# Lecture # 27: Airfoil Aerodynamics – Part 05 : Cambered Airfoil

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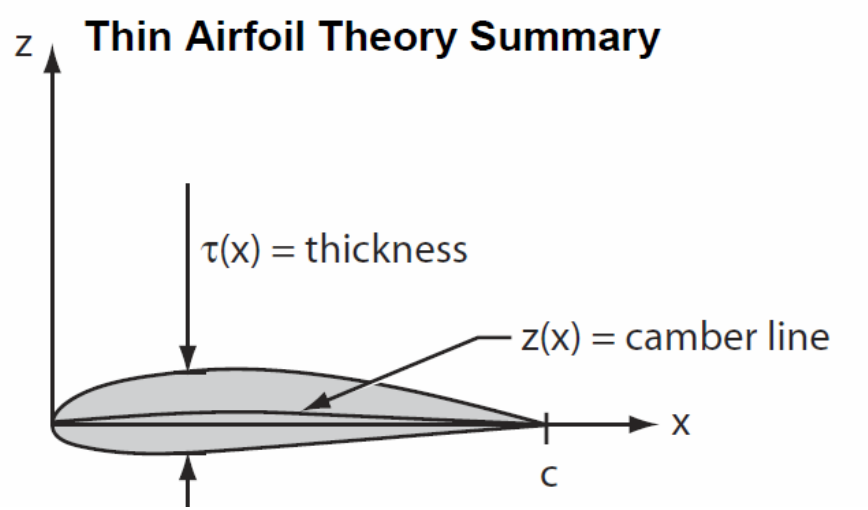
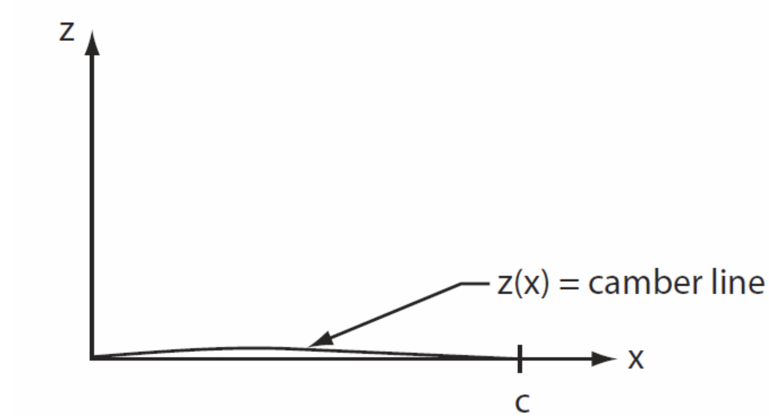
# ❑ Thin Airfoil Theory

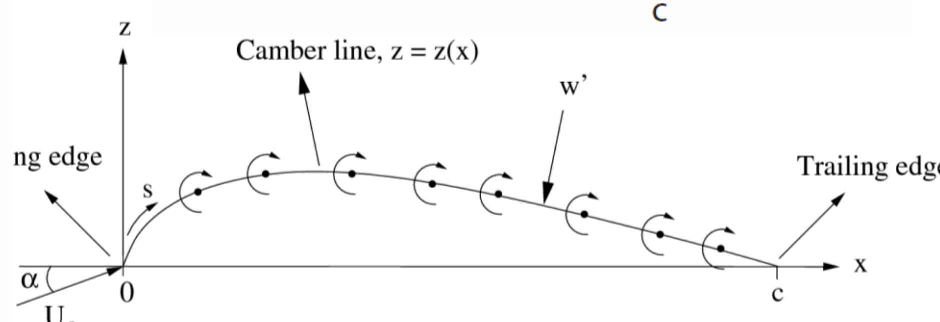
***Principle:***

- ***Replace thin airfoil with the mean camber line (MCL) because of the small thickness and camber of the airfoil***
- ***MCL assumed to be a streamline of the flow around the thin airfoil.***
- ***To force the MCL to be a streamline, the sum of all velocity components normal to the MCL must be equal to zero.***

$$V_{\infty,n} + V_{vortex-induced,n} = 0$$



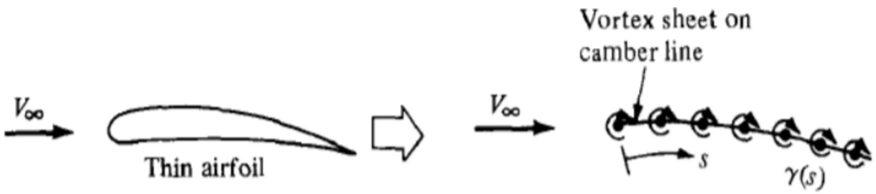




(b) Vorticity distribution on the camber line of the airfoil

# ❑ Thin Airfoil Theory

- *To force the mean camber line to be a streamline, the sum of all velocity components normal to the mcl must be equal to zero.*

$$V_{\infty,n} + V_{vortex-induced,n} = 0$$



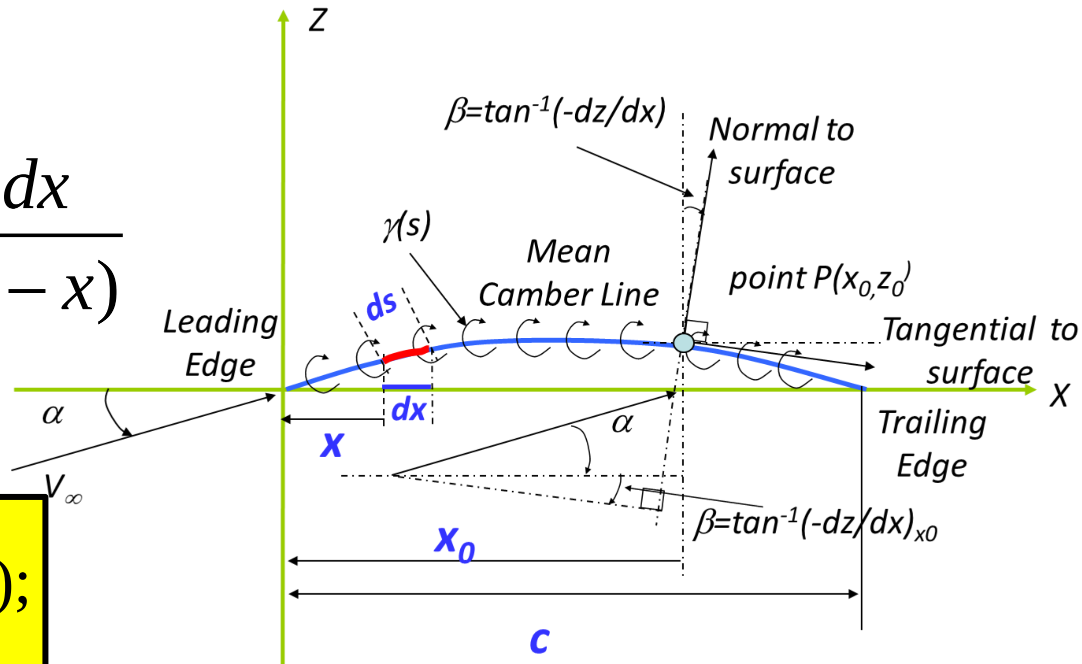

- ***Since***

$$V_{\infty,n} = V_\infty(\alpha - \frac{dz}{dx});$$

$$V_{vortex-induced,n} = -\int_0^c \frac{\gamma(x)dx}{2\pi(x_0 - x)}$$

- ***Therefore:***

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(x)dx}{(x_0 - x)} = V_\infty(\alpha - \frac{dz}{dx});$$



# ❑ Thin Airfoil Theory – Cambered Airfoil

❑ ***The integral equation of thin airfoil theory:***

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(x)dx}{(x_0 - x)} = V_\infty(\alpha - \frac{dz}{dx});$$

- ***To solve the integral equation, we first make a transformation:***

$$x = \frac{c}{2}(1-\cos\theta); \quad \theta = (0,\pi)$$

$$If \;\; \theta = 0 \quad \Rightarrow \quad x = 0, \quad at\ Leading\ \ Edge$$

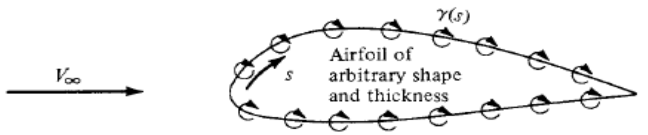
$$If \;\; \theta = \pi \quad \Rightarrow \quad x = c, \quad at\ Trailing\ \ Edge$$

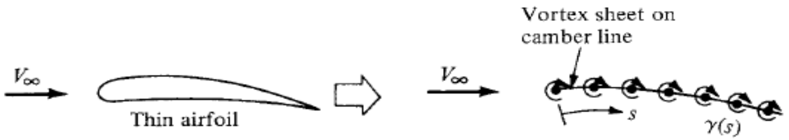
*Then*, $x_0$ is a point on $x$ , corresponding to $\theta_0$

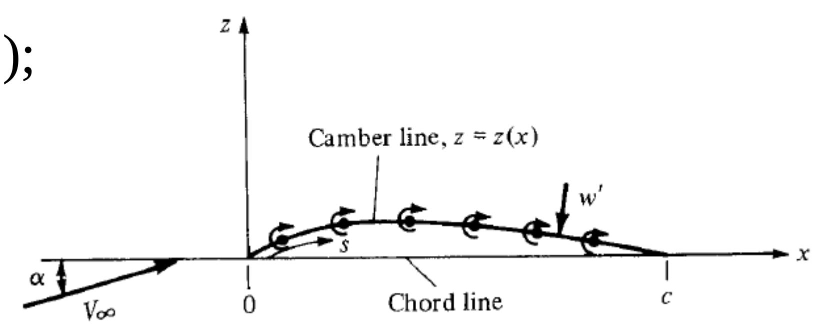
$$Then: \quad x = \frac{c}{2}(1-\cos\theta_0);$$

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\bullet c \bullet \sin\theta d\theta}{\frac{c}{2}(1-\cos\theta_0) - \frac{c}{2}(1-\cos\theta)} = V_\infty(\alpha - \frac{dz}{dx});$$

$$\Rightarrow \frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta d\theta}{\cos\theta - \cos\theta_0} = V_\infty(\alpha - \frac{dz}{dx});$$



# Thin Airfoil Theory

- **The integral equation of thin airfoil theory:**

$$\frac{1}{2\pi}\int_0^{\pi}\frac{\gamma(\theta)\sin\theta d\theta}{\cos\theta-\cos\theta_0}=V_\infty\left(\alpha-\frac{dz}{dx}\right);$$

- **Solution of the integral equation for a symmetrical airfoil:**

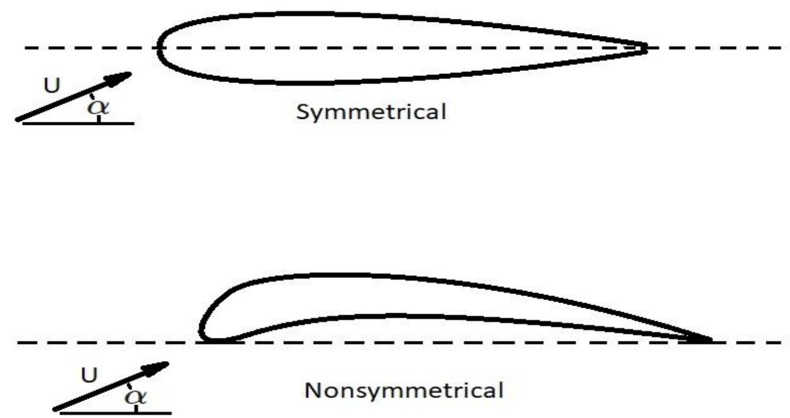
**Since it is symmetrical airfoil, therefore:**

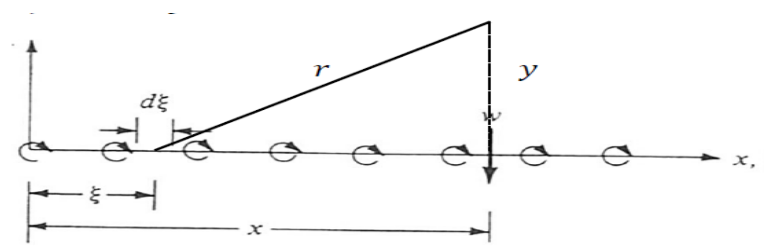
$$\frac{dz}{dx}=0$$

$$\frac{dz}{dx}=0\Rightarrow\frac{1}{2\pi}\int_0^{\pi}\frac{\gamma(\theta)\sin\theta d\theta}{\cos\theta-\cos\theta_0}=V_\infty\alpha;$$

*The solution will be:*

$$\gamma(\theta)=2\alpha V_\infty\frac{1+\cos\theta}{\sin\theta}$$



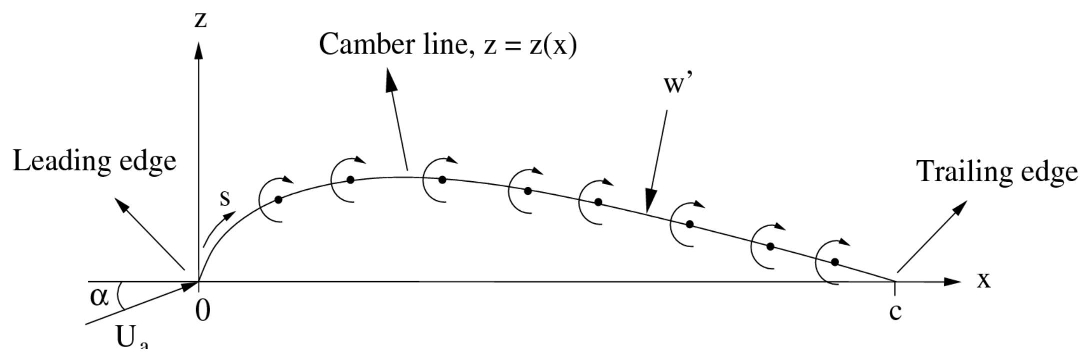



# Thin Airfoil Theory for Cambered Airfoils

- ***The integral equation of thin unsymmetrical airfoil.***

$$\frac{1}{2\pi}\int_0^{\pi}\frac{\gamma(\theta)\sin\theta d\theta}{\cos\theta-\cos\theta_0}=(V_\infty\alpha-\frac{dz}{dx})$$

$$\frac{dz}{dx}\neq 0$$




- ***Solution of the equations:***

$$\gamma(\theta)=2V_\infty[A_0\frac{1+\cos\theta}{\sin\theta}+\sum_{n=1}^{\infty}A_n\sin(n\theta)]$$

# Thin Airfoil Theory for Cambered Airfoils

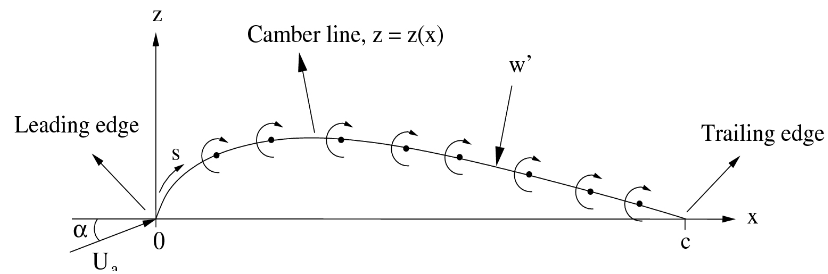
$$\frac{1}{2\pi}\int_0^{\pi}\frac{\gamma(\theta)\sin\theta d\theta}{\cos\theta-\cos\theta_0}=V_\infty(\alpha-\frac{dz}{dx})$$

## Verification of the solution:

$$\gamma(\theta)=2V_\infty(A_0\frac{1+\cos\theta}{\sin\theta}+\sum_{n=1}^{\infty}A_n\sin(n\theta)]$$

$$\Rightarrow\frac{1}{2\pi}\int_0^{\pi}\frac{\gamma(\theta)\sin\theta d\theta}{\cos\theta-\cos\theta_0}=\frac{1}{2\pi}\int_0^{\pi}\frac{2V_\infty[A_0(\frac{1+\cos\theta}{\sin\theta})+\sum_{n=1}^{\infty}A_n\sin(n\theta)]\sin\theta d\theta}{\cos\theta-\cos\theta_0}$$

$$=\frac{V_\infty}{\pi}[\int_0^{\pi}\frac{A_0(1+\cos\theta)}{\cos\theta-\cos\theta_0}d\theta+\int_0^{\pi}\frac{\sum_{n=1}^{\infty}A_n\sin(n\theta)\sin\theta}{\cos\theta-\cos\theta_0}d\theta]$$




## For the first term:

$$\because\quad\int_0^{\pi}\frac{\cos(n\theta)d\theta}{\cos\theta-\cos\theta_0}=\frac{\pi\sin(n\theta_0)}{\sin\theta_0}$$

$$\therefore\int_0^{\pi}\frac{A_0(1+\cos\theta)}{\cos\theta-\cos\theta_0}d\theta=\int_0^{\pi}\frac{A_0}{\cos\theta-\cos\theta_0}d\theta+\int_0^{\pi}\frac{A_0\cos\theta}{\cos\theta-\cos\theta_0}d\theta$$

$$=A_0[\frac{\pi\sin(0\bullet\theta_0)}{\sin\theta_0}+\frac{\pi\sin(1\cdot\theta_0)}{\sin\theta_0}]=A_0\pi$$

# Thin Airfoil Theory for Cambered Airfoils

## Verification of the solution:

$$\gamma(\theta) = 2V_\infty [A_0 \frac{1+\cos\theta}{\text{sins}\,\theta} + \sum_{n=1}^{\infty} A_n \sin(n\theta)]$$

$$\Rightarrow \frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta d\theta}{\cos\theta - \cos\theta_0} = \frac{V_\infty}{\pi}[\int_0^\pi \frac{A_0(1+\cos\theta)}{\cos\theta-\cos\theta_0}d\theta + \int_0^\pi \frac{\sum_{n=1}^{\infty} A_n \sin(n\theta)\sin\theta}{\cos\theta-\cos\theta_0}d\theta]$$

## For the 2nd term in the above equation:

$$\int_0^\pi \frac{\sin(n\theta)\sin\theta d\theta}{\cos\theta-\cos\theta_0} = -\pi\cos(n\theta_0)$$

$$\therefore \int_0^\pi \frac{\sum_{n=1}^{\infty} A_n \sin(n\theta)\sin\theta}{\cos\theta-\cos\theta_0}d\theta] = -\pi\sum_{n=1}^{\infty} A_n\cos(n\theta_0)$$

*Therefore*:

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta d\theta}{\cos\theta - \cos\theta_0} = \frac{V_\infty}{\pi}[A_0\pi - \pi\sum_{n=1}^{\infty} A_n\cos(n\theta_0)]$$

$$= V_\infty[A_0 - \sum_{n=1}^{\infty} A_n\cos(n\theta_0)]$$

## Therefore, the controlling equation will be:

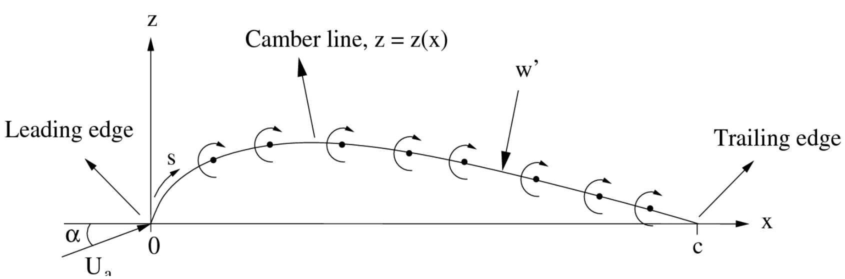
$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\sin\theta d\theta}{\cos\theta - \cos\theta_0} = V_\infty(\alpha - \frac{dz}{dx})$$

$$\Rightarrow V_\infty[A_0 - \sum_{n=1}^{\infty} A_n\cos(n\theta_0)] = V_\infty(\alpha - \frac{dz}{dx})$$

$$\Rightarrow \frac{dz}{dx} = (\alpha - A_0) + \sum_{n=1}^{\infty} A_n\cos(n\theta_0)$$

# ❑ Thin Airfoil Theory for Cambered Airfoils

$$\frac{dz}{dx} = (\alpha - A_0) + \sum_{n=1}^{\infty} A_n \cos(n\theta_0)$$

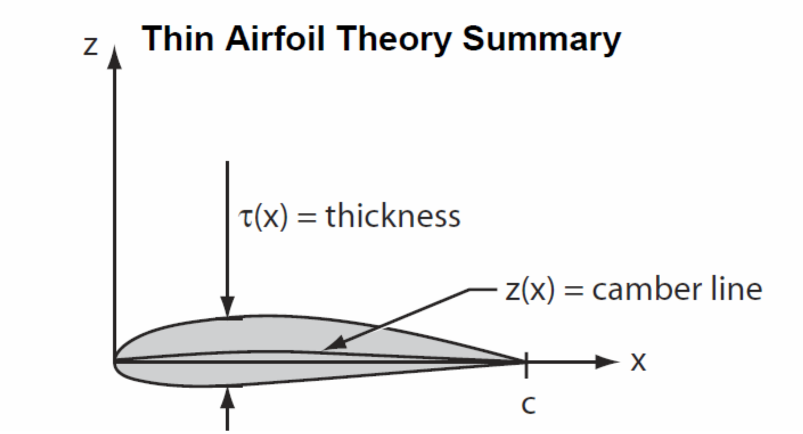



- ❑ ***From Fourie's series expression***

$$\frac{dz}{dx} = f(\theta) = B_0 + \sum_{n=1}^{\infty} B_n \cos(n\theta_0)$$

$$B_0 = \frac{1}{\pi}\int_0^{\pi} f(\theta)d\theta;$$

$$B_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos(n\theta)d\theta; \quad n = 1, 2, \ldots\infty$$




- ❑ ***Therefore:***

$$A_0 = \alpha - B_0 = \alpha - \frac{1}{\pi}\int_0^{\pi} f(\theta)d\theta = \alpha - \frac{1}{\pi}\int_0^{\pi} \frac{dz}{dx}d\theta$$

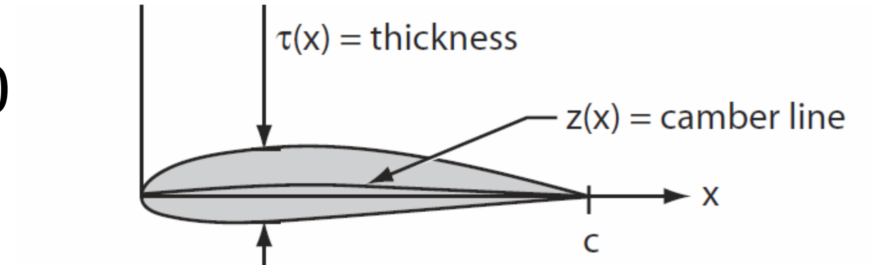
$$A_n = B_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos(n\theta)d\theta = \frac{2}{\pi}\int_0^{\pi} \frac{dz}{dx}\cos(n\theta)d\theta$$

# Thin Airfoil Theory

- **The integral equation of thin airfoil theory:**

$$\frac{1}{2\pi}\int_0^{\pi}\frac{\gamma(\theta)\sin\theta d\theta}{\cos\theta-\cos\theta_0}=V_\infty(\alpha-\frac{dz}{dx});$$

- **For a symmetrical airfoil:** $\frac{dz}{dx}=0$

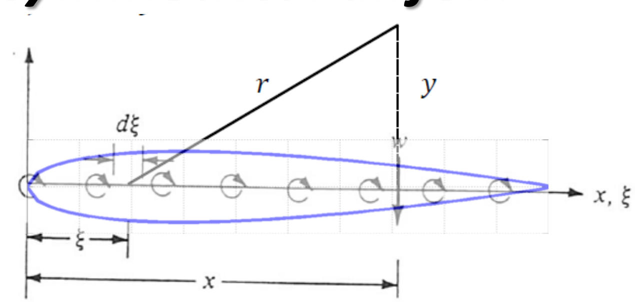



- **The integral equation of thin airfoil theory can be simplified as:**

$$\frac{1}{2\pi}\int_0^{\pi}\frac{\gamma(\theta)\sin\theta d\theta}{\cos\theta-\cos\theta_0}=V_\infty\alpha;$$


- **Solution of the integral equation for a symmetrical airfoil:**

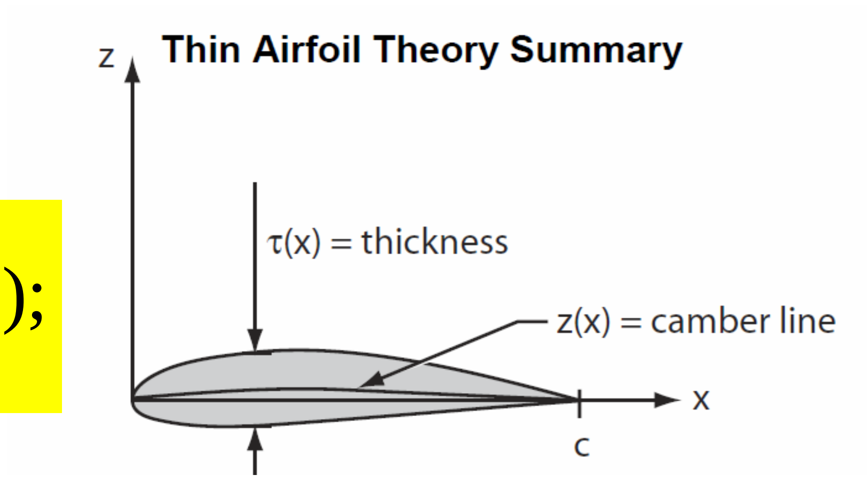
$$\gamma(\theta)=2\alpha V_\infty\frac{1+\cos\theta}{\sin\theta}$$



# ❑ Thin Airfoil Theory for Cambered Airfoils

❑ ***The integral equation for a cambered airfoil:***

$$\frac{1}{2\pi}\int_0^{\pi}\frac{\gamma(\theta)\sin\theta d\theta}{\cos\theta-\cos\theta_0}=V_\infty(\alpha-\frac{dz}{dx});$$

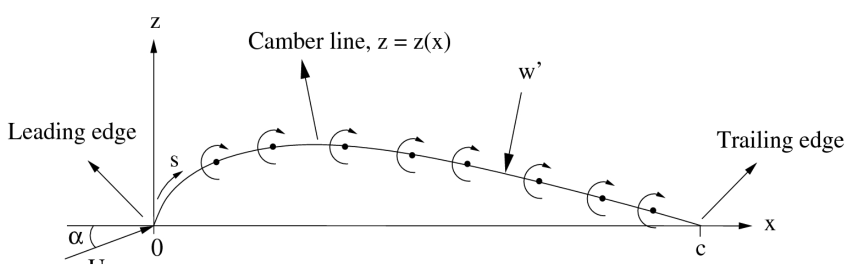



❑ ***The solution for a cambered airfoil will be:***

$$\gamma(\theta)=2V_\infty[A_0\frac{1+\cos\theta}{\sin\theta}+\sum_{n=1}^{\infty}A_n\sin(n\theta)]$$

$$A_0=\alpha-\frac{1}{\pi}\int_0^{\pi}\frac{dz}{dx}d\theta$$

$$A_n=\frac{2}{\pi}\int_0^{\pi}f(\theta)\cos(n\theta)d\theta=\frac{2}{\pi}\int_0^{\pi}\frac{dz}{dx}\cos(n\theta)d\theta$$



# ❑ Thin Airfoil Theory – Cambered Airfoil

❑ *Is **Kutta condition** satisfied at TE?*

$$\gamma(\theta) = 2V_\infty[A_0 \frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin(n\theta)]$$

$$\text{at airfoil LE}: \quad \theta = \pi$$

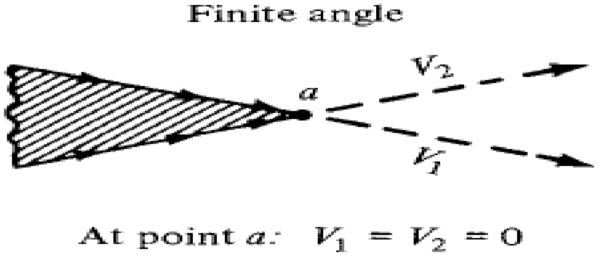
$$\Rightarrow \gamma(\pi) = 2V_\infty A_0 \frac{1+\cos\pi}{\sin\pi} = \frac{0}{0}$$

*Therefore:*

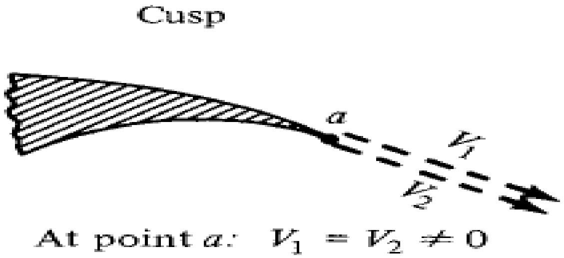
$$\gamma(\theta)\Big|_{\theta\to\pi} = 2V_\infty A_0 \left.\frac{\dfrac{d(1+\cos\theta)}{d\theta}}{\dfrac{d(\sin\theta)}{d\theta}}\right|_{\theta\to\pi}$$

$$= 2V_\infty A_0 \left.\frac{-\sin\theta}{\cos\theta}\right|_{\theta\to\pi} = 2V_\infty A_0 \frac{0}{-1} = 0$$

$$\Rightarrow \text{Kutta condition is satisfied at TE!}$$




- *Case #1*




- *Case #2*

- In relation to the vortex sheet discontinuity

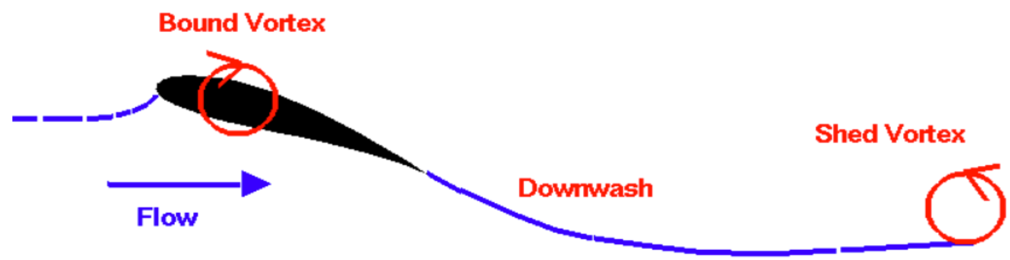
$$\gamma(\text{TE}) = V_2 - V_1$$

$$\gamma(\text{TE}) = 0$$

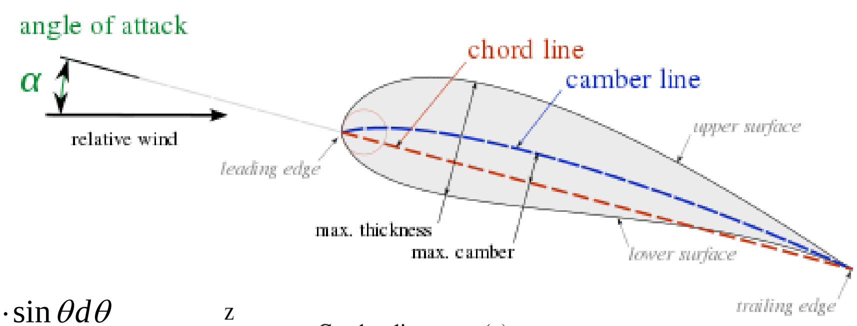
# Thin Airfoil Theory – Cambered Airfoil

- ***The Kutta-Joukowski Lift Theorem:***

$$L' = \rho V_\infty \Gamma$$




## *Lift generated by the cambered airfoil*




$$\Gamma = \int_{LE}^{TE} \gamma(x)dx$$

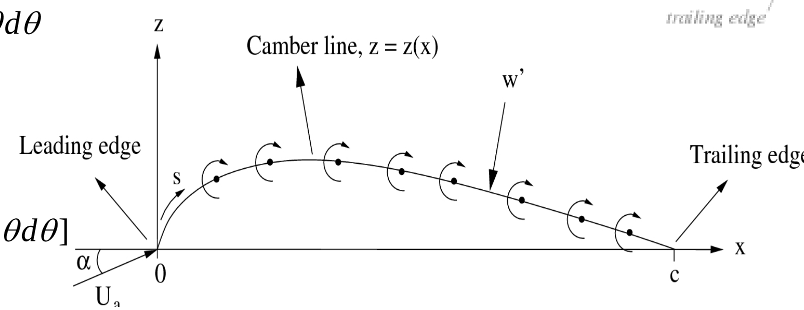
$$\gamma(\theta) = 2V_\infty[A_0 \frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin(n\theta)]$$

$$\because x = \frac{c}{2}(1-\cos\theta) \Rightarrow dx = \frac{c}{2}\sin\theta d\theta$$

$$\therefore \Gamma = \int_{LE}^{TE} \gamma(x)dx = \int_0^{\pi} \gamma(\theta)\cdot\frac{c}{2}\cdot\sin\theta\cdot d\theta = \frac{c}{2}\cdot\int_0^{\pi} 2V_\infty[A_0\frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin(n\theta)]\cdot\sin\theta d\theta$$

$$= V_\infty c\cdot\{\int_0^{\pi} A_0 \frac{1+\cos\theta}{\sin\theta}\cdot\sin\theta d\theta + \int_0^{\pi}[\sum_{n=1}^{\infty} A_n \sin(n\theta)]\cdot\sin\theta d\theta\}$$

$$= V_\infty c\{A_0(\theta+\sin\theta)\Big|_0^{\pi} + \int_0^{\pi}[\sum_{n=1}^{\infty} A_n \sin(n\theta)]\cdot\sin\theta d\theta\} = V_\infty c[A_0\pi + \sum_{n=1}^{\infty} A_n \int_0^{\pi}\sin(n\theta)\cdot\sin\theta d\theta]$$




$$\because \int_0^{\pi}\sin(n\theta)\cdot\sin\theta d\theta] = \Bigg| \begin{matrix} \frac{\pi}{2} & if & n=1 \\ 0 & if & n\neq 1 \end{matrix}$$

$$\therefore \Gamma = V_\infty c[A_0\pi + \frac{\pi}{2}A_1]$$

# ❑ Thin Airfoil Theory – Cambered Airfoil

## ❑ *Lift coefficient of cambered airfoil*

$$L=\rho V_\infty \Gamma=\rho V_\infty{}^2 c[A_0\pi+\frac{\pi}{2}A_1]$$

$$C_L=\frac{L}{\frac{1}{2}\rho V_\infty{}^2 c}=\frac{\rho V_\infty{}^2 c[A_0\pi+\frac{\pi}{2}A_1]}{\frac{1}{2}\rho V_\infty{}^2 c}=2\pi[A_0+A_1/2]$$

$$\because A_0=\alpha+B_0=\alpha-\frac{1}{\pi}\int_0^\pi \frac{dz}{dx}d\theta;\quad A_1=\frac{2}{\pi}\int_0^\pi \frac{dz}{dx}\cos(\theta)d\theta$$

$$\therefore C_L=2\pi[\alpha-\frac{1}{\pi}\int_0^\pi \frac{dz}{dx}(1-\cos\theta)d\theta]$$

$$\alpha_{L0}=\frac{1}{\pi}\int_0^\pi \frac{dz}{dx}(1-\cos\theta)d\theta$$

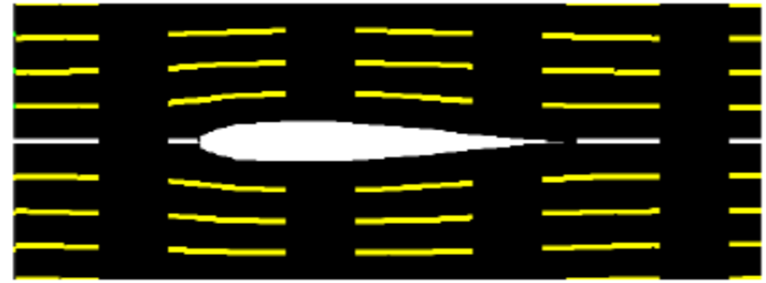
$$C_L=2\pi[\alpha-\alpha_{L0}]$$

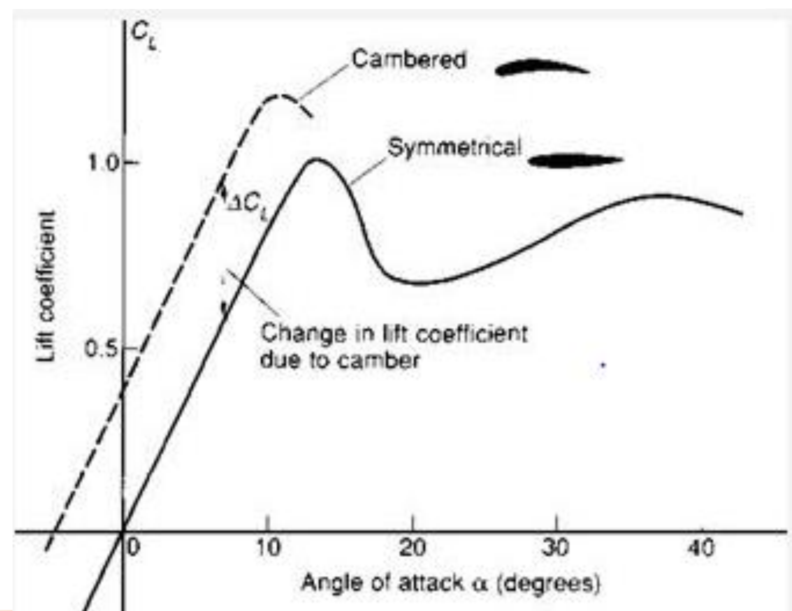
- ***The value of $\alpha_{L0}$ will be determined if the MCL is given for an airfoil, which is not a function of $\alpha$.***
- ***The slope of the Lift coefficient profile is still $2\pi$.***

$$C_L=2\pi[\alpha-\alpha_{L0}]$$

$$\alpha_{L0}=\frac{1}{\pi}\int_0^\pi \frac{dz}{dx}(1-\cos\theta)d\theta$$

$$\frac{dC_L}{d\alpha}=2\pi$$

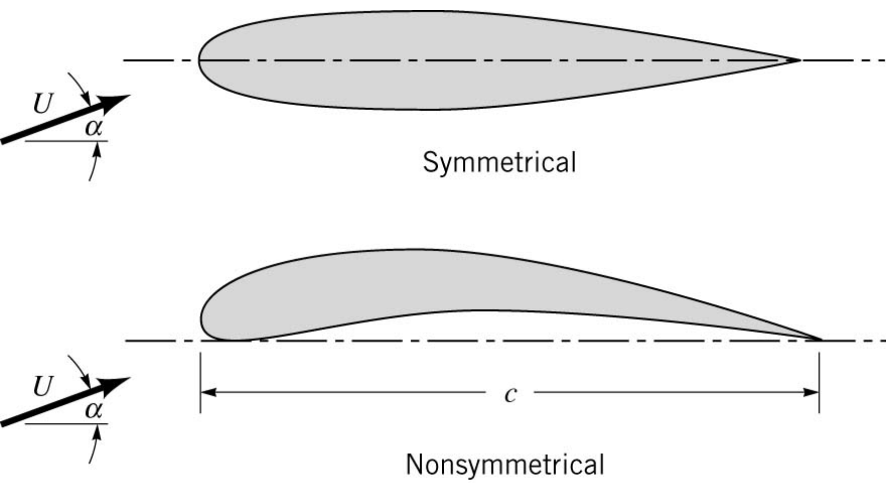


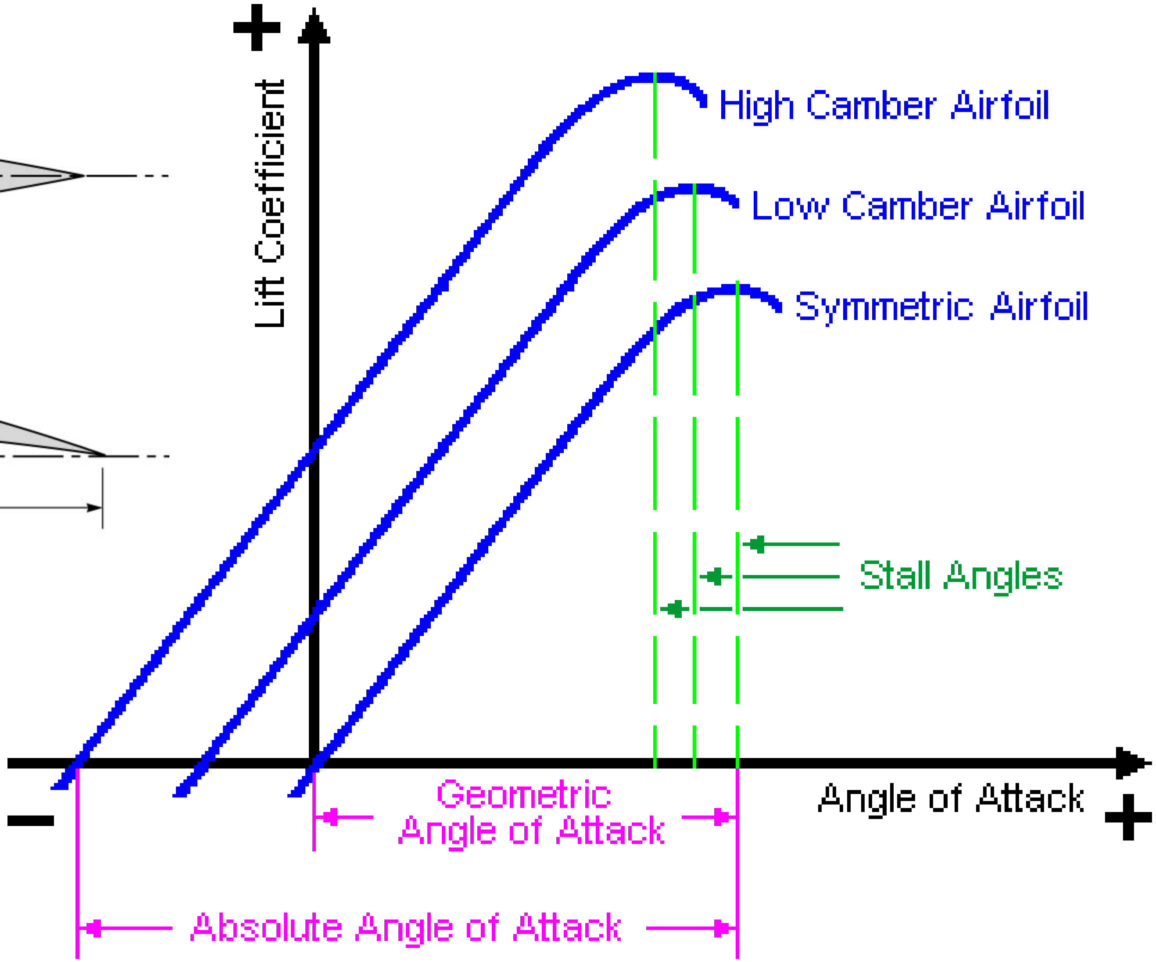


# ❑ Thin Airfoil Theory – Cambered Airfoil

$$C_L = 2\pi[\alpha - \alpha_{L0}]$$

$$\alpha_{L0} = \frac{1}{\pi}\int_0^{\pi}\frac{dz}{dx}(1-\cos\theta)d\theta$$



# Thin Airfoil Theory – Cambered Airfoil

## Moment about the airfoil leading edge

$$M_{LE} = -\int_{LE}^{TE} x dL; \quad dL = \rho V_\infty \gamma(x) dx$$

$$\Rightarrow M_{LE} = -\int_{LE}^{TE} x \cdot \rho V_\infty \gamma(x) dx$$

$$\because x = \frac{c}{2}(1-\cos\theta); \quad \gamma(\theta) = 2\alpha V_\infty \frac{1+\cos\theta}{\sin\theta}$$

$$\therefore dx = \frac{c}{2}\sin\theta d\theta$$

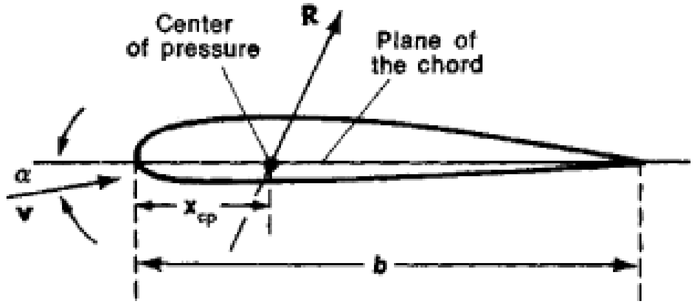
$$\Rightarrow M_{LE} = -\int_{LE}^{TE} x \rho V_\infty \gamma(x) dx$$

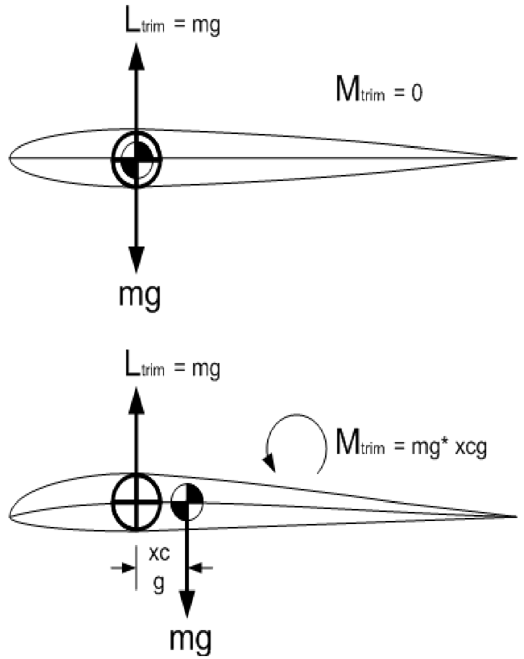
$$= -\int_{LE}^{TE} \frac{c}{2}(1-\cos\theta)\cdot \rho V_\infty \cdot 2V_\infty [A_0 \frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin(n\theta)]\cdot \frac{c}{2}\sin\theta d\theta$$

*Therefore:*

$$C_{M,LE} = \frac{M_{LE}}{\frac{1}{2}\rho V_\infty^2 c^2} = -A_0\frac{\pi}{2} - A_1\frac{\pi}{2} + A_2\frac{\pi}{4}$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} \frac{dz}{dx}\cos(n\theta) d\theta$$

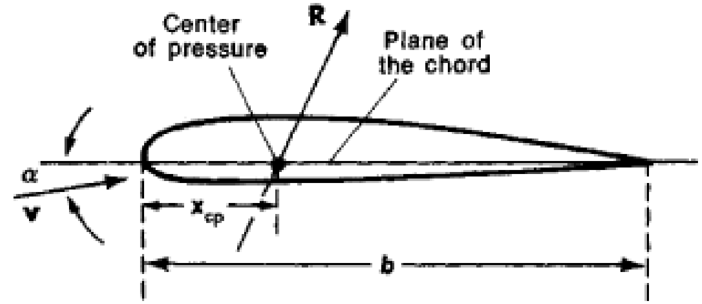



- ***No dependency on $\alpha$ for aerodynamic center***

# Thin Airfoil Theory – Cambered Airfoil

## Moment about the airfoil leading edge




*Therefore:*

$$\because C_L = 2\pi A_0 + A_1\pi;$$

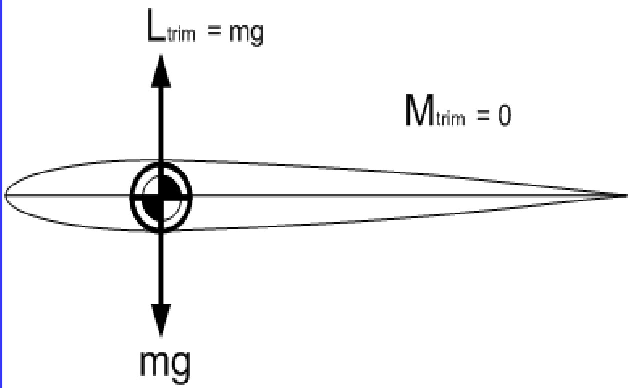
$$\therefore C_{M,LE} = -A_0\frac{\pi}{2} - A_1\frac{\pi}{2} + A_2\frac{\pi}{4}$$

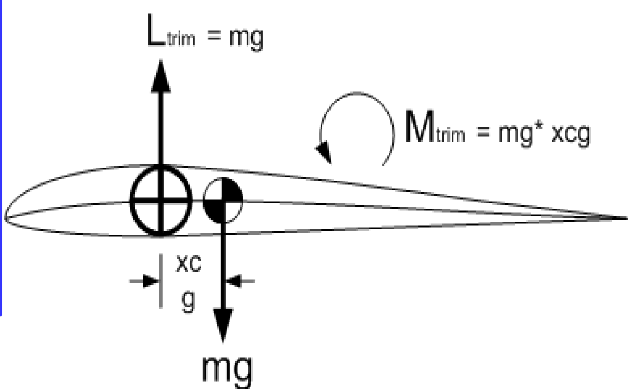
$$= -[\frac{2\pi A_0 + \pi A_1}{4} + \frac{\pi}{4}(A_1 - A_2)] = -[\frac{C_L}{4} + \frac{\pi}{4}(A_1 - A_2)]$$

$$\because C_{M,LE} = C_{M,c/4} - \frac{C_L}{4}$$

$$\therefore C_{M,c/4} = -\frac{\pi}{4}(A_1 - A_2)$$

$$x_{CP} = \frac{1}{4}[c + \frac{\pi c}{C_L}(A_1 - A_2)]$$



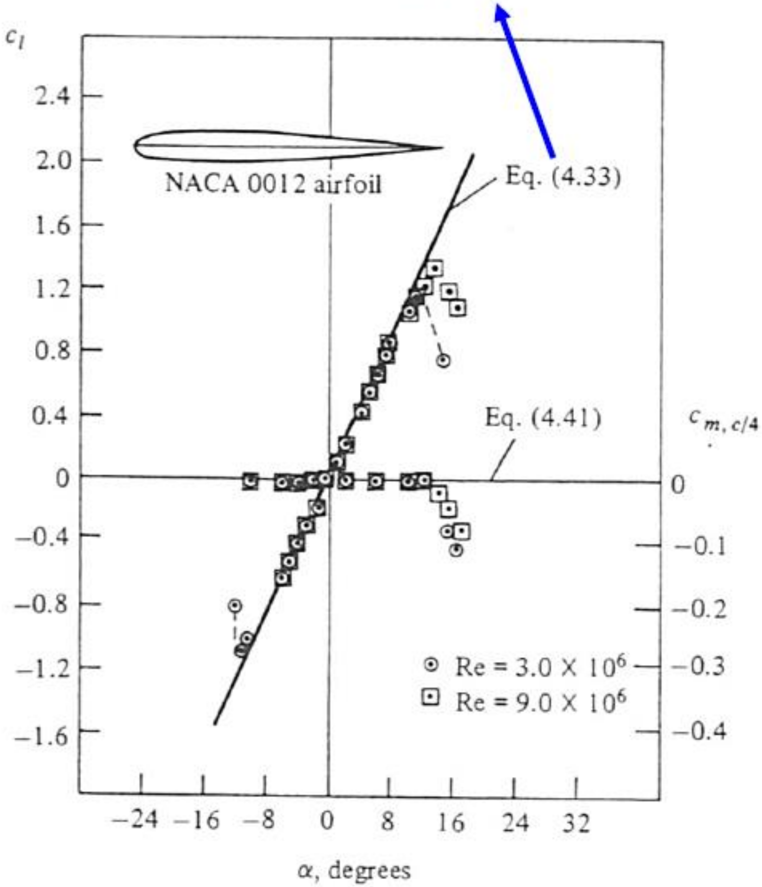




- ***No dependency on $\alpha$ for aerodynamic center***

# ❑ Thin Airfoil Theory – Cambered Airfoil

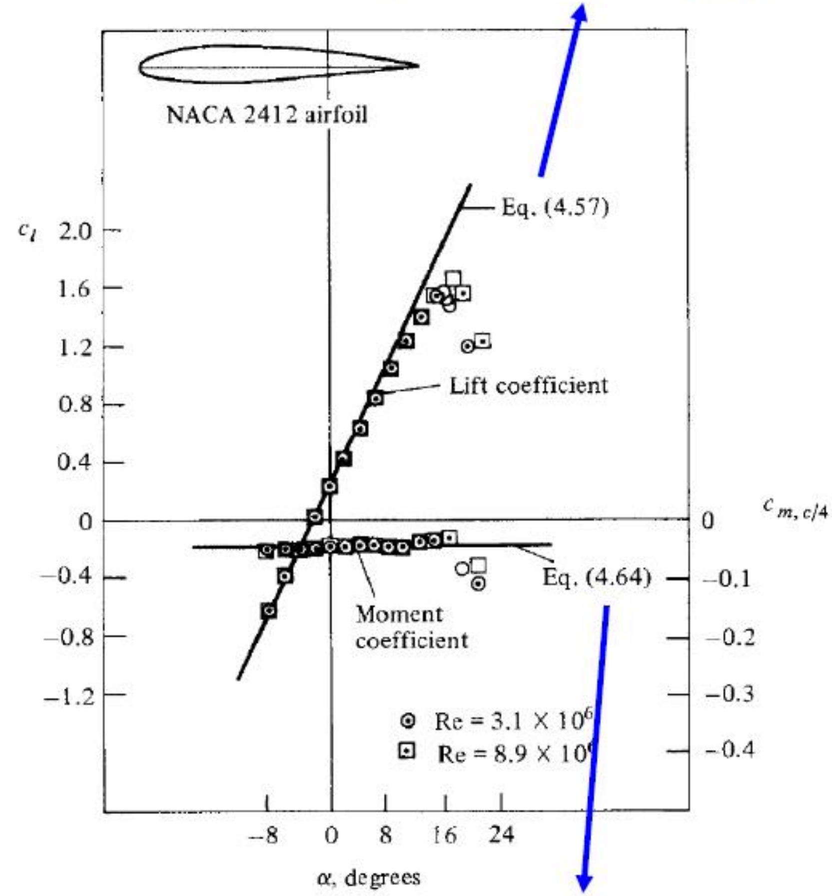
## ❑ *Center of Lift or center of pressure on an airfoil*

$$c_l = 2\pi\alpha$$




$$C_{M,c/4} = 0$$

$$c_l = 2\pi(\alpha - \alpha_{L=0})$$




$$C_{M,c/4} = -\frac{\pi}{4}(A_1 - A_2)$$