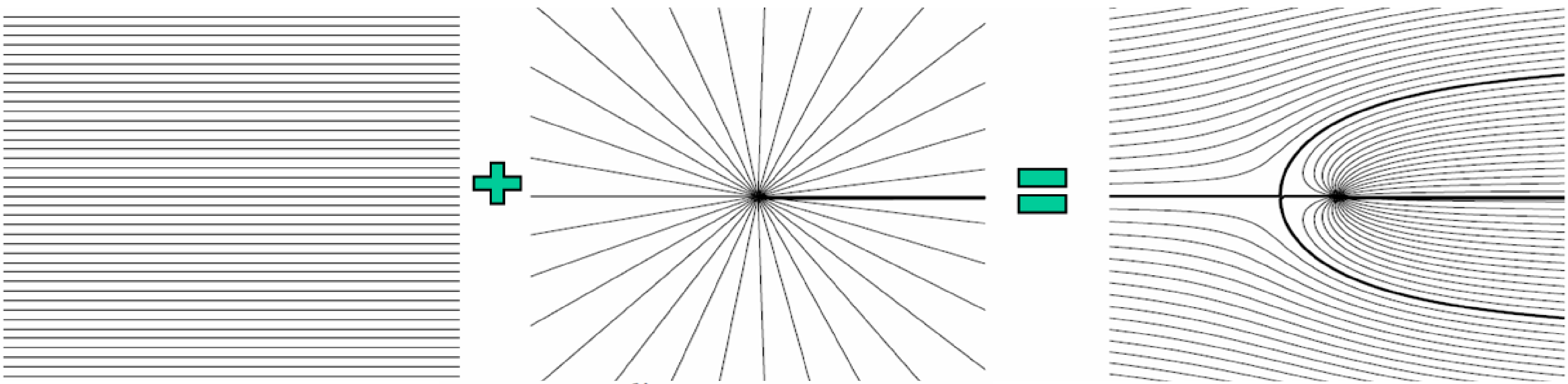
$$V_\theta = -\frac{\Gamma}{2\pi r}$$

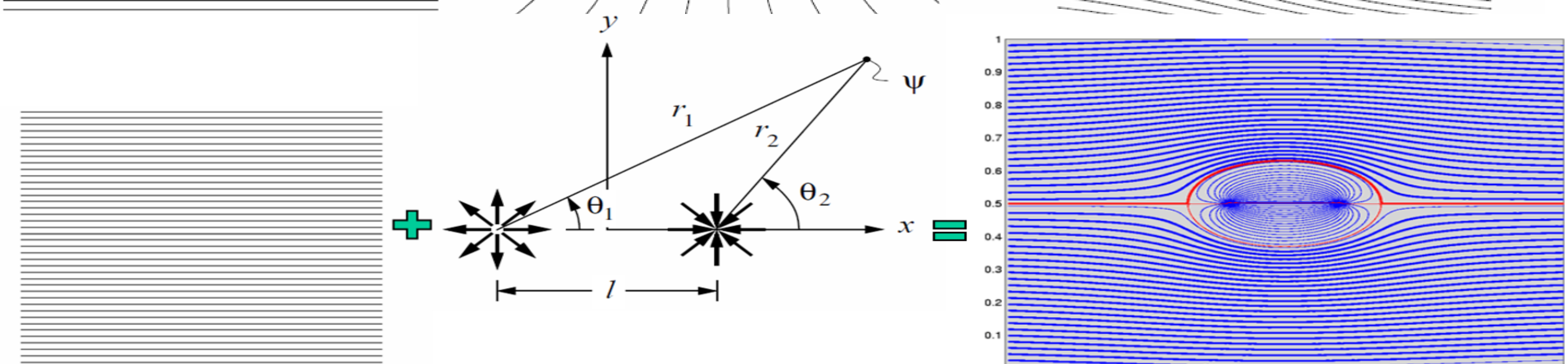




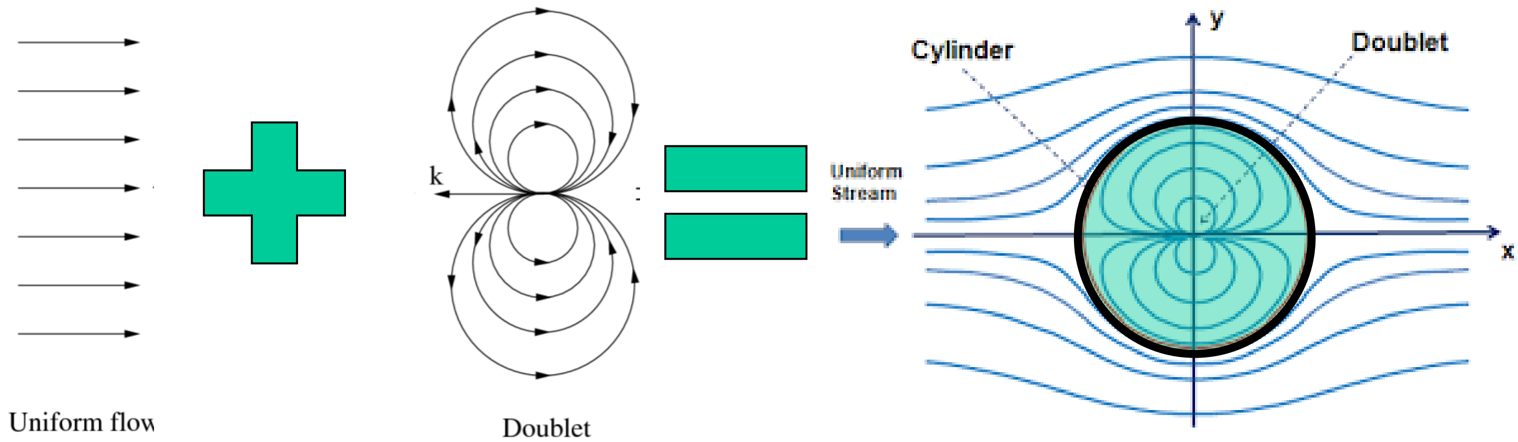
Superposition of Basic Flows



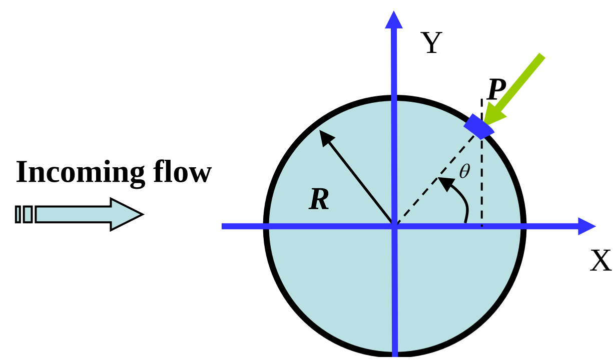




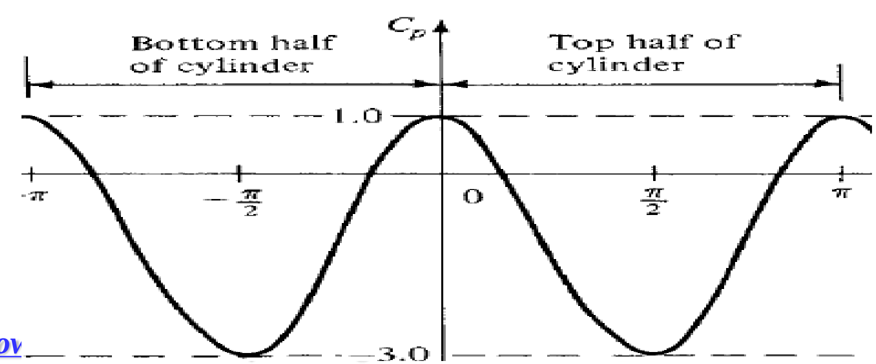
Superposition of Basic Flows



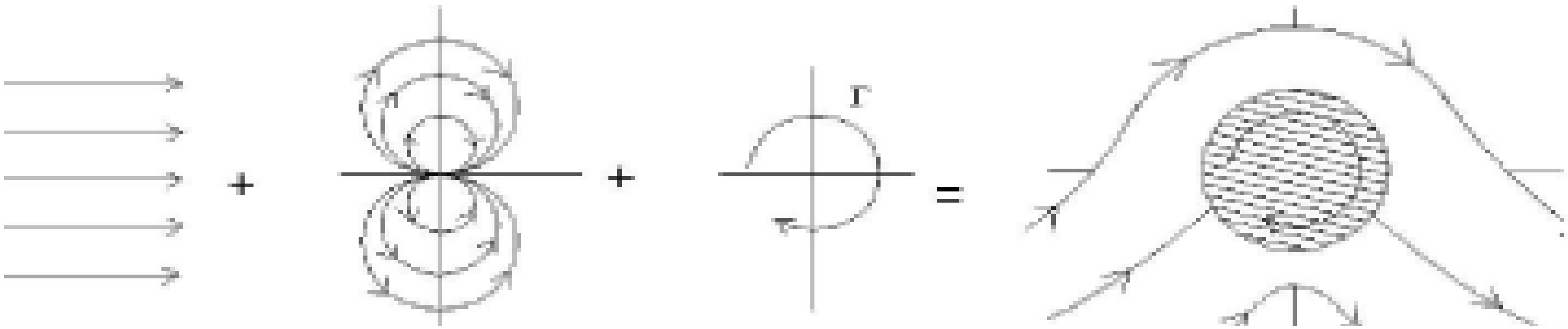
$$\left.\begin{aligned}\phi &= V_\infty r\cos\theta\left(1+\frac{R^2}{r^2}\right)\\ \psi &= V_\infty r\sin\theta\left(1-\frac{R^2}{r^2}\right)\\ V_r &= V_\infty\cos\theta\left(1-\frac{R^2}{r^2}\right)\\ V_\theta &= -V_\infty\sin\theta\left(1+\frac{R^2}{r^2}\right)\end{aligned}\right\}(r\geq R)$$



$$\boxed{C_p = 1 - 4\sin^2\theta}$$

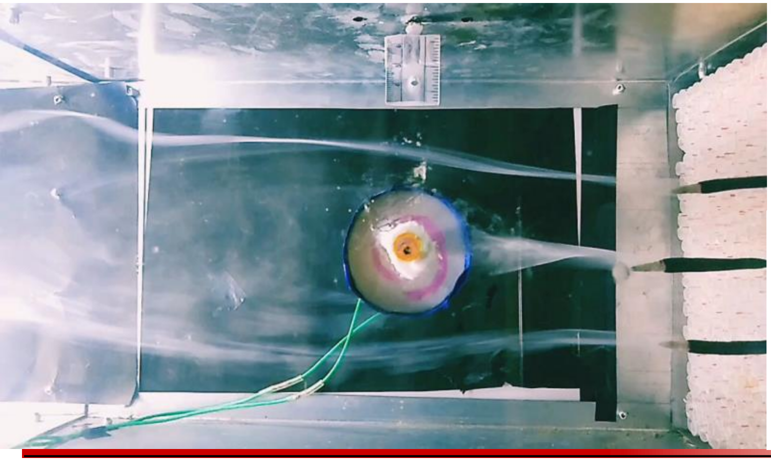


Superposition of Basic Flows



The parameters for lifting flow over a cylinder are as follow (spinning cylinder):

Quantity	Non-lifting flow over a cylinder	Vortex of Strength Γ	Combination
ψ	$V_\infty r \sin\theta(1-\frac{R^2}{r^2})$	$\frac{\Gamma}{2\pi}\ln\frac{r}{R}$	$V_\infty r \sin\theta(1-\frac{R^2}{r^2})+\frac{\Gamma}{2\pi}\ln\frac{r}{R}$
ϕ	$V_\infty r \cos\theta(1+\frac{R^2}{r^2})$	$-\frac{\Gamma}{2\pi}\theta$	$V_\infty r \cos\theta(1+\frac{R^2}{r^2})-\frac{\Gamma}{2\pi}\theta$
V_r	$V_\infty \cos\theta(1-\frac{R^2}{r^2})$	0	$V_\infty \cos\theta(1-\frac{R^2}{r^2})$
V_θ	$-V_\infty \sin\theta(1+\frac{R^2}{r^2})$	$-\frac{\Gamma}{2\pi r}$	$-V_\infty \sin\theta(1+\frac{R^2}{r^2})-\frac{\Gamma}{2\pi r}$



Kutta-Joukowski theorem

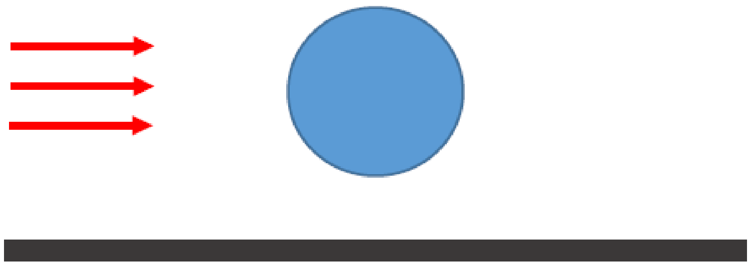
$$L' = \rho V_\infty \Gamma$$

- ***What Is The Magnus Force?***

❑ Notes on Superposition of Basic Flows

Bounded and unbounded flows

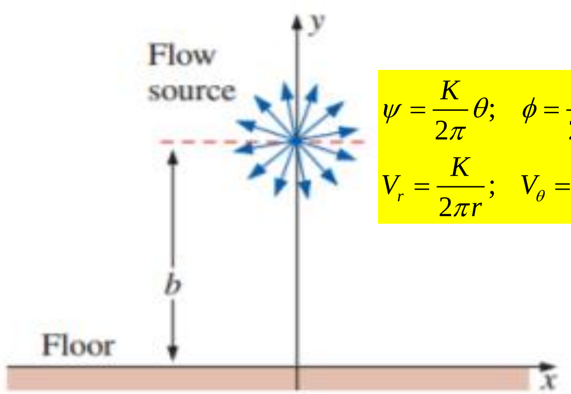
- The solution of flow over a circular cylinder represents an unbounded flow (i.e., flow approaches free stream far away from the object)
- There are many cases, where we are interested in the flow near a solid surface (single wall, corners, ...)
- In such cases, the basic unbounded potential flow solutions can be modified for wall effects by the method of images.



❑ Notes on Superposition of Basic Flows

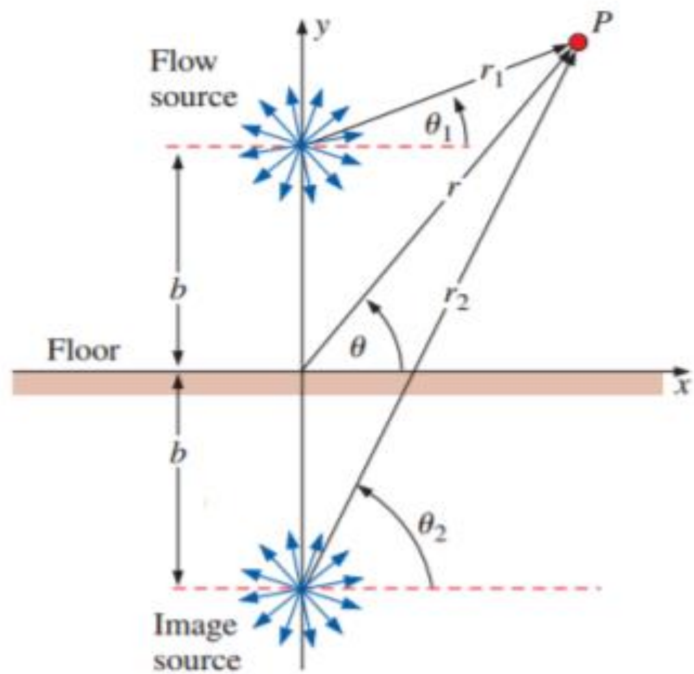
A source near a wall:

- *Consider a single source with strength K at a distant $\hbar$ from a plane wall.*
- *To create the induced flow over the wall, the wall surface should represent a streamline*
- *Now, imagine a second source with the same strength of K and on the opposite side of the wall (image of the primary source).*
- *Write the induced velocity at an arbitrary point on the x axis (wall)*



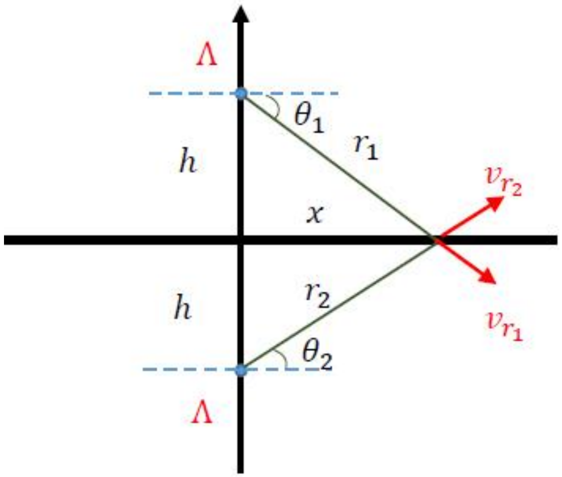
$$\psi = \frac{K}{2\pi}\theta; \quad \phi = \frac{K}{2\pi}\ln r$$

$$V_r = \frac{K}{2\pi r}; \quad V_\theta = 0$$



Notes on Superposition of Basic Flows

Source near a wall- method of images



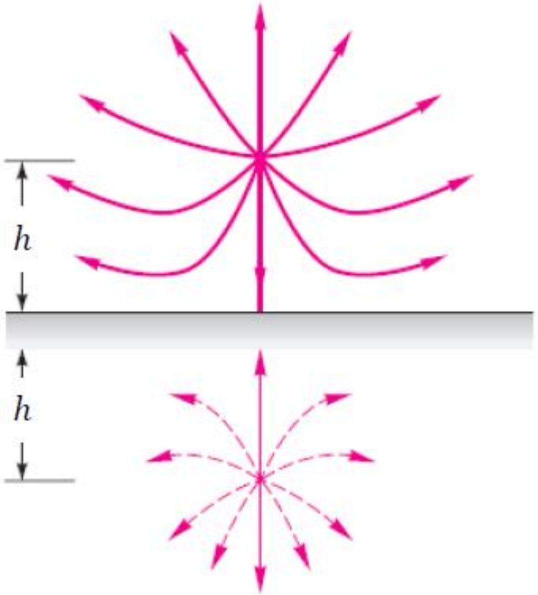
By symmetry, $\theta_1 = \theta_2$ and $r_1 = r_2$, therefore

$$u = 2V_r \cos\theta_1 = \frac{K \bullet r_1 \cos\theta_1}{\pi r_1^2} = \frac{K \bullet x}{\pi(x^2 + h^2)}$$

$$v = 0$$

This applies to any point on the x-axis, which means x-axis is a streamline!

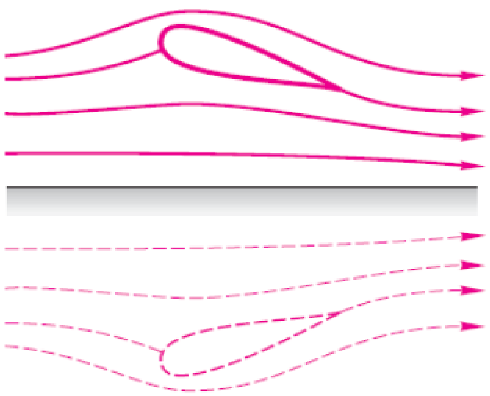
- Therefore, by adding a fictitious image of the primary source, the flow of the source near a wall is modeled.



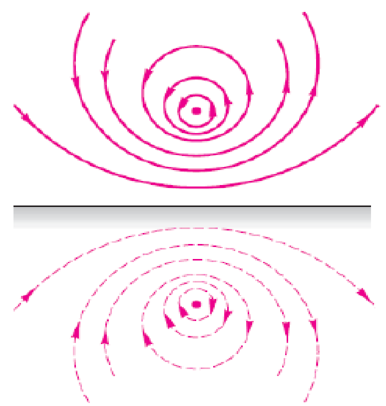
❑ Notes on Superposition of Basic Flows

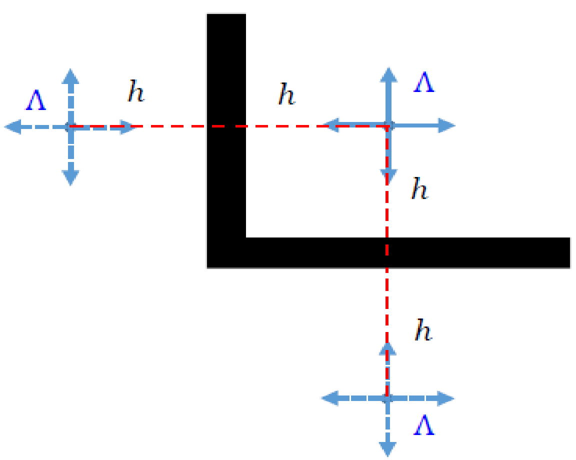
Method of images

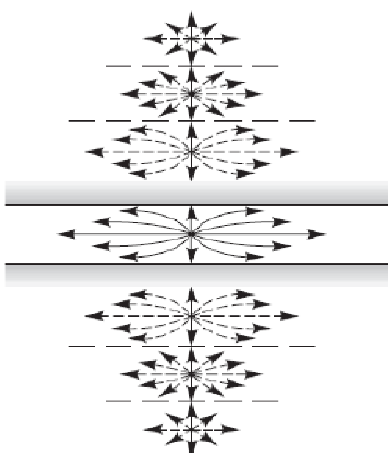
- We can apply the same process for other potential flows (elementary or more complicated flows)



- Corners or multiple walls, require multiple images







Vortex pairs of opposite sign create a plane surface in the middle

Notes on Superposition of Basic Flows

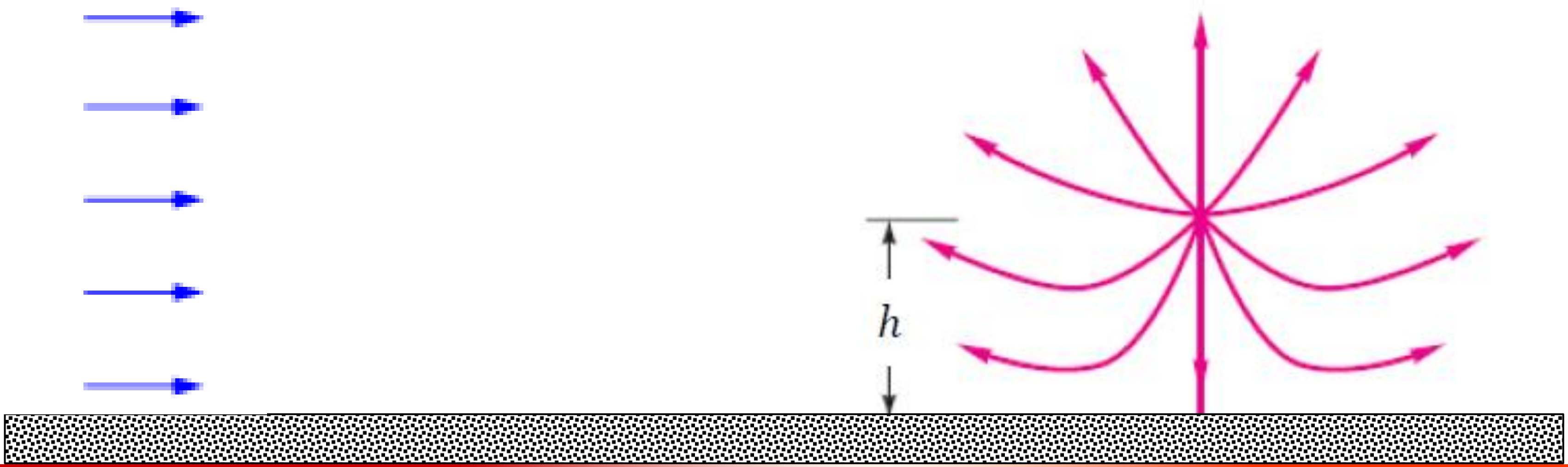
Example:

- *Consider a uniform flow approaching a sources with strength **K** that is located at distance h from a solid wall.*
  - *Please find the location and value of the maximum velocity on the wall.*
  - *Find the pressure on the x-axis.*
  - *What is the total force acting on the ground*

$$\psi = \frac{K}{2\pi}\theta; \quad \phi = \frac{K}{2\pi}\ln r$$

$$V_r = \frac{K}{2\pi r}; \quad V_\theta = 0$$

$U_\infty; \; P_\infty$



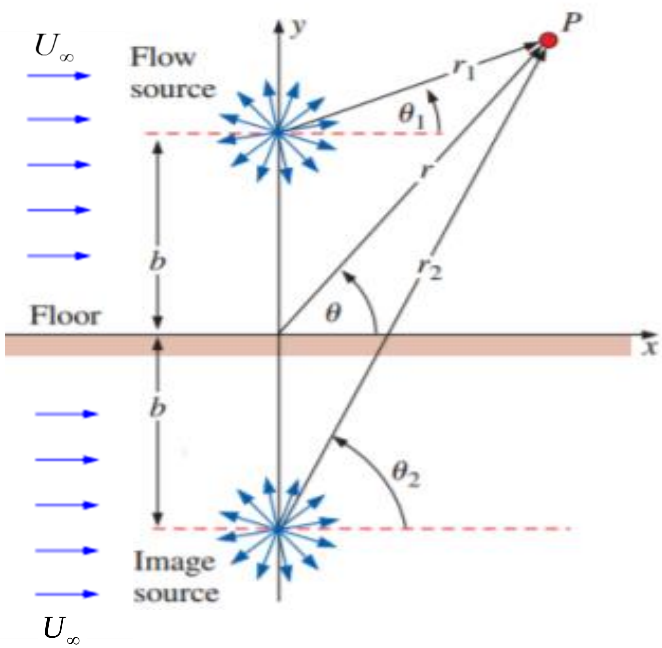
❑ Notes on Superposition of Basic Flows

Source near a wall- method of images

By symmetry, $\theta_1 = \theta_2$ and $r_1 = r_2$, therefore

$$u = U_\infty + 2V_r \cos\theta_1 = U_\infty + \frac{K \bullet r_1 \cos\theta_1}{\pi r_1^2} = U_\infty + \frac{K \bullet x}{\pi(x^2 + h^2)}$$

$$v = 0$$



- ***To find the maximum velocity on the wall:***

$$\frac{du}{dx} = 0 \Rightarrow \frac{du}{dx} = \frac{K}{\pi(x^2 + h^2)} + \frac{-2Kx^2}{\pi(x^2 + h^2)^2} = 0$$

$$\Rightarrow \frac{x^2 + h^2 - 2x^2}{\pi(x^2 + h^2)^2} = 0 \Rightarrow x = \pm h$$

$$u_{\max} = U_\infty \pm \frac{K \bullet h}{\pi(h^2 + h^2)} = U_\infty \pm \frac{K}{2\pi h}$$

Notes on Superposition of Basic Flows

- ***Based on Bernoulli's Equation***

$$P_\infty + \frac{1}{2}\rho U_\infty{}^2 = P(x) + \frac{1}{2}\rho u(x)^2$$

$$\because \quad u(x) = U_\infty + \frac{K \bullet x}{\pi(x^2+h^2)}$$

$$\therefore \quad P(x) = P_\infty + \frac{1}{2}\rho U_\infty{}^2 - \frac{1}{2}\rho u(x)^2$$

$$= P_\infty + \frac{1}{2}\rho U_\infty{}^2 - \frac{1}{2}\rho [U_\infty + \frac{K \bullet x}{\pi(x^2+h^2)}]^2$$

$$= P_\infty + \frac{1}{2}\rho U_\infty{}^2 - \frac{1}{2}\rho [U_\infty{}^2 + 2U_\infty \frac{K \bullet x}{\pi(x^2+h^2)} + \frac{K^2 \bullet x^2}{\pi^2(x^2+h^2)^2}]$$

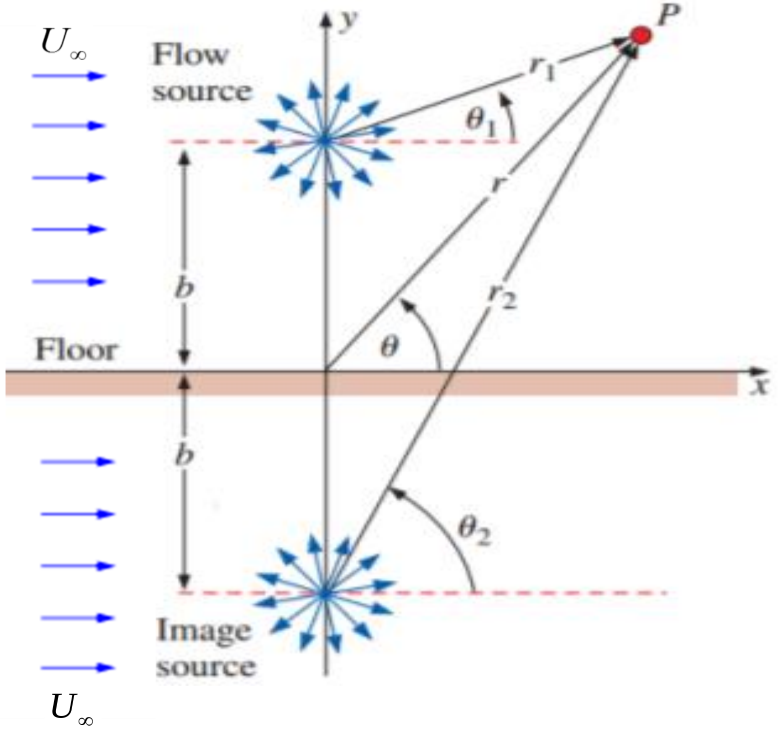
$$= P_\infty - \frac{1}{2}\rho [2U_\infty \frac{K \bullet x}{\pi(x^2+h^2)} + \frac{K^2 \bullet x^2}{\pi^2(x^2+h^2)^2}]$$

The pressure on the ground will be

$$P(x) = P_\infty - \frac{1}{2}\rho [2U_\infty \frac{K \bullet x}{\pi(x^2+h^2)} + \frac{K^2 \bullet x^2}{\pi^2(x^2+h^2)^2}]$$

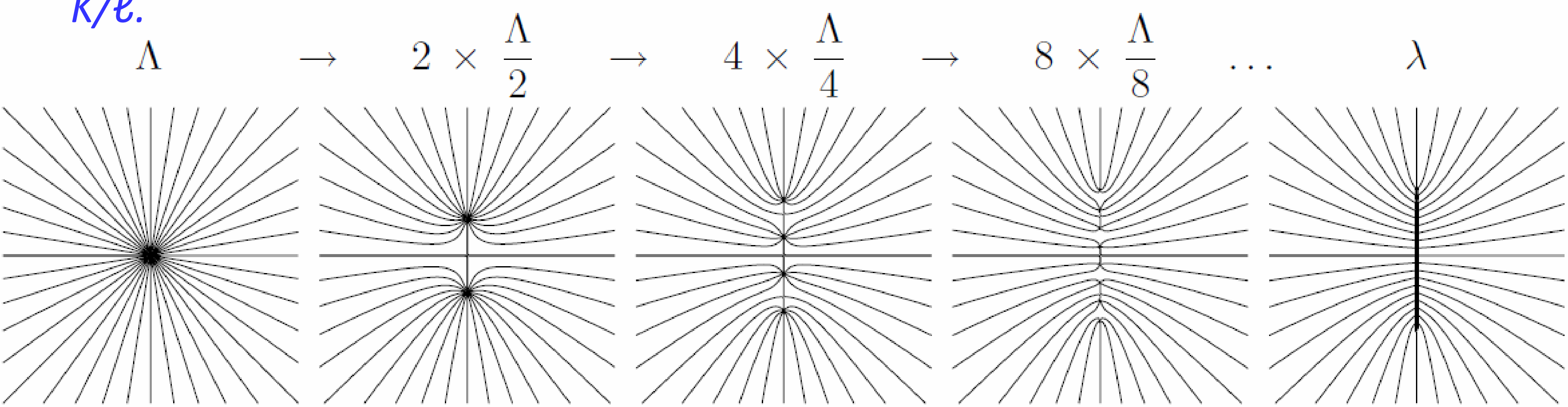
The total force acting on the ground will be

$$F_{total} = \int_{-\infty}^{+\infty} P(x)dx = \int_{-\infty}^{+\infty} \left\{ P_\infty - \frac{1}{2}\rho [2U_\infty \frac{K \bullet x}{\pi(x^2+h^2)} + \frac{K^2 \bullet x^2}{\pi^2(x^2+h^2)^2} \right\} dx$$

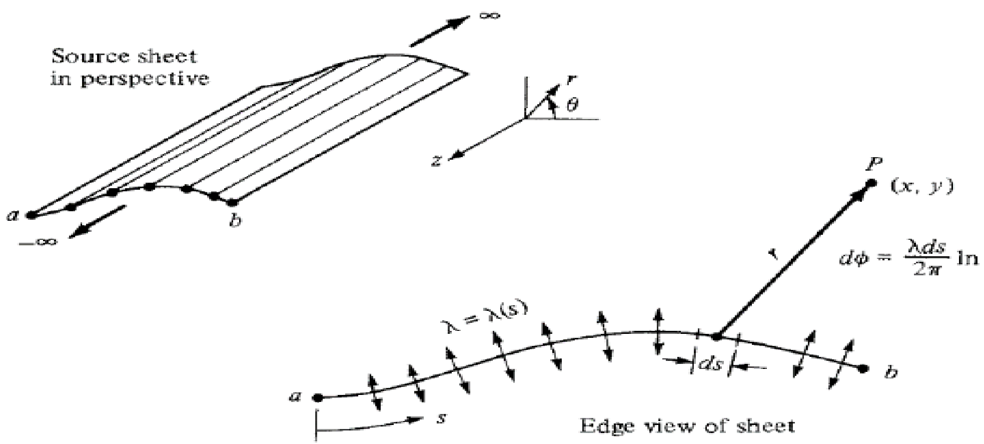


❑ Superposition of Basic Flows

- *Consider a sequence of flows where a single source of strength K is repeatedly subdivided into smaller sources which are evenly distributed along a line segment of length ℓ. The limit of this subdivision process is a source sheet of strength λ = K/ℓ.*



- ***Concept of source sheet:***



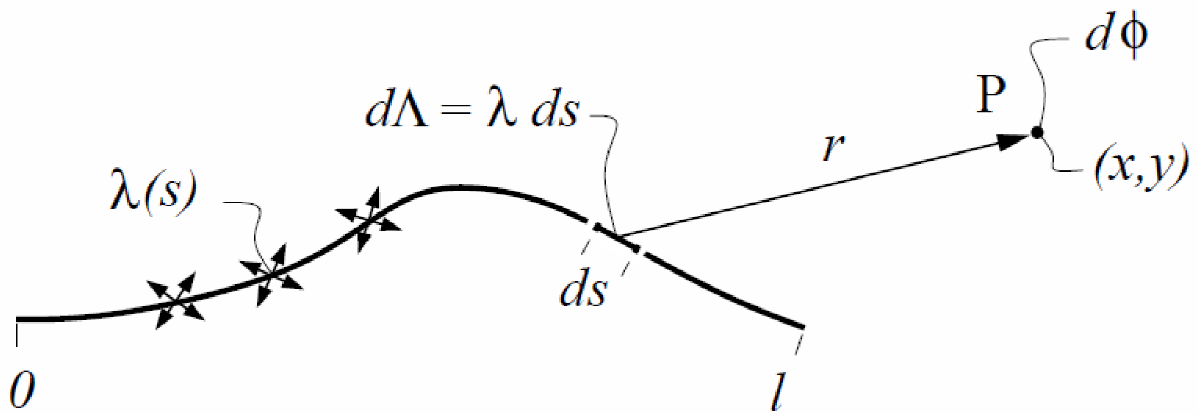
Superposition of Basic Flows

Properties

Consider an infinitesimal length ds of the sheet. The infinitesimal source strength of that piece is $d\Lambda = \lambda\, ds$, and the corresponding potential at some field point P at (x, y) is

$$d\phi = \frac{d\Lambda}{2\pi} \ln r = \frac{\lambda}{2\pi} \ln r \, ds$$

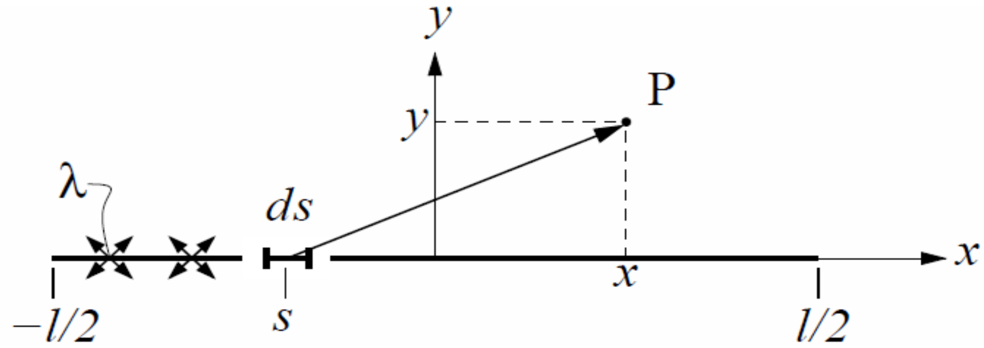
where r is the distance between point (x, y) and the point on the sheet.



The potential of the entire sheet at point P is then obtained by integrating the infinitesimal contributions all along the sheet.

$$\phi(x, y) = \int_0^{\ell} \frac{\lambda}{2\pi} \ln r \, ds$$

Superposition of Basic Flows



The potential and the velocity components at point P are given by

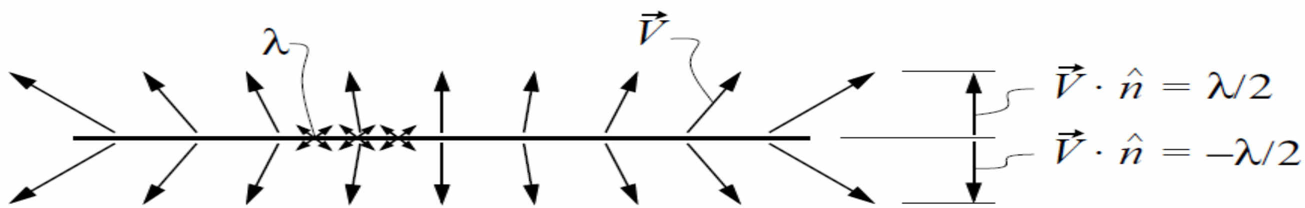
$$\phi(x,y) = \frac{\lambda}{2\pi}\int_{-\ell/2}^{\ell/2} \ln\sqrt{(x-s)^2+y^2}\ ds \tag{1}$$

$$u(x,y) = \frac{\partial\phi}{\partial x} = \frac{\lambda}{2\pi}\int_{-\ell/2}^{\ell/2} \frac{\partial}{\partial x}\left[\ln\sqrt{(x-s)^2+y^2}\right] ds = \frac{\lambda}{2\pi}\int_{-\ell/2}^{\ell/2} \frac{x-s}{(x-s)^2+y^2}\, ds$$

$$v(x,y) = \frac{\partial\phi}{\partial y} = \frac{\lambda}{2\pi}\int_{-\ell/2}^{\ell/2} \frac{\partial}{\partial y}\left[\ln\sqrt{(x-s)^2+y^2}\right] ds = \frac{\lambda}{2\pi}\int_{-\ell/2}^{\ell/2} \frac{y}{(x-s)^2+y^2}\, ds$$

After the necessary integration, we find that

$$v(x,0^+) = \frac{\lambda}{2} \quad , \qquad v(x,0^-) = -\frac{\lambda}{2} \qquad \text{(flat, isolated source sheet)}$$

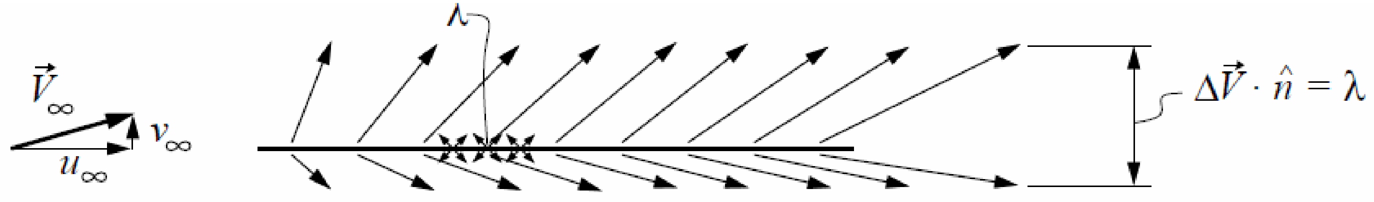


The normal velocity is then simply a constant $\lambda/2$ directed outward.

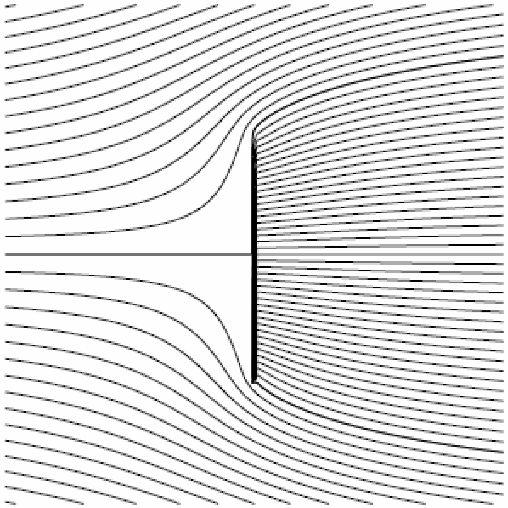
❑ Superposition of Basic Flows

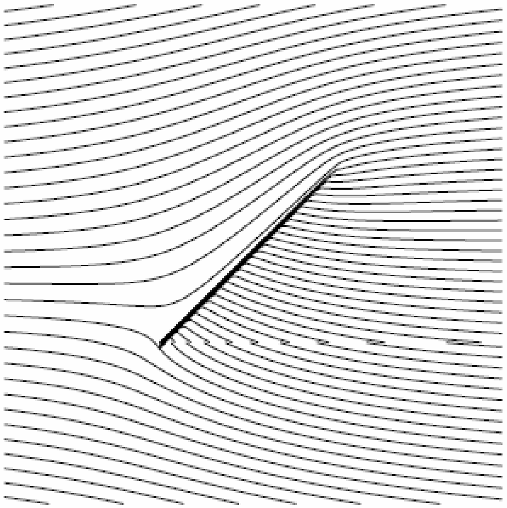
- If the source sheet is immersed in an incoming freestream, we will have

$$v(x,0^+) = \frac{\lambda}{2} + v_\infty \quad , \quad v(x,0^-) = -\frac{\lambda}{2} + v_\infty$$



- Flowfields shown in the figures below can be achieved by suitably adjusting the sheet's strength λ of source sheets superimposed on a uniform flow to the right.

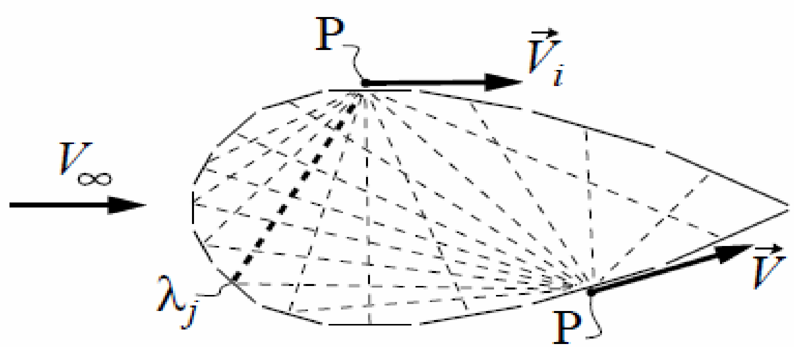


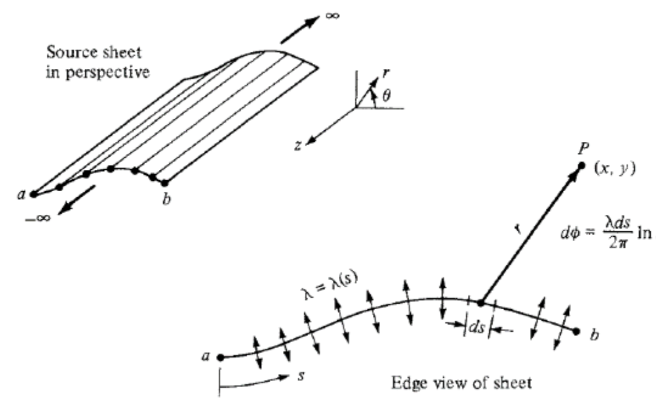


❑ Superposition of Basic Flows

$$\left(\vec{V}\cdot\hat{n}\right)_i \;=\; A_{ij}\,\lambda_j \;+\; \vec{V}_\infty\cdot\hat{n}_i$$

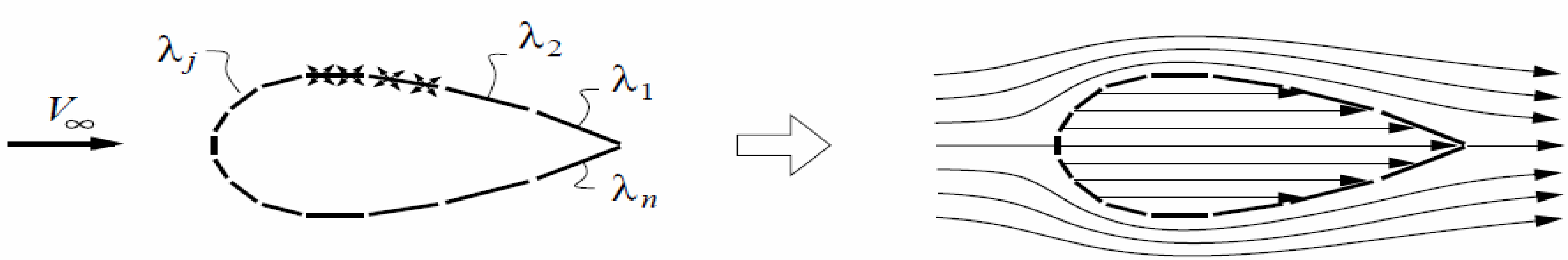
where A_{ij} is called the *aerodynamic influence matrix*, which can be computed once the geometry of all the panels is decided.





Requiring that $\vec{V}\cdot\hat{n} = 0$ for each of the n panel midpoints gives the following.

$$A_{ij}\,\lambda_j \;=\; -\vec{V}_\infty\cdot\hat{n}_i$$



uniform flow + n source panels

resulting flowfield

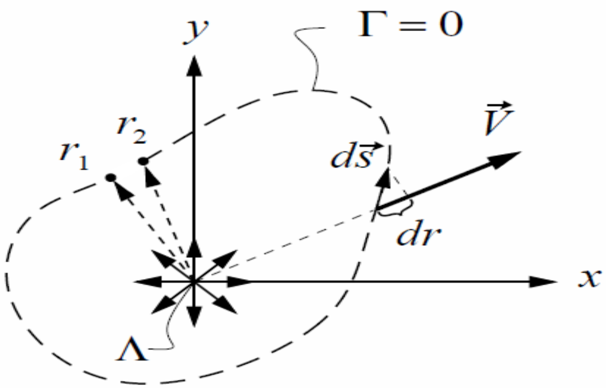
❑ Superposition of Basic Flows

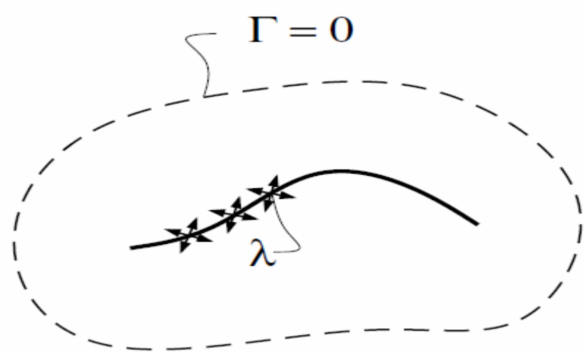
Limitations of Source Sheets

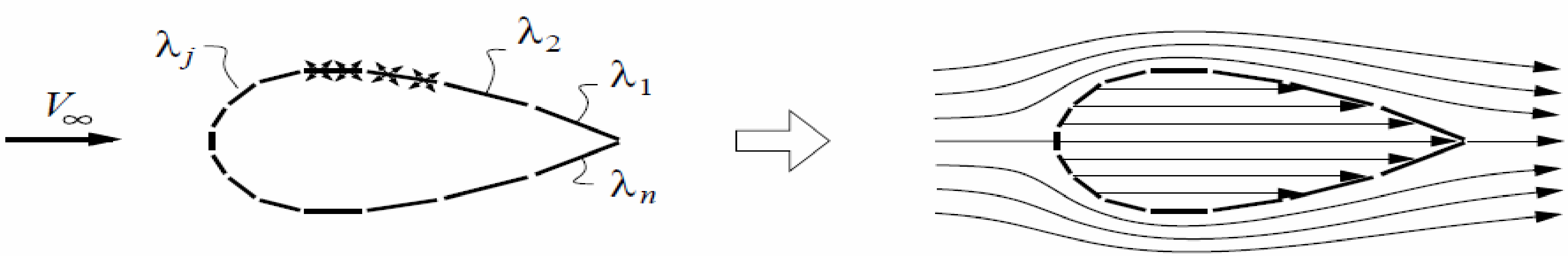
A point source has zero circulation about any circuit. Evaluating Γ using its definition we have

$$\Gamma \equiv -\oint \vec{V} \cdot d\vec{s} = -\oint V_r \, dr = -\int_{r_1}^{r_2} \frac{\Lambda}{2\pi r} \, dr = -\frac{\Lambda}{2\pi} (\ln r_2 - \ln r_1) = 0$$

which gives zero simply because $r_1 = r_2$ for any closed circuit, whether the origin is enclosed or not. A source sheet, which effectively consists of infinitesimal sources, must have zero circulation as well.







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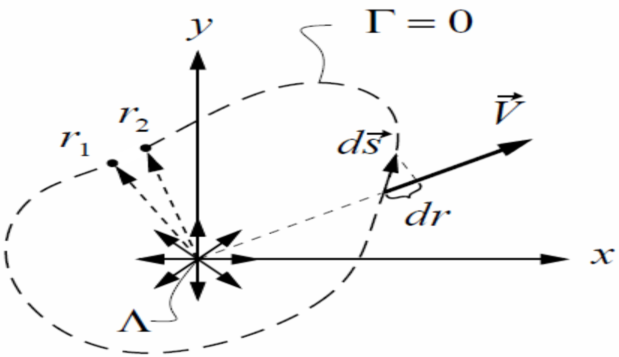
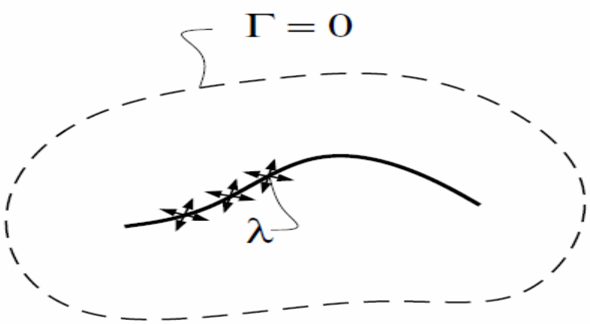
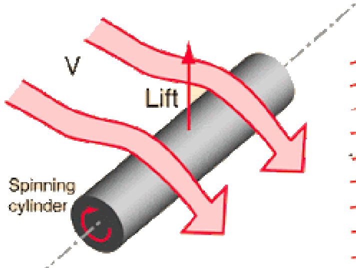
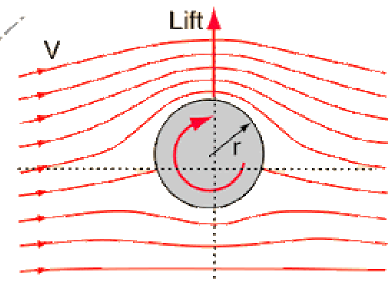




- ***The Kutta-Joukowski Lift Theorem* states the lift per unit length of a spinning cylinder is equal to the density (ρ) of the air times the strength of the rotation (Γ) times the velocity (V) of the air.**

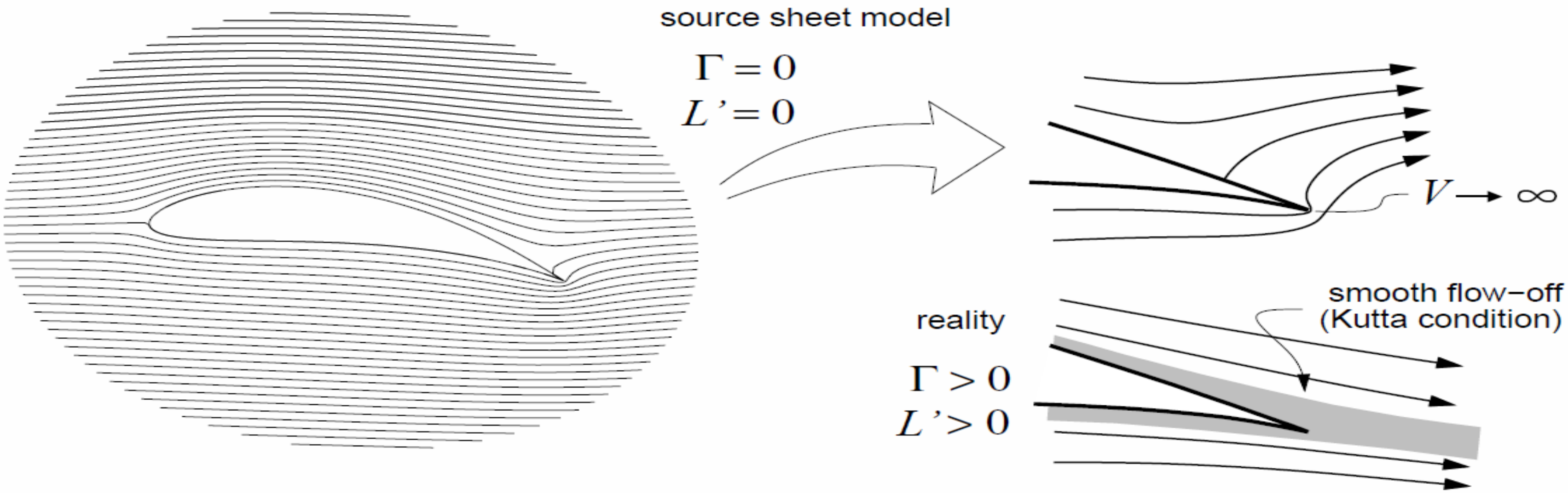
$$\boxed{L' = \rho V_\infty \Gamma}$$





❑ Superposition of Basic Flows

This limitation is illustrated if we use source panels to model a flow expected to produce lift, such as that on an airfoil at an angle of attack. Examination of the streamlines reveals that the rear dividing streamline leaves the airfoil off one surface as shown in the figure. The model also predicts an infinite velocity going around the sharp trailing edge.



- *This use of source sheets in this manner to represent a flow is the basis of the panel method, which is widely used to compute the flow about aerodynamic bodies of arbitrary shape.*
- *The approach of using panel method with source sheets is suitable only for non-lifting bodies such as fuselages.*
- *For airfoils, wings, and other lifting bodies, vortices must be added in some form to enable circulation to be represented.*

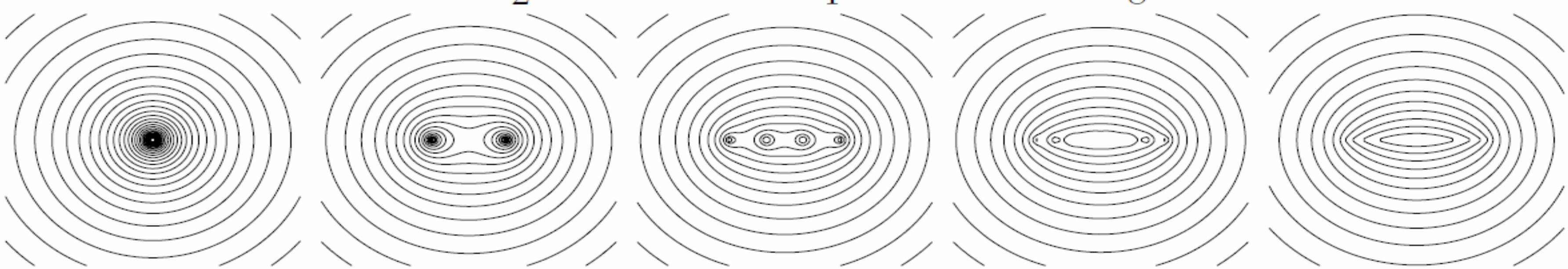
❑ Superposition of Basic Flows

Vortex Sheets

Definition

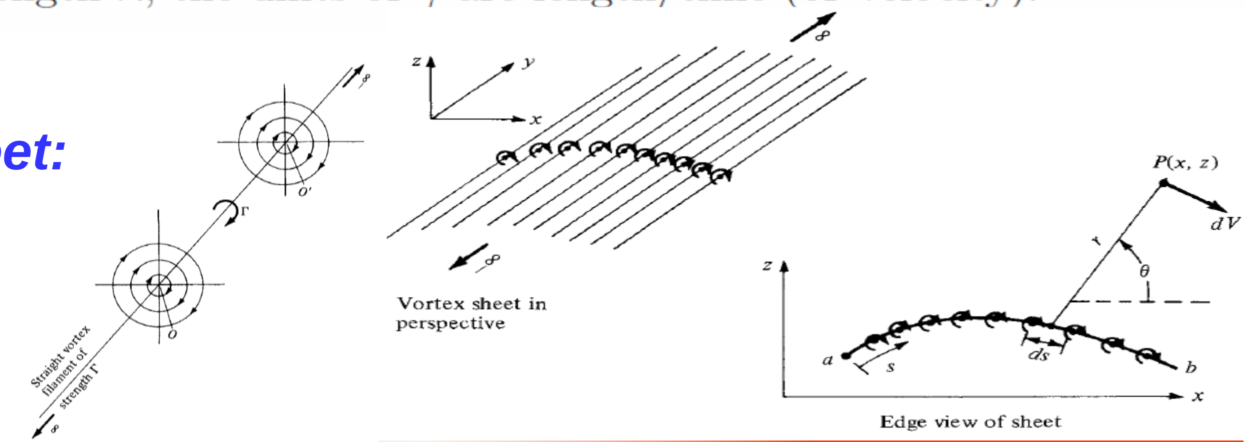
Consider a sequence of flows where a single vortex of strength Γ is repeatedly subdivided into smaller vortices which are evenly distributed along a line segment of length ℓ. The limit of this subdivision process is a *vortex sheet* of strength $\gamma = \Gamma/\ell$.

$$\Gamma \quad \rightarrow \quad 2 \times \frac{\Gamma}{2} \quad \rightarrow \quad 4 \times \frac{\Gamma}{4} \quad \rightarrow \quad 8 \times \frac{\Gamma}{8} \quad \ldots \quad \gamma$$



Like with the source sheet strength λ, the units of γ are length/time (or velocity).

- ***Concept of source sheet:***

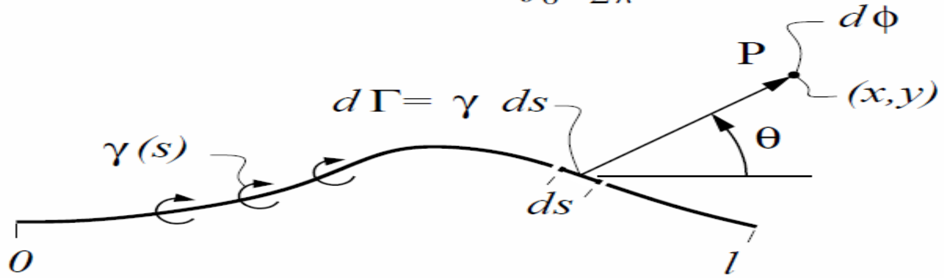


❑ Superposition of Basic Flows

Properties

The analysis of the vortex sheet closely follows that of the source sheets. The potential of the vortex sheet at point P is

$$\phi(x,y) \;=\; -\int_0^\ell \frac{\gamma}{2\pi}\,\theta\;ds$$



For a straight vortex sheet extending from $(-\ell/2, 0)$ to $(\ell/2, 0)$, with a constant strength γ, the potential and the velocity components at point P are given by

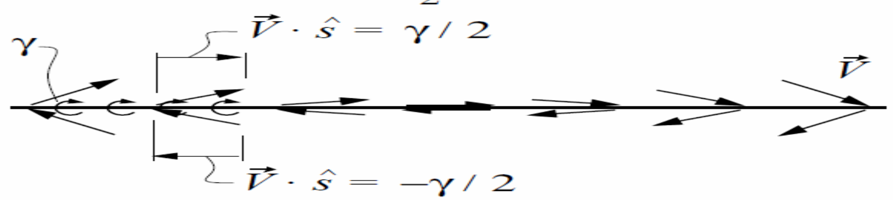
$$\phi(x,y) \;=\; \frac{\gamma}{2\pi}\int_{-\ell/2}^{\ell/2} -\arctan\frac{y}{x-s}\;ds$$

$$u(x,y) \;=\; \frac{\partial\phi}{\partial x} \;=\; \frac{\gamma}{2\pi}\int_{-\ell/2}^{\ell/2}\frac{\partial}{\partial x}\left[-\arctan\frac{y}{x-s}\right]ds \;=\; \frac{\gamma}{2\pi}\int_{-\ell/2}^{\ell/2}\frac{y}{(x-s)^2+y^2}\;ds$$

$$v(x,y) \;=\; \frac{\partial\phi}{\partial y} \;=\; \frac{\gamma}{2\pi}\int_{-\ell/2}^{\ell/2}\frac{\partial}{\partial y}\left[-\arctan\frac{y}{x-s}\right]ds \;=\; \frac{\gamma}{2\pi}\int_{-\ell/2}^{\ell/2}\frac{-x}{(x-s)^2+y^2}\;ds$$

- If we evaluate very close to the sheet, either just above at $y = 0^+$, or just below at $y = 0-$, the tangential velocity becomes very simple.

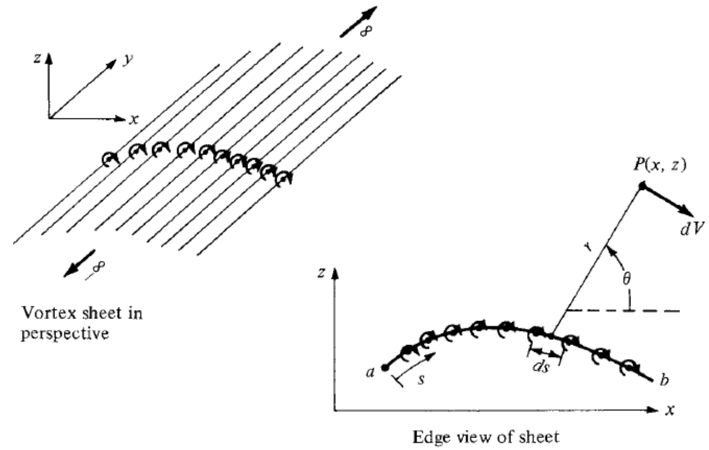
$$u(x,0^+) \;=\; \frac{\gamma}{2} \quad , \qquad u(x,0^-) \;=\; -\frac{\gamma}{2} \qquad \text{(flat, isolated vortex sheet)}$$



Superposition of Basic Flows

Velocity discontinuity

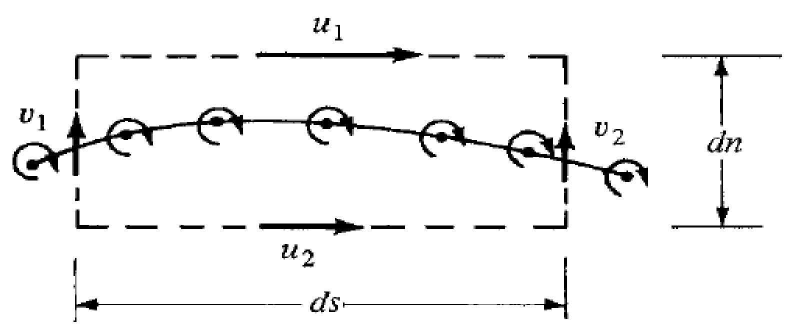
- On a source sheet the normal component of velocity is discontinuous (note 180-degree change in direction)
- On a vortex sheet the tangential component of velocity is discontinuous



Consider circulation over a box with size ds and dn

$$\Gamma = v_2 dn - u_1 ds - v_1 dn + u_2 ds$$
$$\Gamma = (u_2 - u_1) ds + (v_2 - v_1) dn$$

But $\Gamma = \gamma ds$



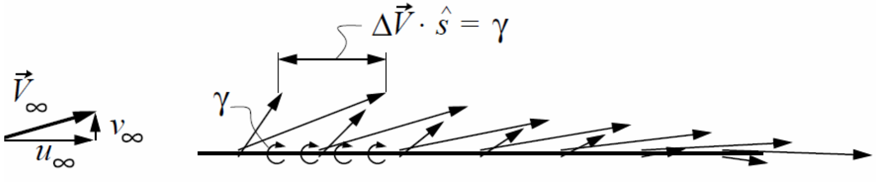
$$\gamma ds = (u_2 - u_1) ds + (v_2 - v_1) dn$$

As $dn \to 0$ then $\gamma ds = (u_2 - u_1) ds \to$

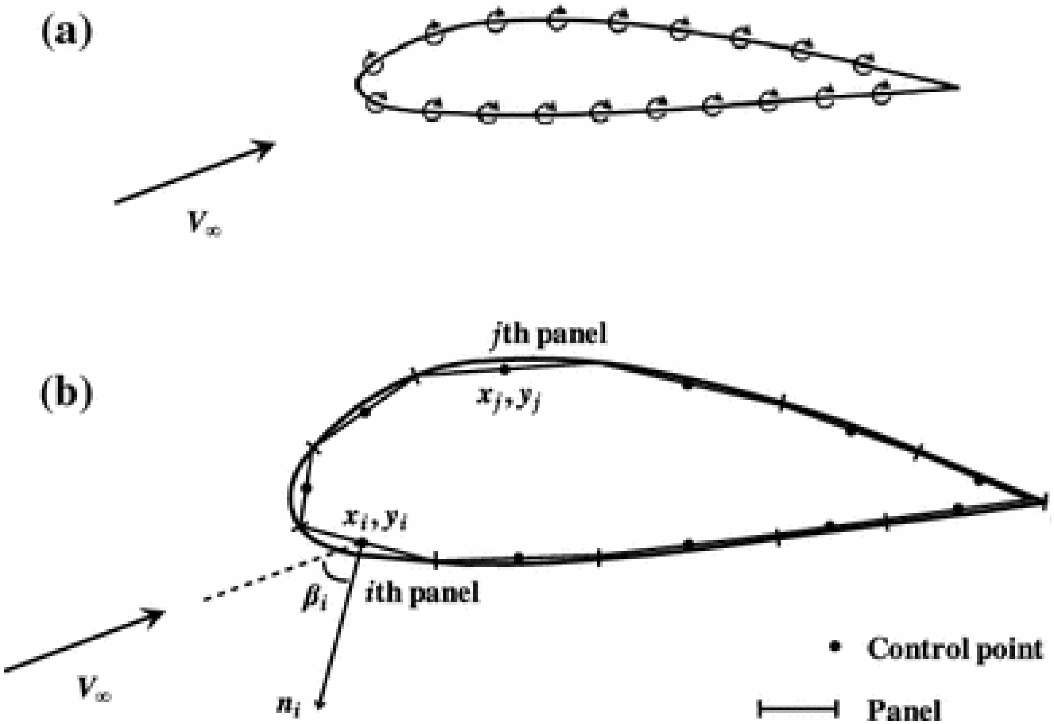
$$\boxed{\gamma = u_2 - u_1}$$

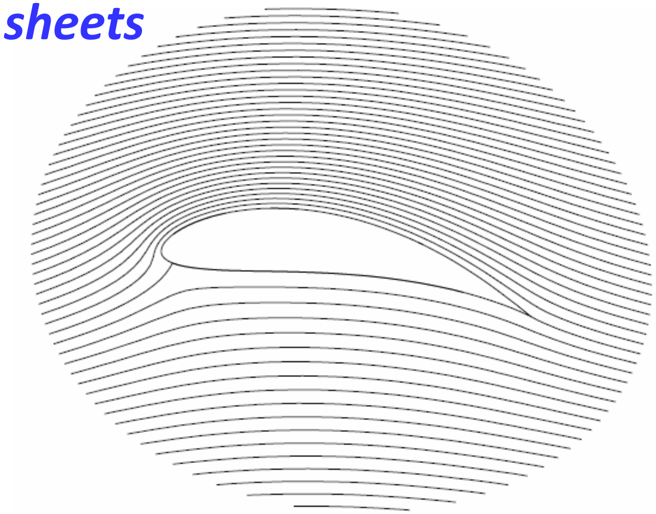
❑ Superposition of Basic Flows

- *The figure shows a vortex sheet exposed to an incoming freestream superimposed. The surface velocity vector pattern is very complicated, but the tangential velocity jump across the sheet is a constant equal to γ at all points.*



- ***The approach of using** panel method with vortex sheets*





The Kutta-Joukowski Lift Theorem:

$$\boxed{L' = \rho V_\infty \Gamma}$$