Lecture # 32: 3D Wing Aerodynamics: Lifting Line Theory – Part #1

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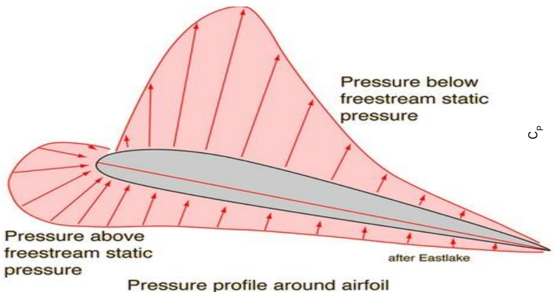
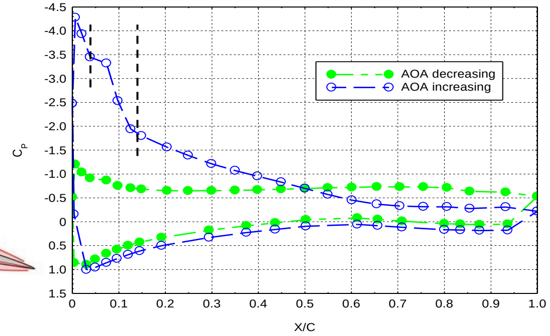
❑ Supersonic Air Travel - NASA & LM

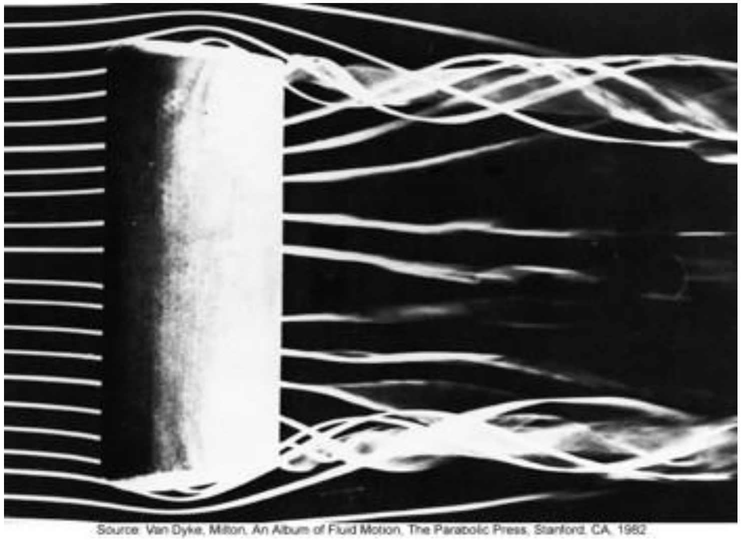
https://www.yahoo.com/news/experimental-aircraft-used-test-supersonic-142048339.html?guccounter=1 on Nov 13, 2021

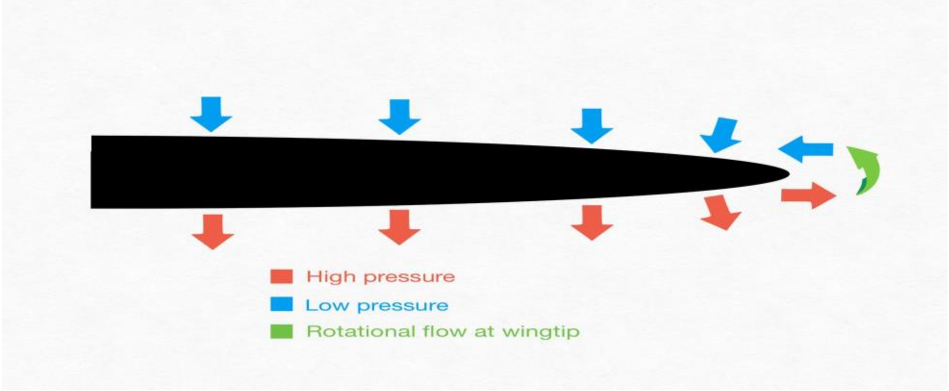
❑ 3D Wing Aerodynamics

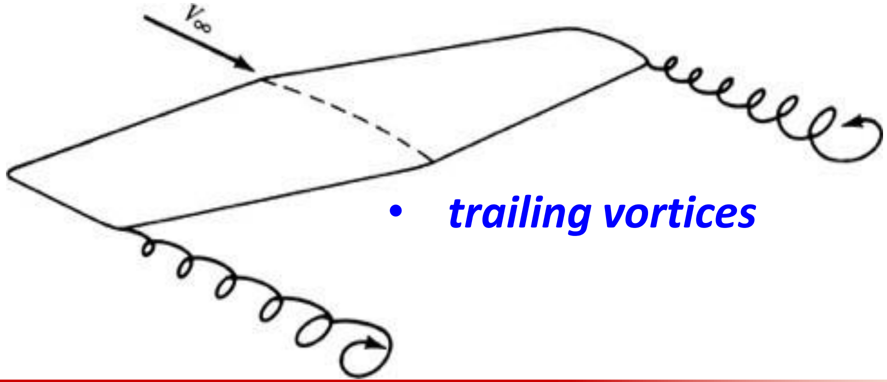
- ***Air flow leaks around wing tips produces a trailing vortex at each wing tip.***



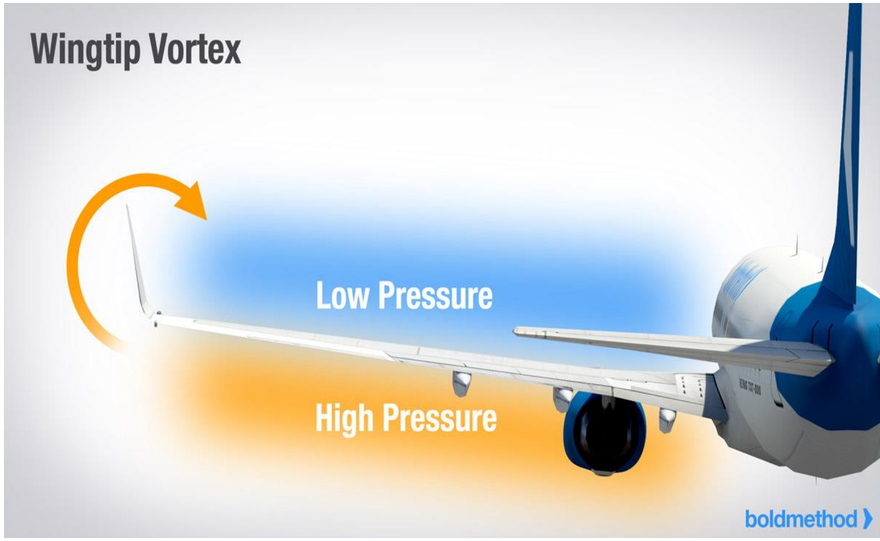






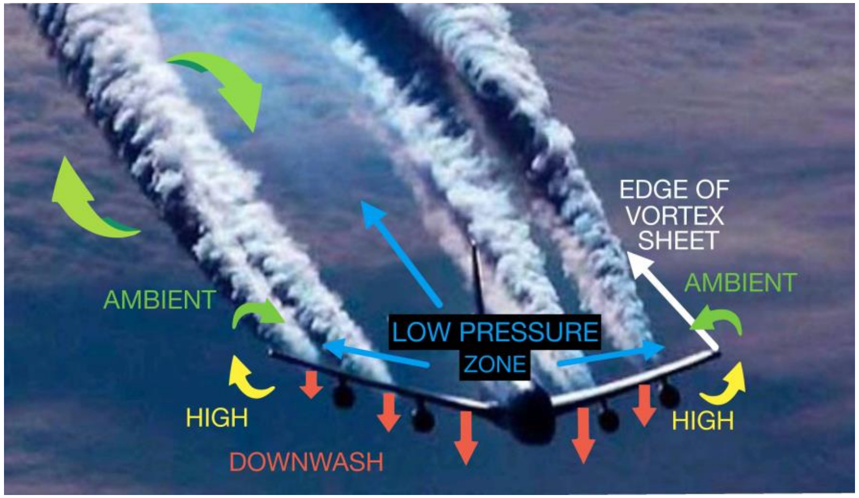


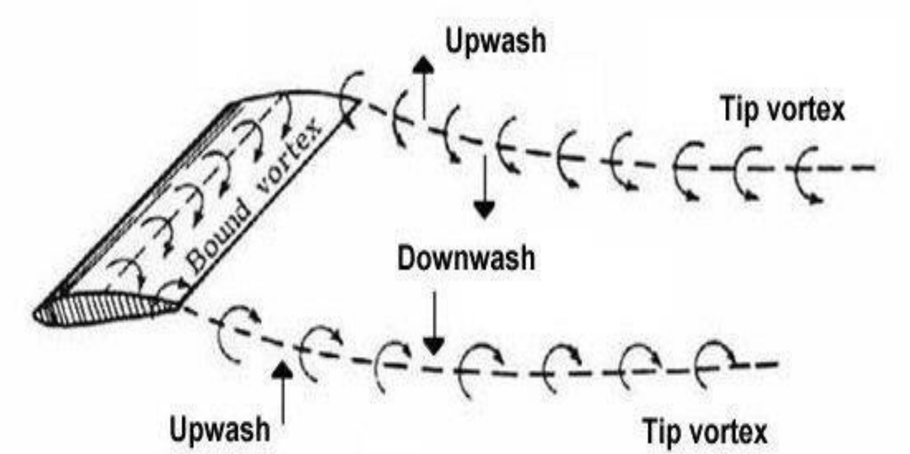
- ***trailing vortices***

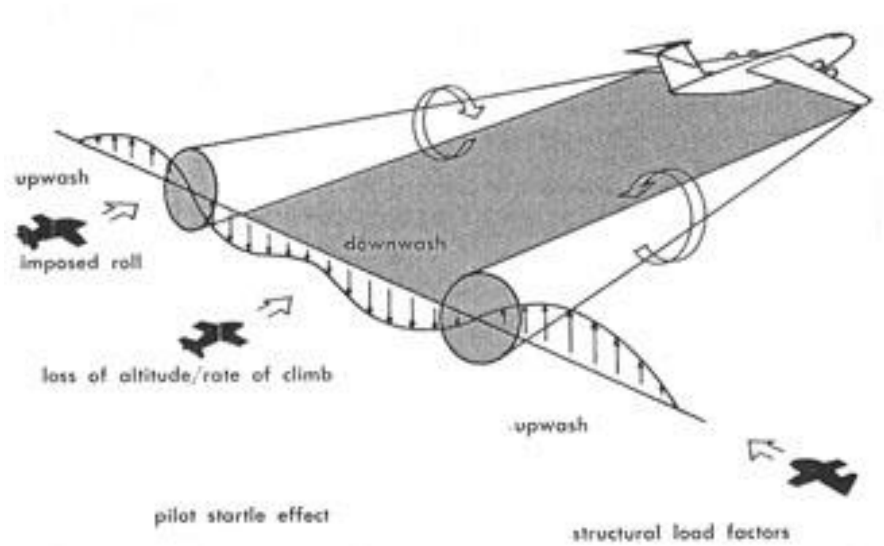


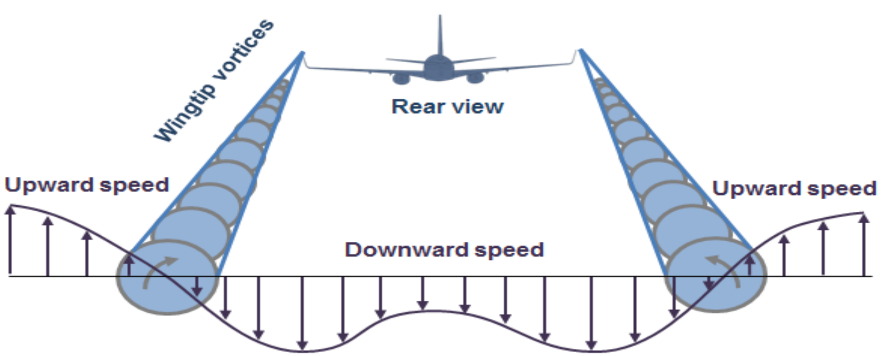
❑ 3D Wing Aerodynamics

- *Trailing vortices at each wing tip would drag the surrounding air inducing a velocity component in the downward direction - downwash.*
- *The downwash combines with the local freestream to create a local relative wind.*



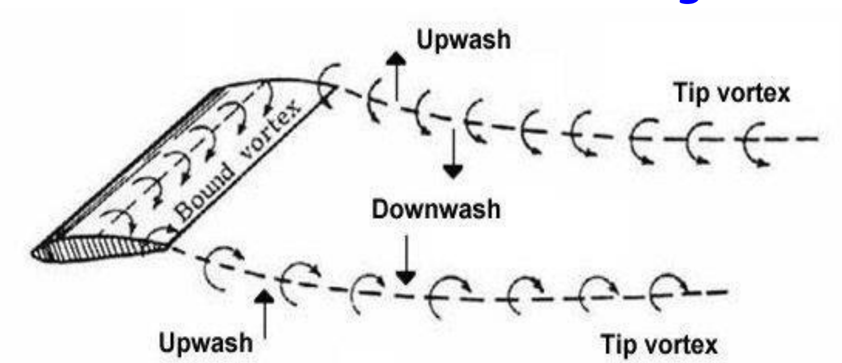


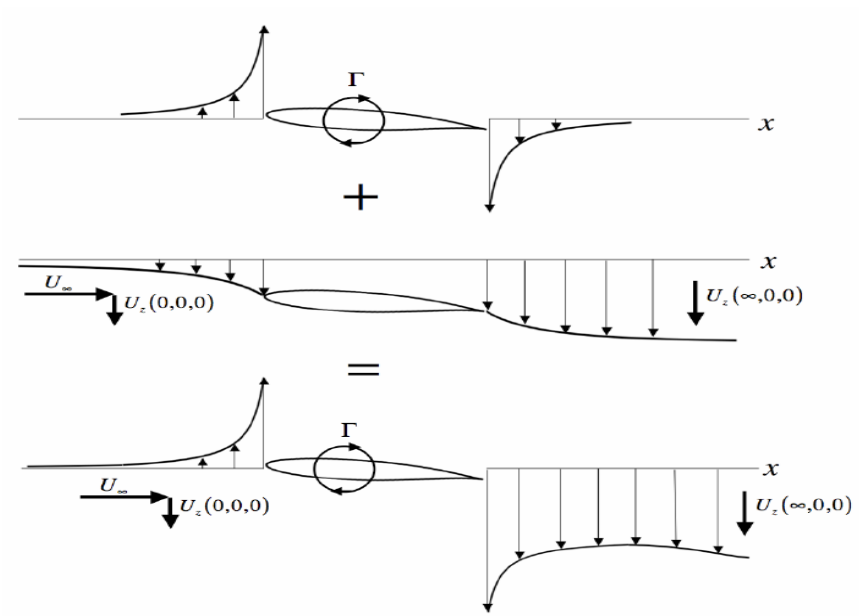




❑ 3D Wing Aerodynamics

- ***Downwash and Induced Drag***





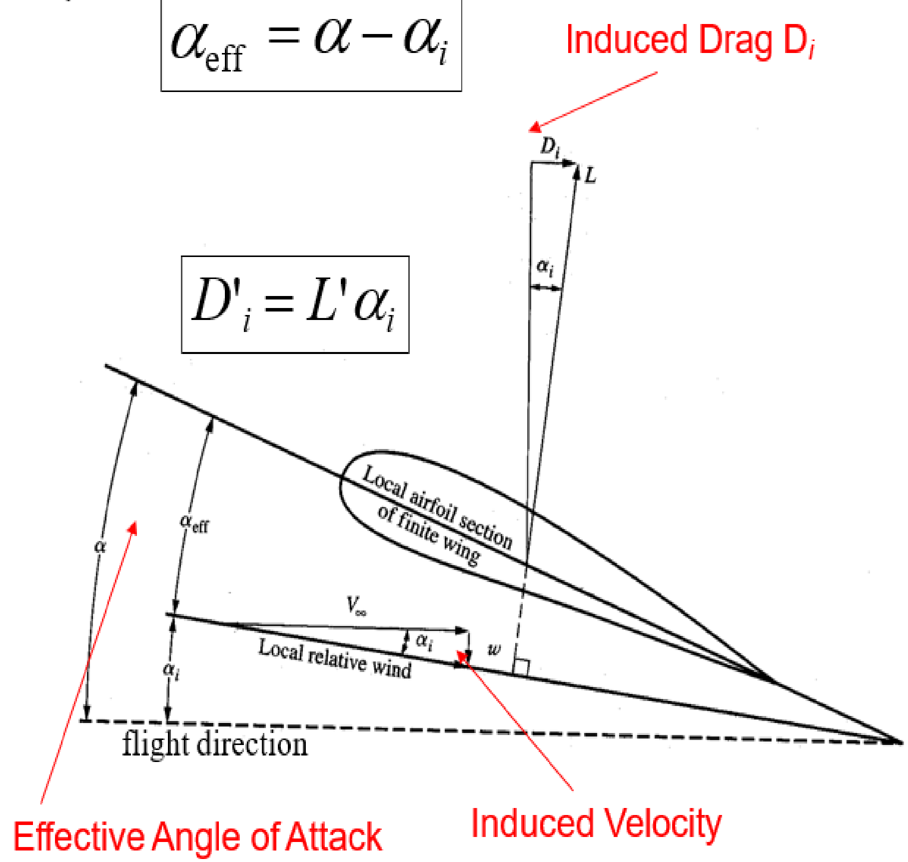
- *Velocity field normal to a wing comprising a transverse bound vortex of circulation Γ plus downwash generated by a semi-infinite system of free vortices in the wake.*

The downwash has two important effects:

- ***The effective angle of attack is reduced to cause lift reduction.***
- ***Induced drag is created due to tilting of the local lift vector.***

$$\alpha_{\text{eff}} = \alpha - \alpha_i$$

$$D'_i = L'\alpha_i$$



❑ 3D Wing Aerodynamics

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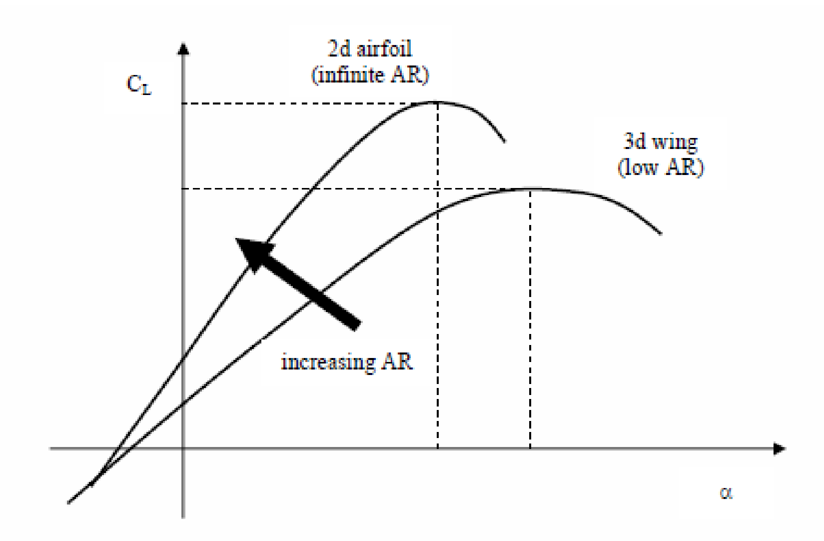
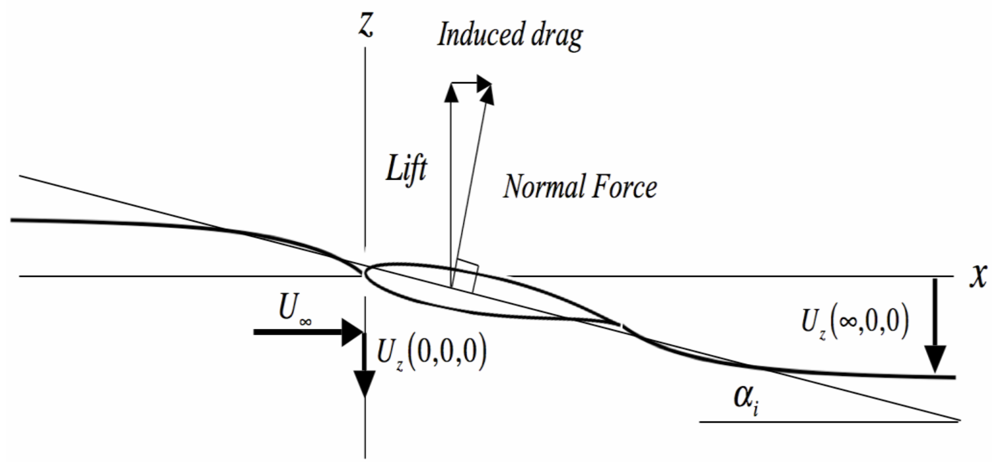
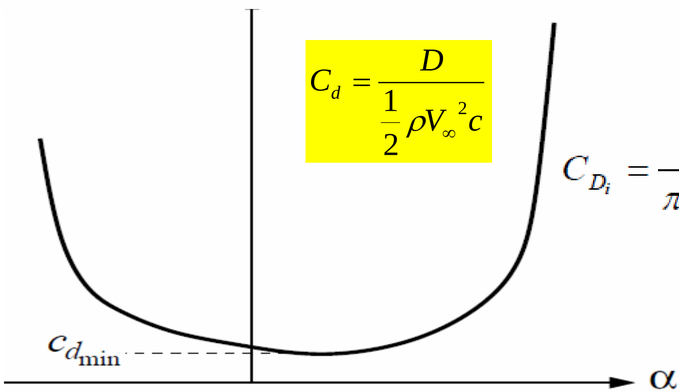


Figure 5.28. The effect of AR on C_L versus angle of attack graph

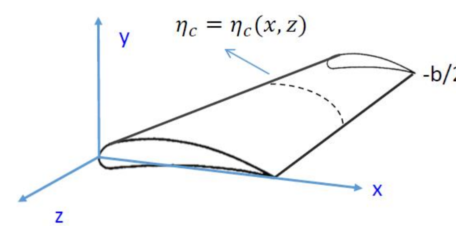


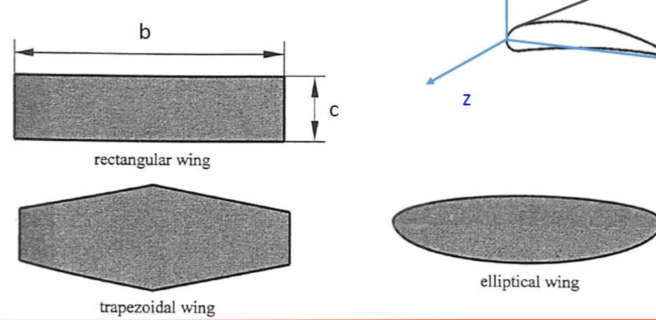
- Wingspan = b, wing area=S
- Aspect ratio

$$AR = b^2/S$$



$$C_{D_i} = \frac{C_L^2}{\pi \cdot e \cdot AR}$$

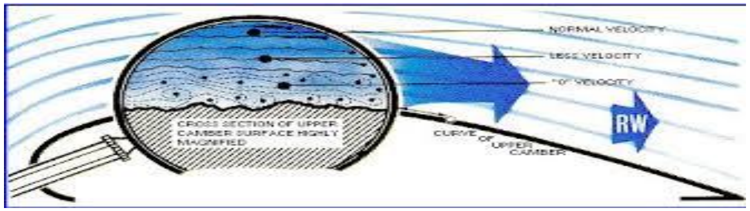


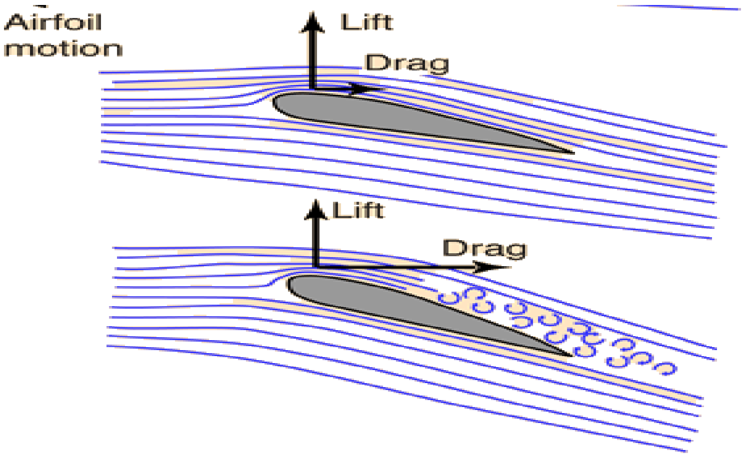


❑ 3D Wing Aerodynamics

❑ *Aerodynamic drags*

- ***Skin friction drag, D_f – drag caused by skin friction.***
- ***Pressure drag, D_p – drag due to flow separation, which causes pressure differences between front and back of the wing.***
- ***Induced drag, D_i – drag due to lift force redirection caused by the induced flow or downwash.***





$$C_d = \frac{D_f + D_p}{q_\infty S}$$

Induced drag coefficient, C_{D_i} - nondimensional induced drag

$$C_{D_i} = \frac{D_i}{q_\infty S}$$

Total drag coefficient, C_D

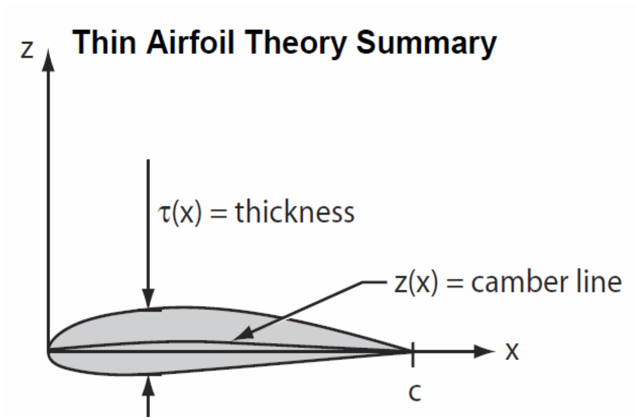
$$C_D = C_d + C_{D_i}$$

Airfoil data (C_d) — Finite wing theory (C_{D_i})

- ***The total drag = friction drag + pressure drag + induced drag.***
  - ***Total drag coefficient $C_D = (D_f + D_p + D_i) / (1/2 \rho V_\infty^2 S)$***

❑ Thin Airfoil Theory

Circulation around thin airfoil

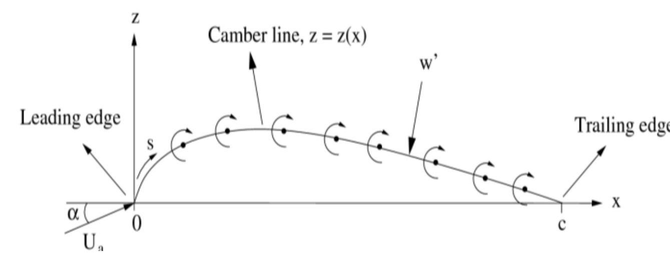


- Recall from thin airfoil solution

$$\gamma(\theta) = 2V_\infty \left[(\alpha - A_0)\frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin n\theta \right]$$

For a flat plate airfoil (i.e., no camber→ $A_0 = A_1 = \cdots = A_n = 0$)

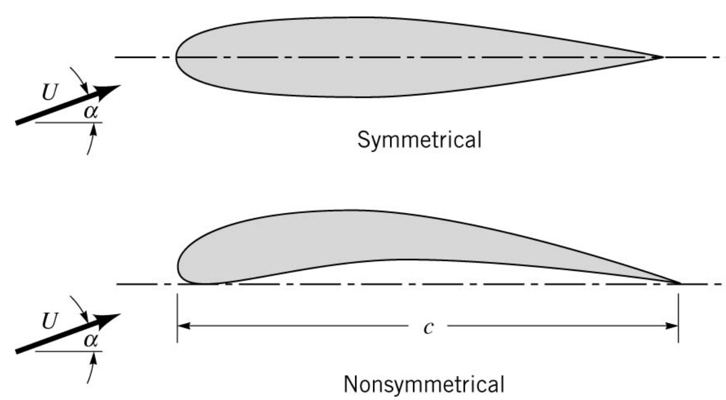
$$\gamma(\theta) = 2V_\infty \alpha \frac{1+\cos\theta}{\sin\theta}$$



The total circulation around the airfoil is

$$\Gamma = \int_0^c \gamma(x)dx = \int_0^\pi \gamma(\theta)\frac{c}{2}\sin\theta \, d\theta$$

$$\Gamma = cV_\infty \alpha \int_0^\pi (1+\cos\theta)d\theta = \pi c V_\infty \alpha$$

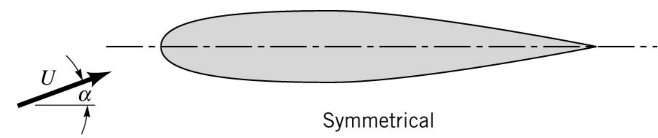


And airfoil lift

$$L' = \rho V_\infty \Gamma$$

❑ Lumped Vortex Model

Lumped vortex element (model)



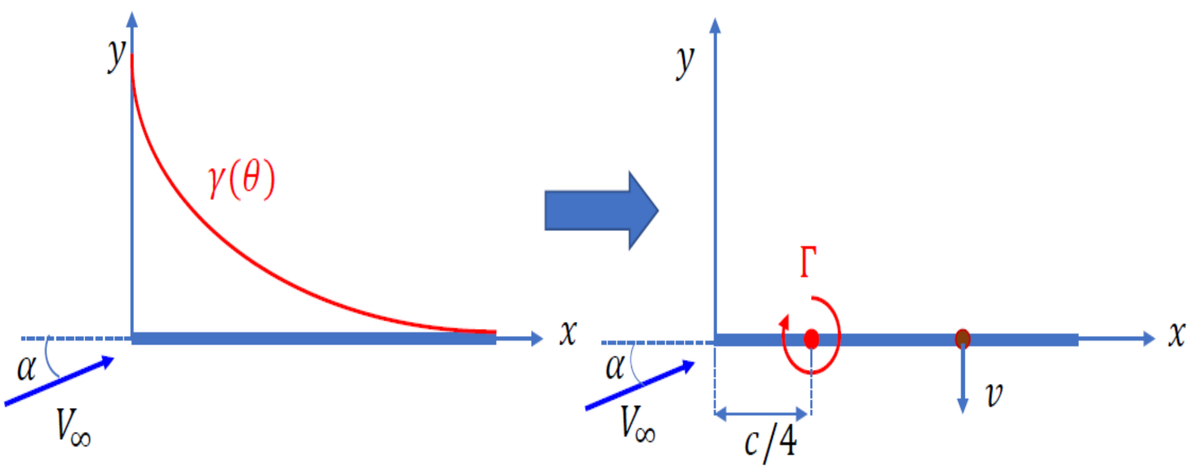
- It is sometimes useful to represent the entire circulation with a single vortex
  - Since lift acts at center of pressure, naturally the vortex should be placed there (i.e., $x = c/4$)

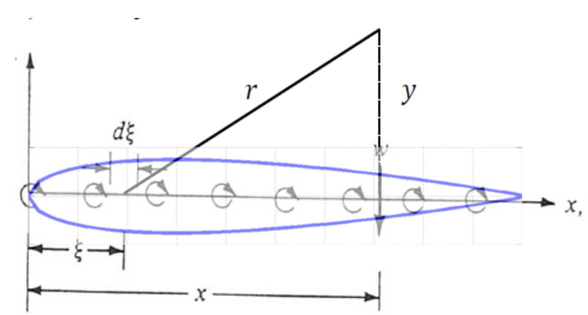
$$\frac{1}{2\pi}\int_0^{\pi}\frac{\gamma(\theta)\sin\theta d\theta}{\cos\theta-\cos\theta_0}=V_\infty\alpha;$$

  - However, with distributed γ, the zero normal velocity is enforced on the surface of airfoil. Where should we enforce this boundary condition for single vortex case?

$$\gamma(\theta)=2\alpha V_\infty\frac{1+\cos\theta}{\sin\theta}$$

$$\Gamma=\alpha V_\infty c\pi$$

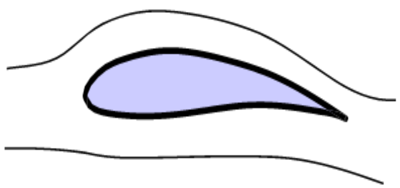


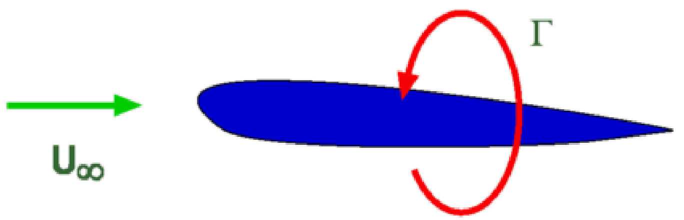


❑ 3D Wing Aerodynamics

▪ *3-D Vortex Theory: the vortex filament*

flow around a real wing $\Rightarrow$ uniform flow + vortices

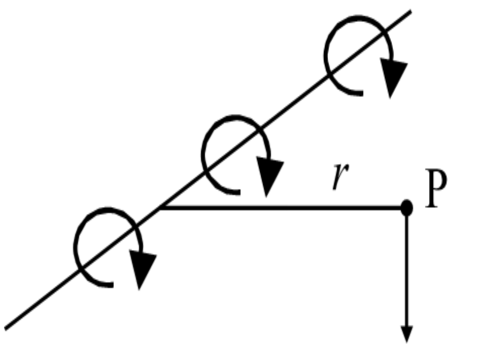




$L = -\rho U_\infty \Gamma$

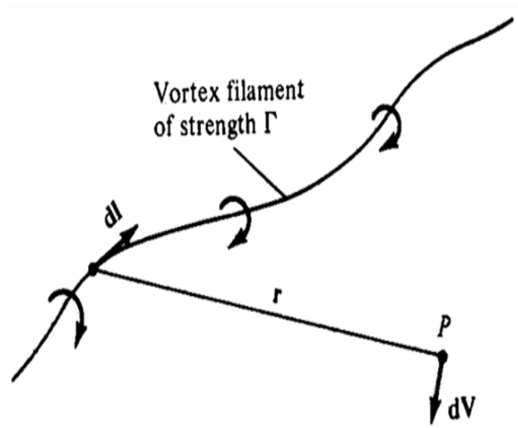
❑ ***2D airfoil aerodynamics***

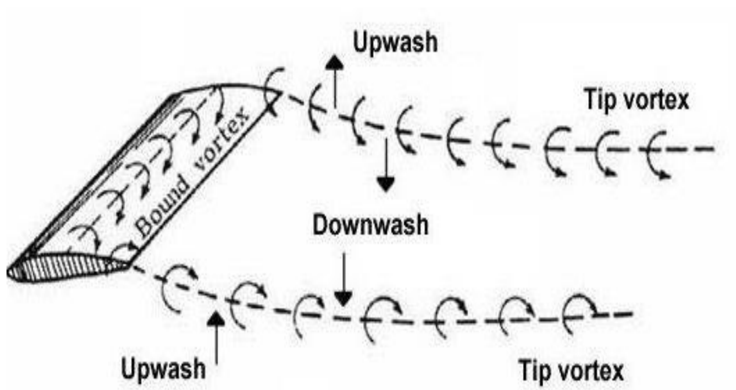
2D: Straight vortex line:



induced velocity $V = \dfrac{\Gamma}{2\pi r}$

3D general: curved vortex line



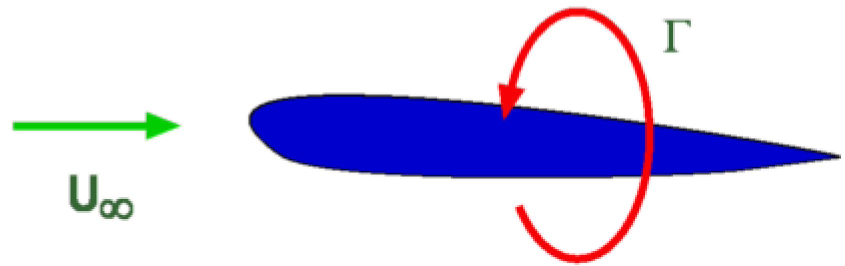


❑ ***3D wing aerodynamics***

Thin Airfoil Theory

The Kutta-Joukowski Lift Theorem:

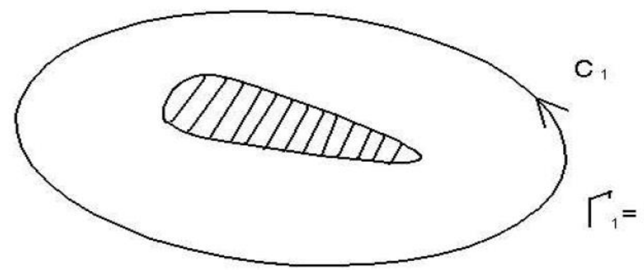
$$L' = \rho V_\infty \Gamma$$



$L = -\rho\, U_\infty\, \Gamma$

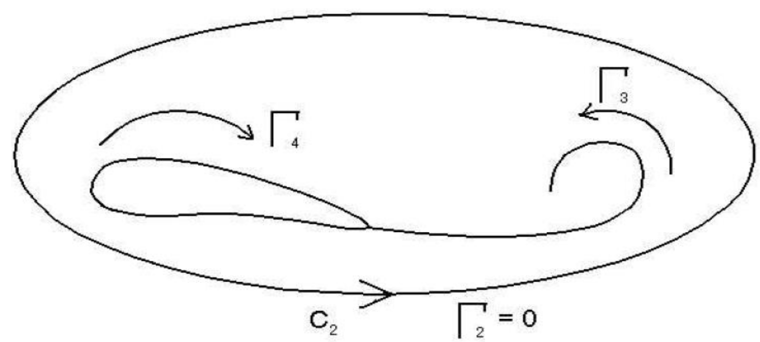
Helmholtz's Vortex theorem **(basic principles of vortex behavior).**

- *The strength of a vortex filament is constant along its length.*
- *A vortex filament cannot end in a fluid; it must extent to the boundaries of the fluid or form a closed path.*

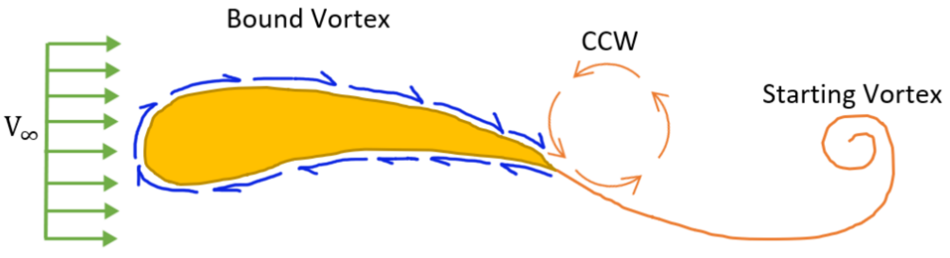


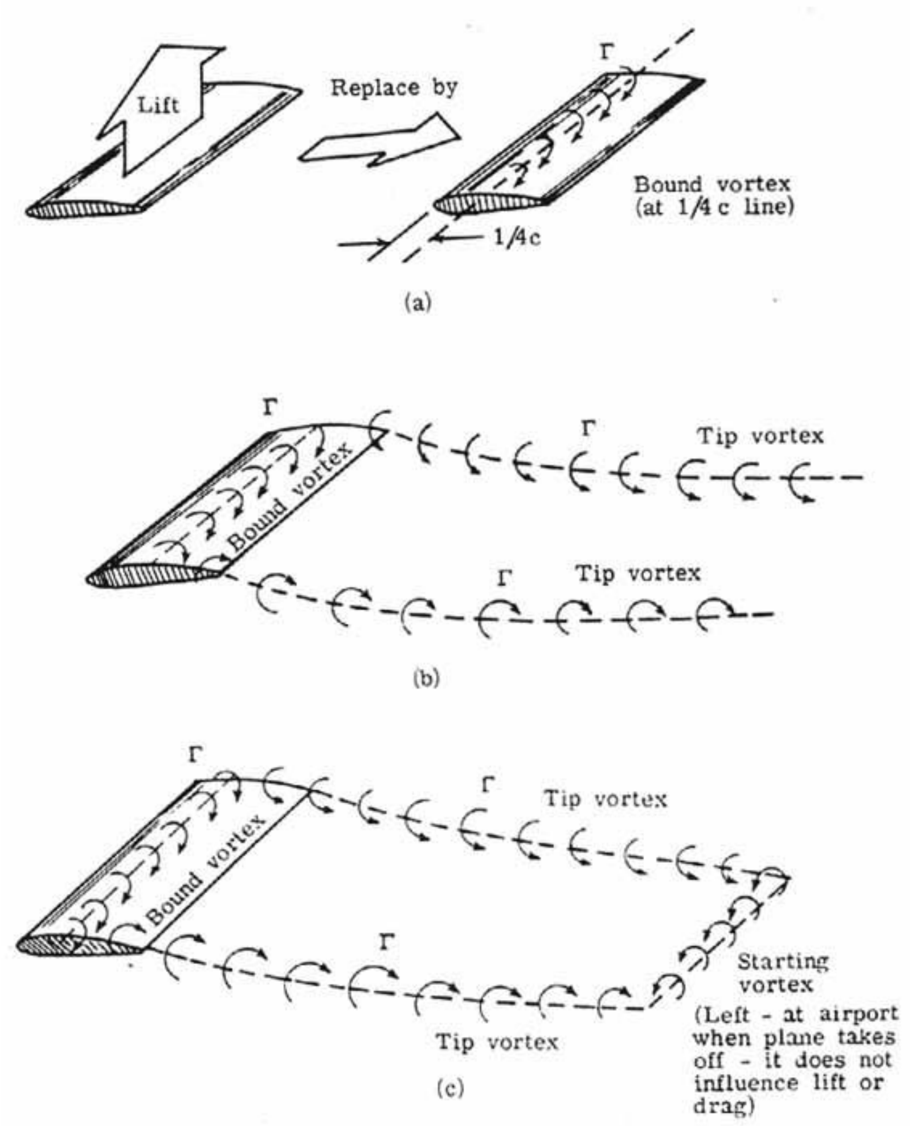
Kelvin's vortex theorem:

- *Circulation around a closed curve formed by a set of continuous fluid elements remains constant as the fluid elements move through the flow:*



❑ Starting Vortex from Shedding from a Wing/Airfoil

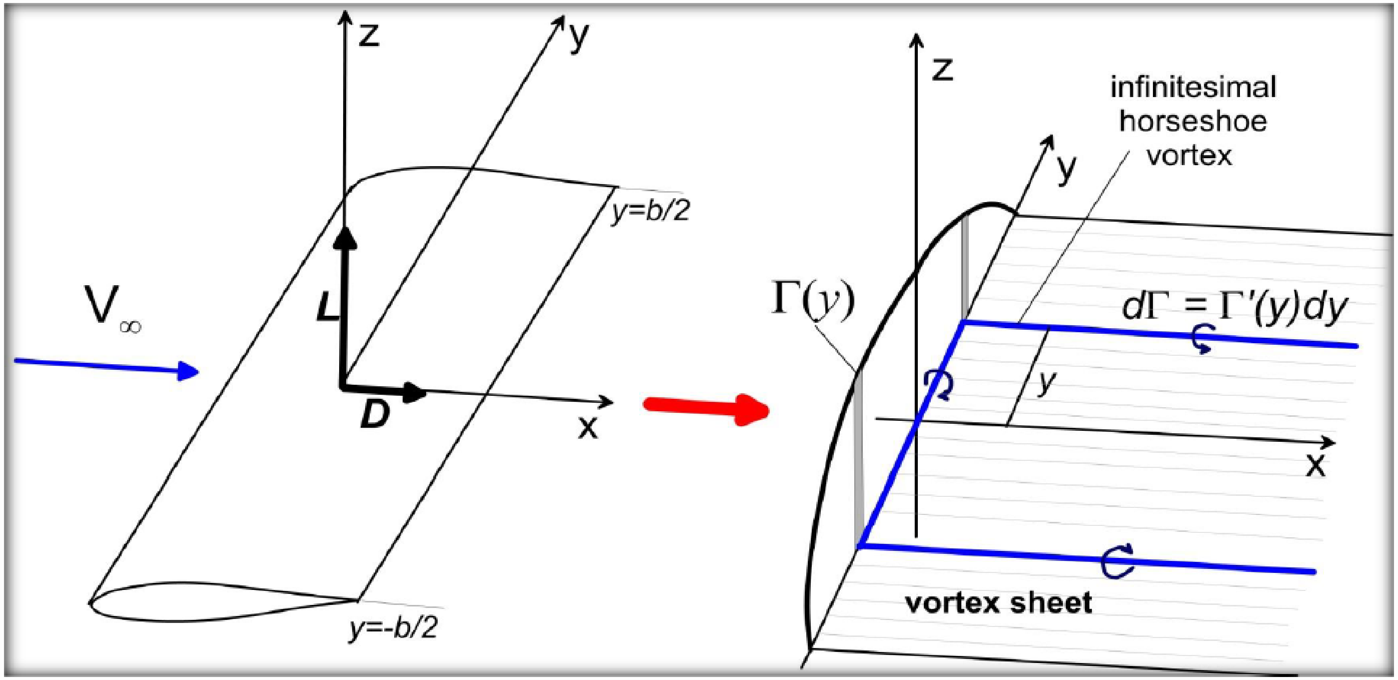




❑ 3D Wing Aerodynamics

Lifting-line model of a finite-span wing

Flow past a wing is modeled by the superposition of the uniform free stream and the velocity induced by a plane vortex sheet "pretending" to be the cortex wave behind the wing.



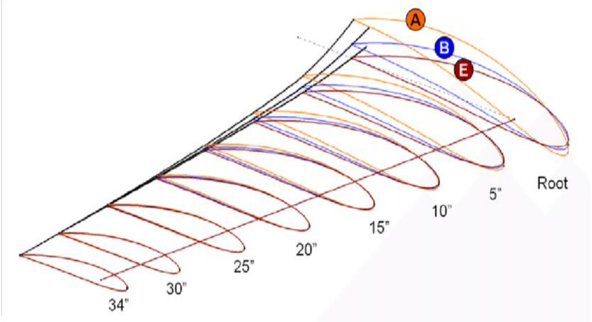
The vortex sheet behind the wing is "woven" from continuum of infinitesimally weak horseshoe vortices. These vortices are "attached" to the lifting line leading to a continuous distribution of circulation along the wing span.

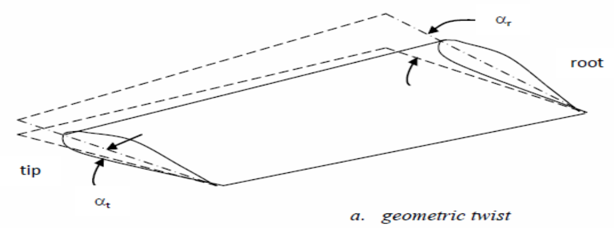
❑ 3D Wing Aerodynamics

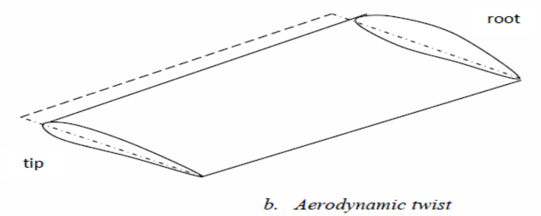
❑ *Lift Distribution on a Finite Wing.*

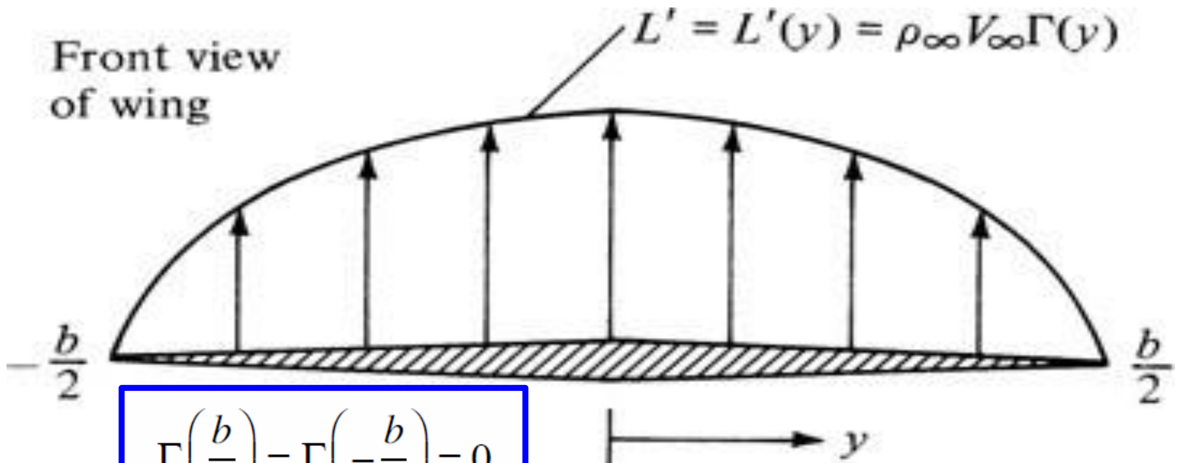
Consider a given spanwise location y_1, where the local chord is c.

- ***The lift per unit span can vary along the span.***
- ***Different spanwise locations can have different angles of attack (geometric twist).***
- ***Wings can also have different airfoil section spanwise (aerodynamic twist).***
- ***Pressure equalization occurs at y = -b/2 and b/2, and consequently there is no lift at these locations.***









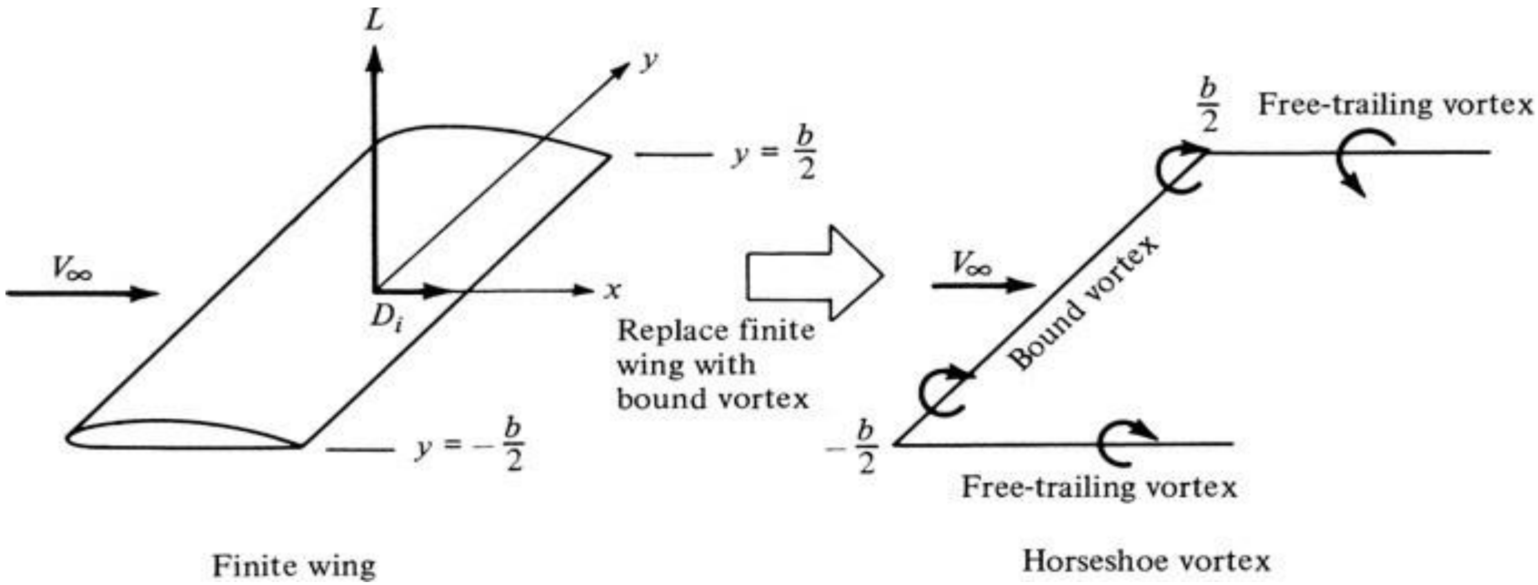
$$\Gamma\left(\frac{b}{2}\right) = \Gamma\left(-\frac{b}{2}\right) = 0$$

- ***Objective is to estimate the lift distribution, total lift and induced drag for the finite wing.***

❑ PRANDTL'S LIFTING LINE THEORY

❑ *Prandtl's Lifting Line Theory*

- *The theory is useful for predicting the aerodynamic characteristics of finite wings.*

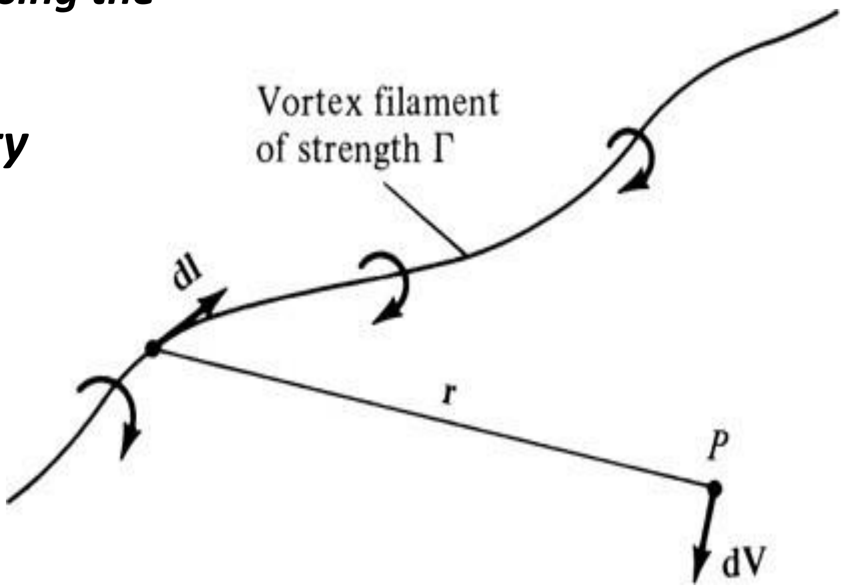


- *The finite wing is replaced with a bound vortex.*
- *Due to Helmholtz's theorem, a vortex filament cannot end in the fluid.*
  - *Therefore, assume the vortex filament continues as two free vortices trailing downstream from the wing tips to infinity.*
- *Bound vortex + Trailing vortices —> Horseshoe Vortex*

❑ *The Vortex Filament Theorem:*

- ***Establish a rational aerodynamic theory for a finite wing.***
- ***The curved filament induces a flow field in the surrounding space.***
- ***Circulation taken about any closed path enclosing the filament is constant.***
- ***Consider a segment dl. It induces a velocity at point, P, equal to:***

$$d\vec{V} = \frac{\Gamma}{4\pi}\frac{d\vec{l}\times\vec{r}}{\left|r^3\right|}$$



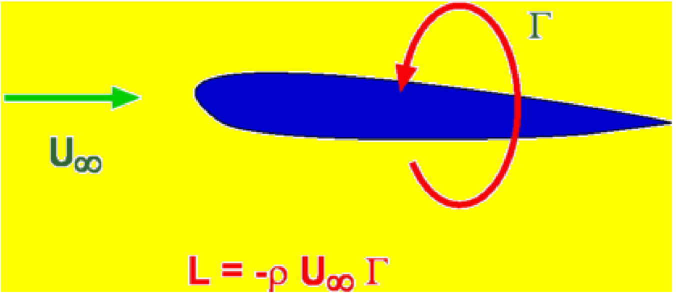
- ***dl** – infinitesimal length along the vortex filament*
- ***r** – radius vector from **dl** to some point in space, P.*
- ***dV** – infinitesimal induced velocity*

- ***Biot-Savart Law***

❑ 3D Wing Aerodynamics

❑ *The Vortex Filament Theorem:*

- ***When a number of vortex filaments are used in conjunction with a uniform free stream, it is possible to synthesize the flow over a finite wing.***



- ***Velocity induced at P by the entire vortex filament is given by:***

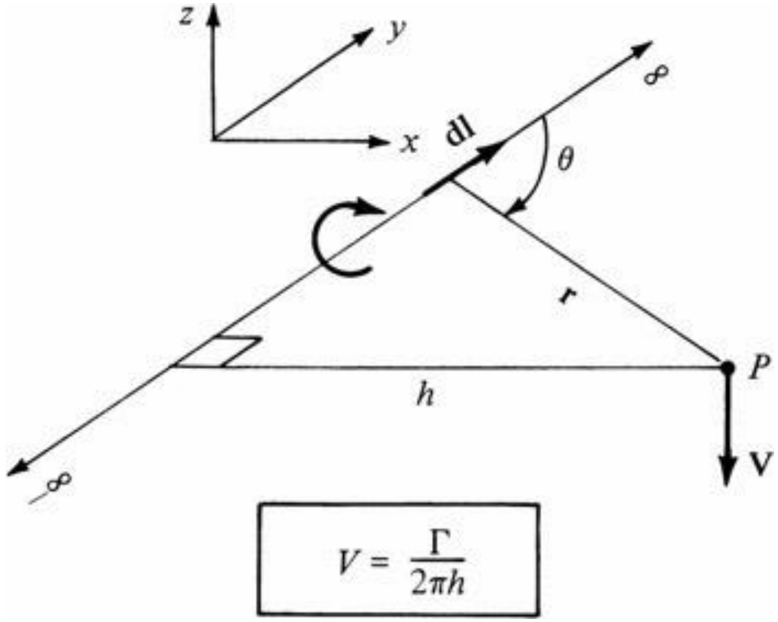
$$d\vec{V} = \frac{\Gamma}{4\pi}\frac{d\vec{l}\times\vec{r}}{\left|r^3\right|} = \vec{V} = \int_{-\infty}^{\infty}\frac{\Gamma}{4\pi}\frac{d\vec{l}\times\vec{r}}{\left|r^3\right|}$$

$$\Rightarrow V = \frac{\Gamma}{4\pi}\int_{-\infty}^{\infty}\frac{\sin\theta\cdot dl}{r^2}$$

$$\because r = \frac{h}{\sin\theta}; \quad l = \frac{h}{\tan\theta}; \quad dl = -\frac{h}{\sin^2\theta}d\theta$$

$$\therefore V = \frac{\Gamma}{4\pi}\int_0^{\pi}\frac{\sin\theta\cdot[-\frac{h}{\sin^2\theta}d\theta]}{(\frac{h}{\sin\theta})^2}$$

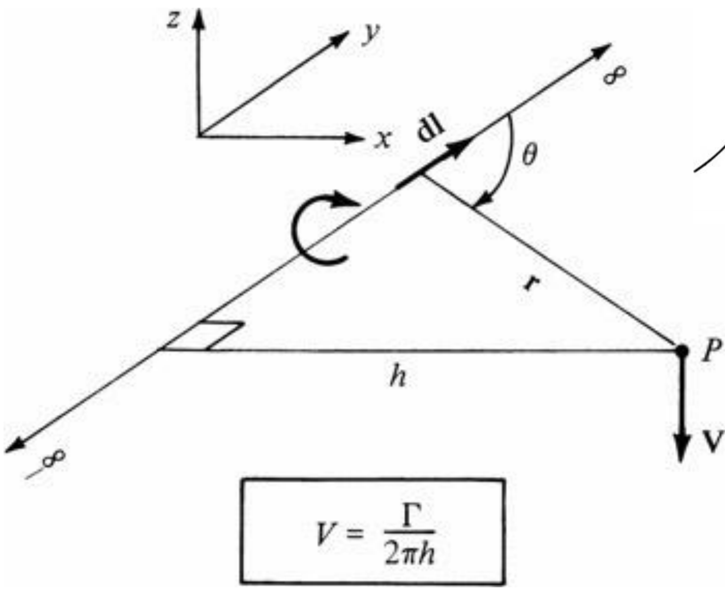
$$= -\frac{\Gamma}{4\pi h}\int_0^{\pi}\sin\theta d\theta = \frac{\Gamma}{2\pi h}$$

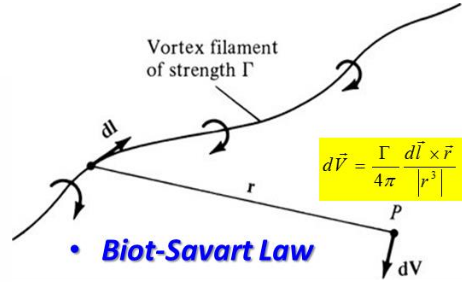


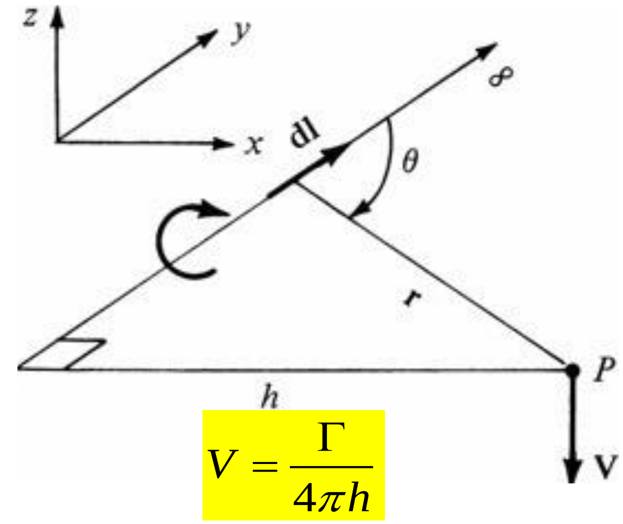
- ***The result is the same as that for a point vortex in a 2D flow.***

❑ 3D Vortex Filament vs. 2D Vortex Flows

❑ 3D Vortex Filament





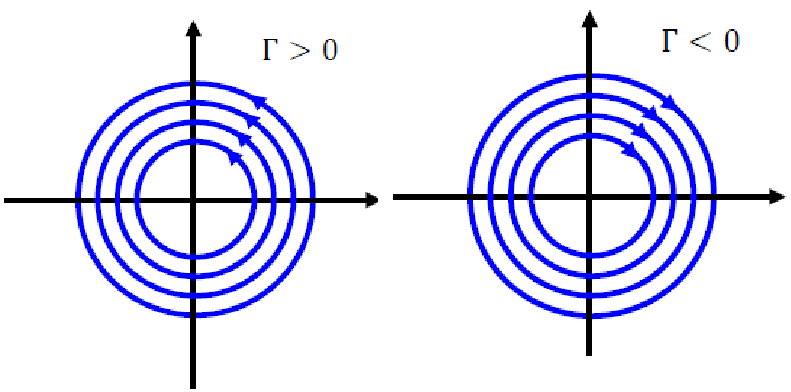


- *Ranging from -∞ to +∞*
- *Ranging from 0 to ∞*

❑ Basic flow # 04: 2D vortex

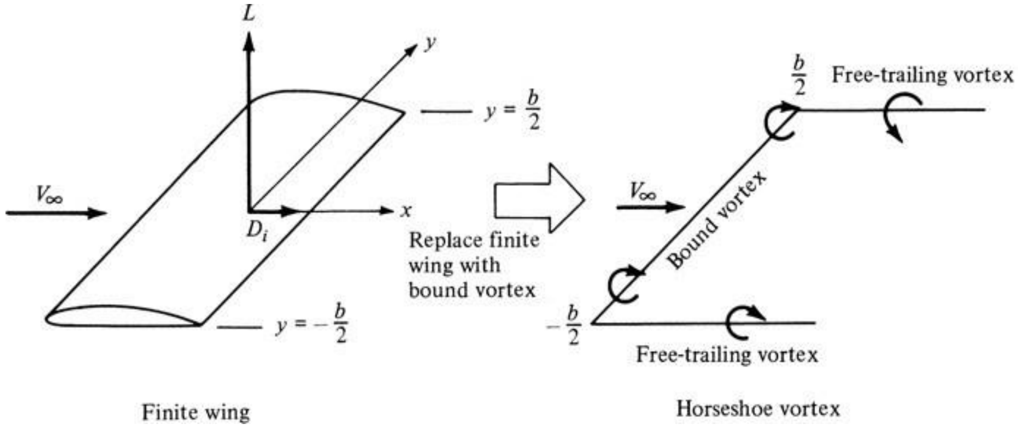
$$\psi = \frac{\Gamma}{2\pi}\ln r; \quad \phi = -\frac{\Gamma}{2\pi}\theta$$

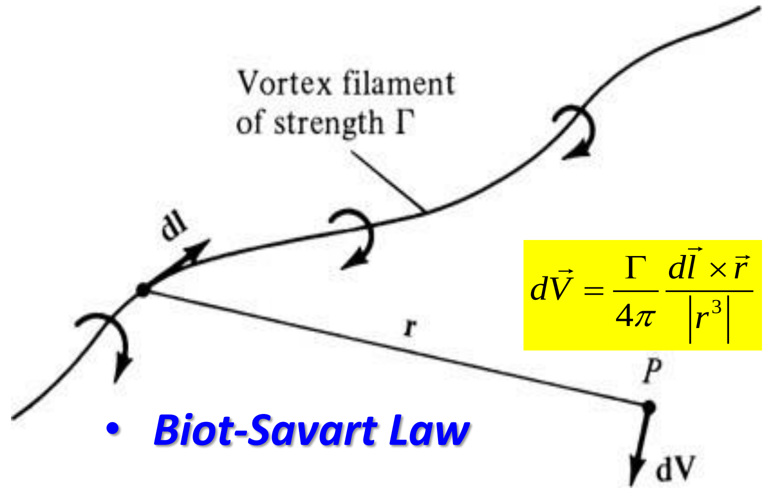
$$V_r = 0; \quad V_\theta = -\frac{\Gamma}{2\pi r}$$



❑ 3D Wing Aerodynamics

❑ *Prandtl's Lifting Line Theory*

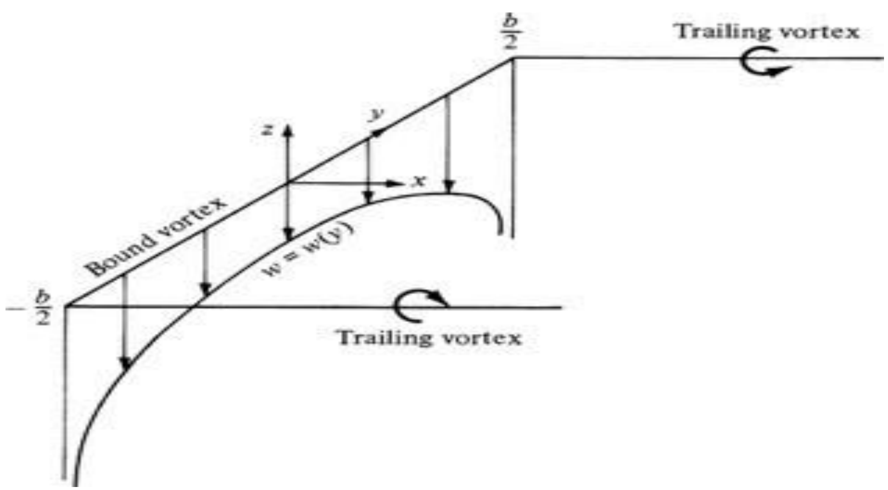




$$dV = \frac{\Gamma}{4\pi}\frac{d\vec{l} \times \vec{r}}{|r^3|}$$

- ***Biot-Savart Law***

- *The Biot-Savart law allows us to determine the downwash along the wing and results in:*



$$w(y) = -\frac{\Gamma}{4\pi\left(b/2 + y\right)} - \frac{\Gamma}{4\pi\left(b/2 - y\right)}$$

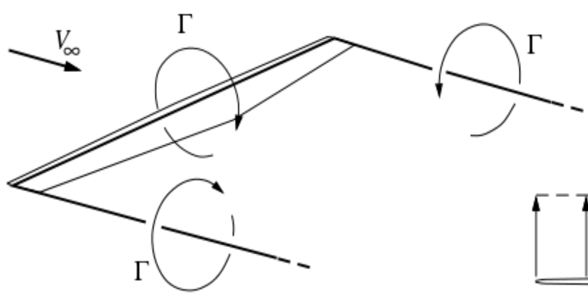
or

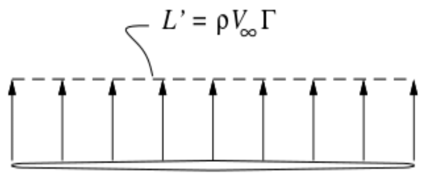
$$w(y) = -\frac{\Gamma}{4\pi}\frac{b}{\left(b/2\right)^2 - y^2}$$

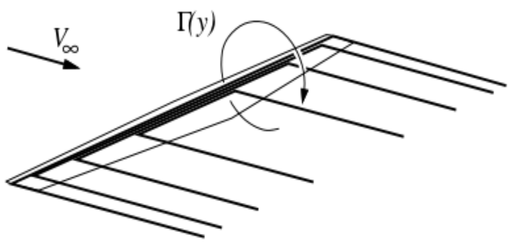
- ***However, the single vortex filament case is not sufficient to describe the physical conditions on the wing since the downwash at the wing tips is infinite!***

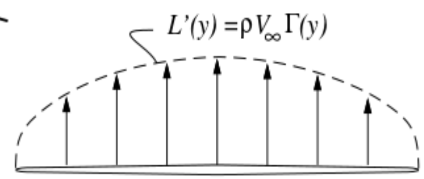
Prandtl's Lifting Line Theory

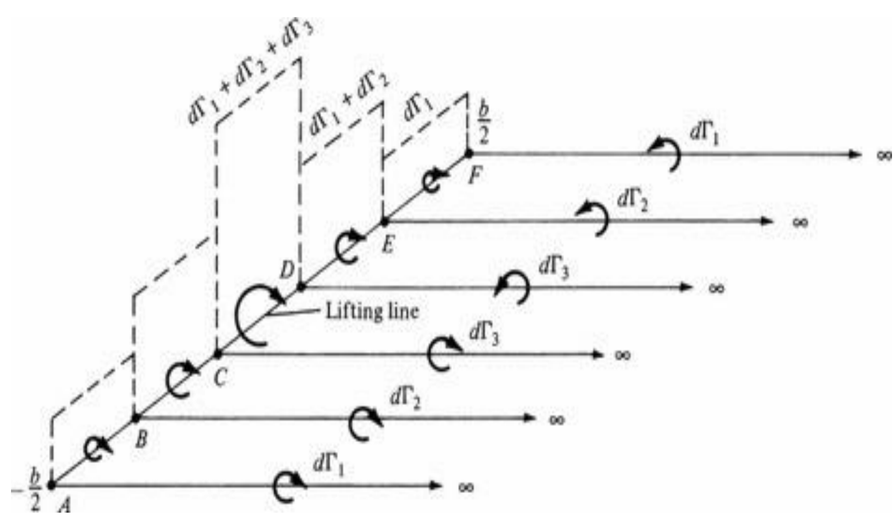
- *Instead of representing the wing by a single horseshoe vortex, superimpose using a large number of horseshoe vortices.*
  - *Each horseshoe with a different length of the bound vortex.*
  - *All bound vortices coincident along a single line - Lifting Line.*
- *The series of trailing vortices represents pairs of vortices.*
  - *Each pair is associated with a given horseshoe vortex.*
- *The strength of each trailing vortex is equal to the change in circulation along the lifting line.*







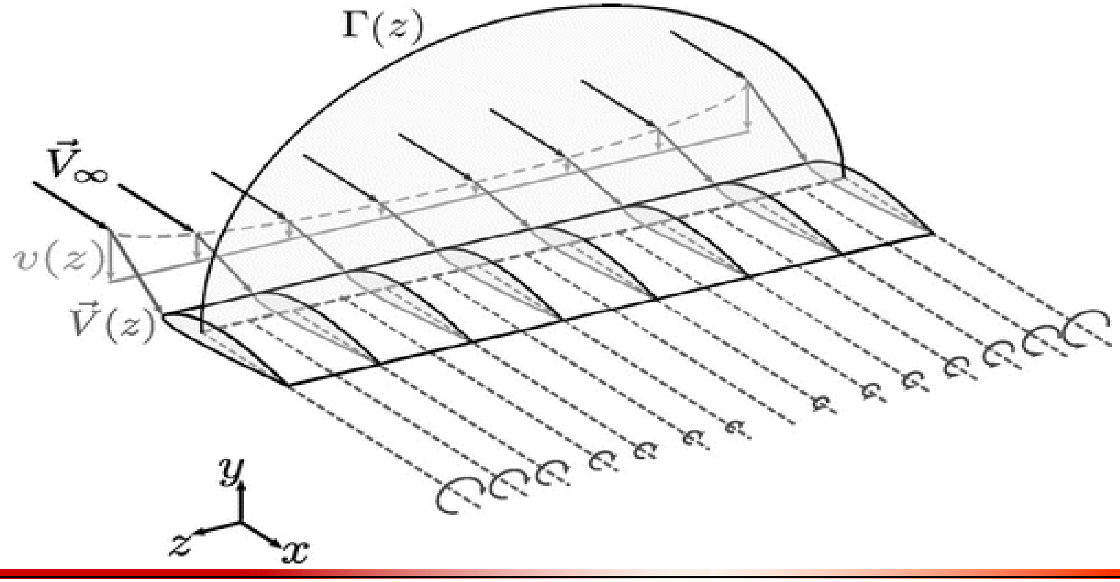




❑ PRANDTL'S LIFTING LINE THEORY

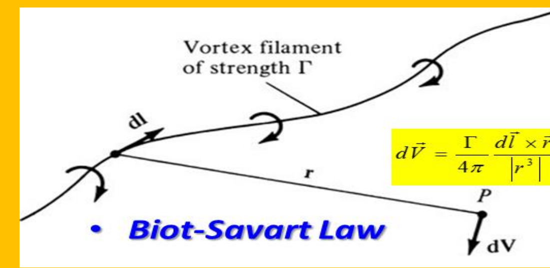
❑ Prandtl's Lifting Line Theory

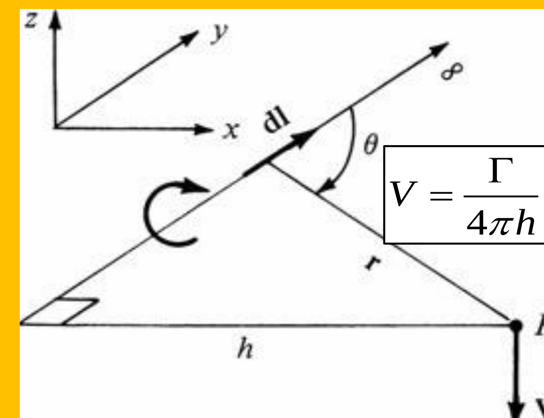
- *Let's extrapolate to the case where an infinite number of horseshoe vortices are superimposed along the lifting line.*
  - *Each horseshoe has vanishingly small strength.*
- *The finite number of trailing vortices in the earlier case have become a continuous vortex sheet.*
- *The total strength of the sheet integrated across the span of the wing is zero (because of pairs of trailing vortices of equal but opposite strengths).*

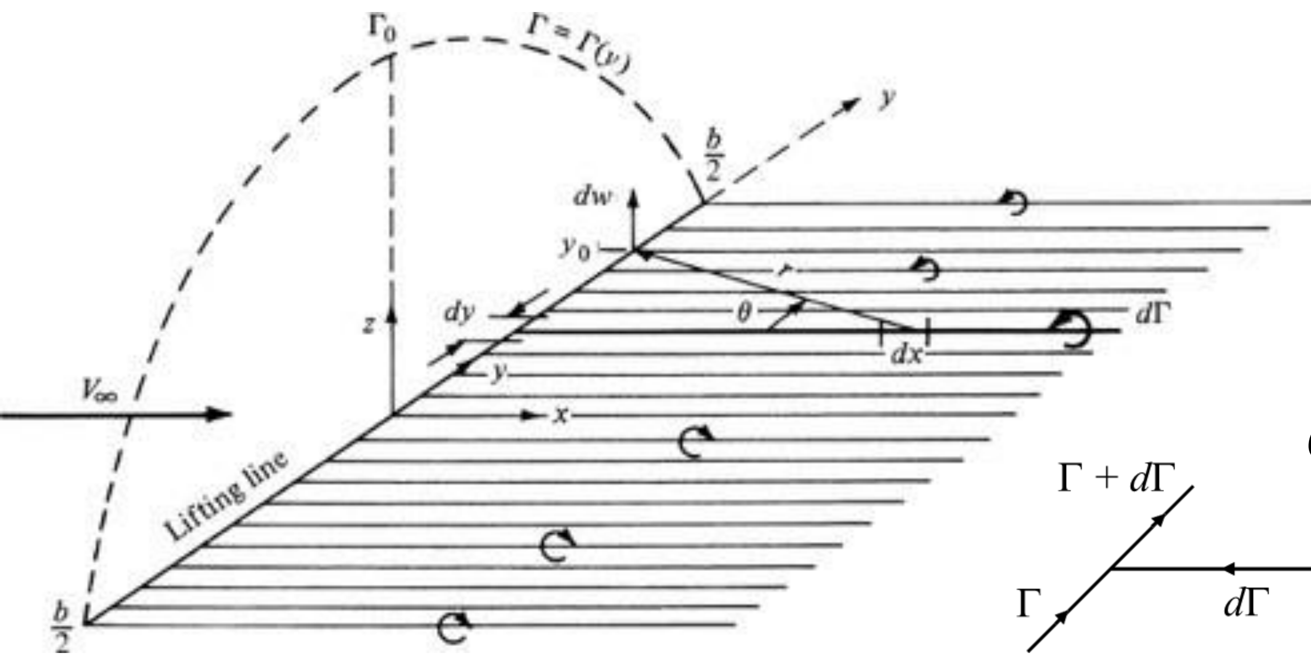


❑ Prandtl's Lifting Line Theory

- *The wing is replaced by a bound vortex with (continuously) varying circulation $\Gamma(y)$*
- *The trailing vortices create a 'vortex wake' in the form of a continuous vortex sheet .*
    - *Local strength of the trailing vortex at position y is given by the change in $\Gamma(y)$: $d\Gamma = (d\Gamma/dy)\,dy$*
    - *the vortex sheet is assumed to remain flat (no deformation)*
- *Validity: good approximation for straight, slender wings at moderate lift*







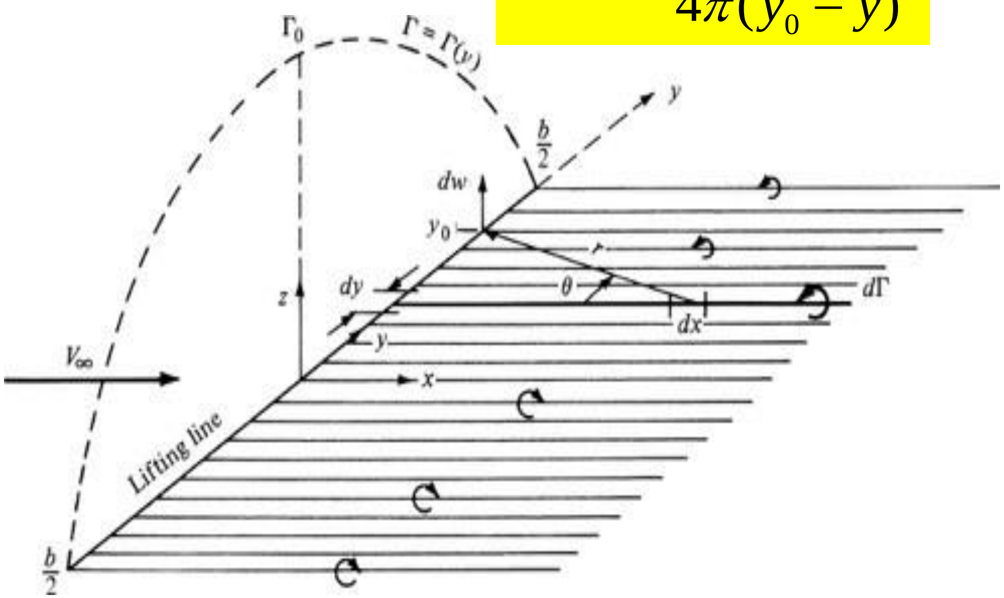
$$dw = -\frac{(d\Gamma/dy) \bullet dy}{4\pi(y_0 - y)}$$

PRANDTL'S LIFTING LINE THEORY

Strength of the trailing vortex at position, y, along the wingspan:

- *Take a small segment of the lifting line, dy, at position y.*
- *The change in circulation of the lifting line over the segment is:* $d\Gamma = (d\Gamma/dy)dy$.
- *This is equal to the strength of the trailing vortex.*
- *The contribution dw to the induced velocity at position.*

$$dw = -\frac{(d\Gamma/dy)\cdot dy}{4\pi(y_0 - y)}$$



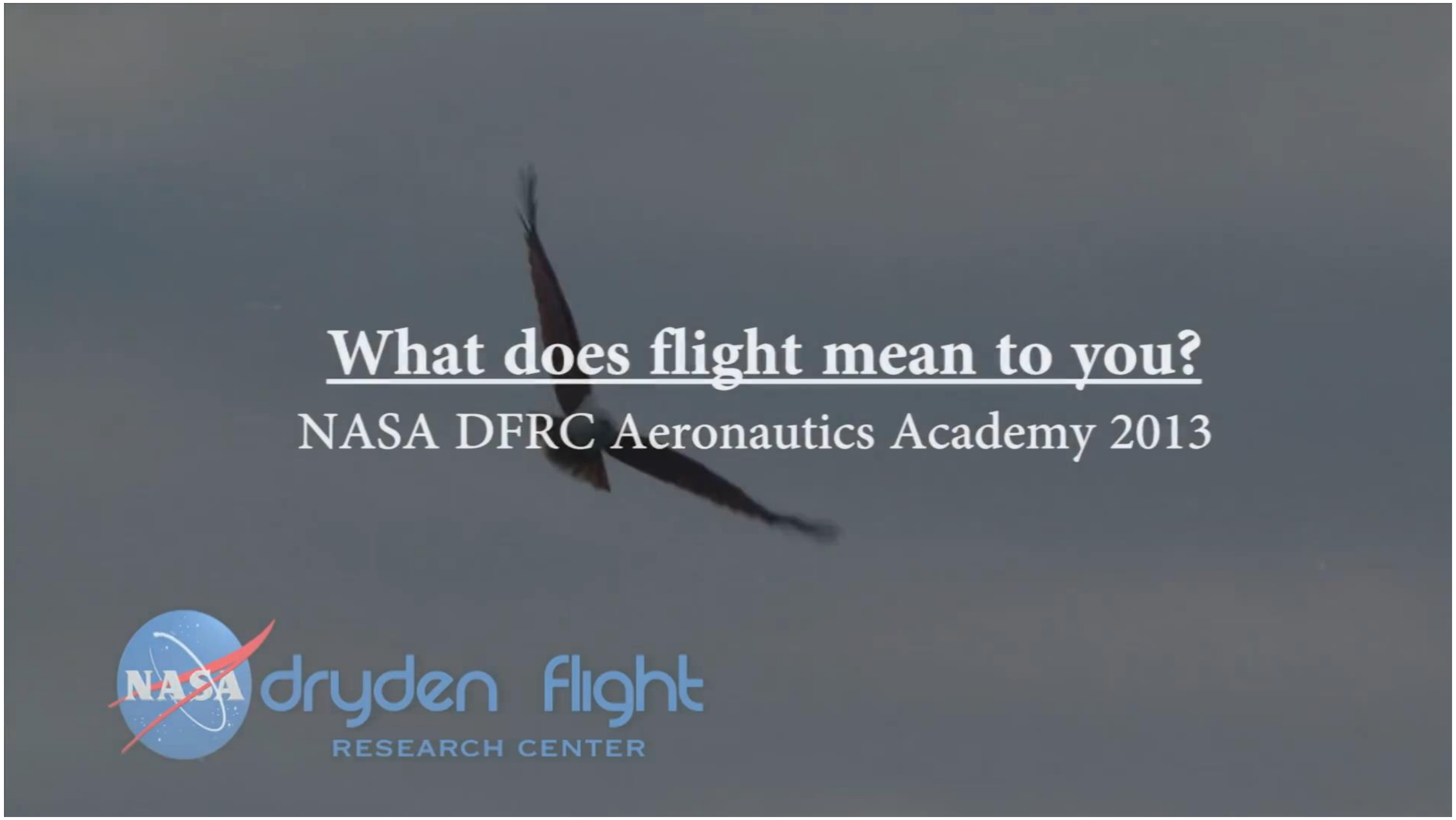
Total velocity at position y_0 induced by the entire wake vortices will be:

$$W(y_0) = \int_{-b/2}^{b/2} dw = -\int_{-b/2}^{b/2} \frac{(d\Gamma/dy)}{4\pi(y_0 - y)} dy$$

❑ PRANDTL'S LIFTING LINE THEORY

Proving Prandtl- With A Twist!

- *A group of college aerospace engineering students in the 2012-2013 Aeronautics Academy at NASA's Dryden Flight Research Center have proven German aerodynamicist Ludwig Prandtl's theory on how to overcome one of the thorny problems of flight.*



https://www.youtube.com/watch?v=Hr0I6wBFGpY&t=135s