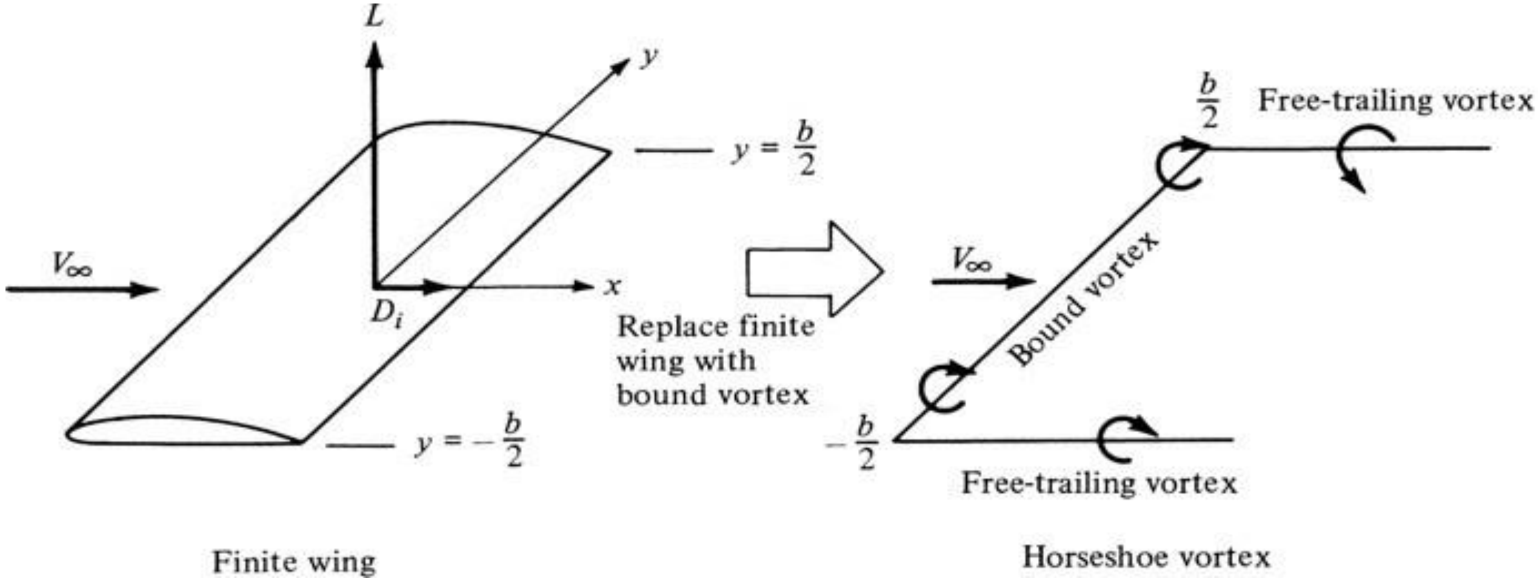
Lecture # 33: 3D Wing Aerodynamics: Lifting Line Theory – Part #2

Dr. Hui HU
Department of Aerospace Engineering
Iowa State University, 2251 Howe Hall, Ames, IA 50011-2271
Tel: 515-294-0094 / Email: huhui@iastate.edu

❑ PRANDTL'S LIFTING LINE THEORY

❑ *Prandtl's Lifting Line Theory*

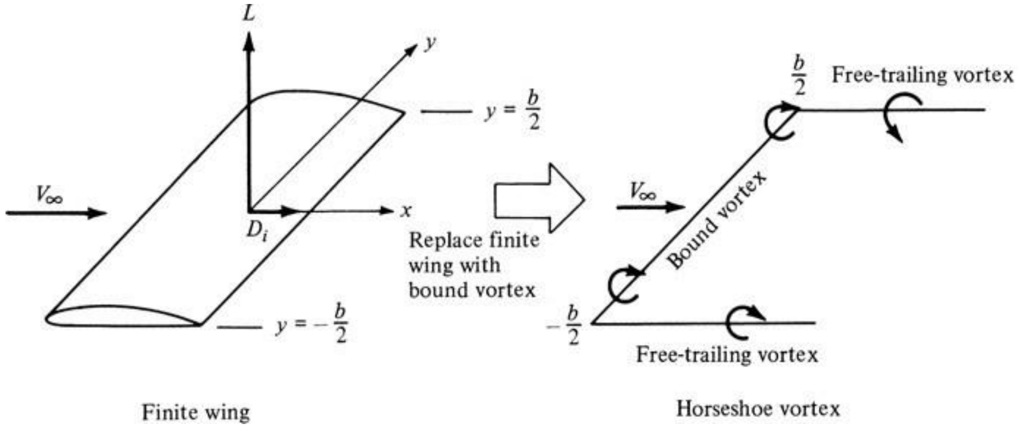
- *The theory is useful for predicting the aerodynamic characteristics of finite wings.*

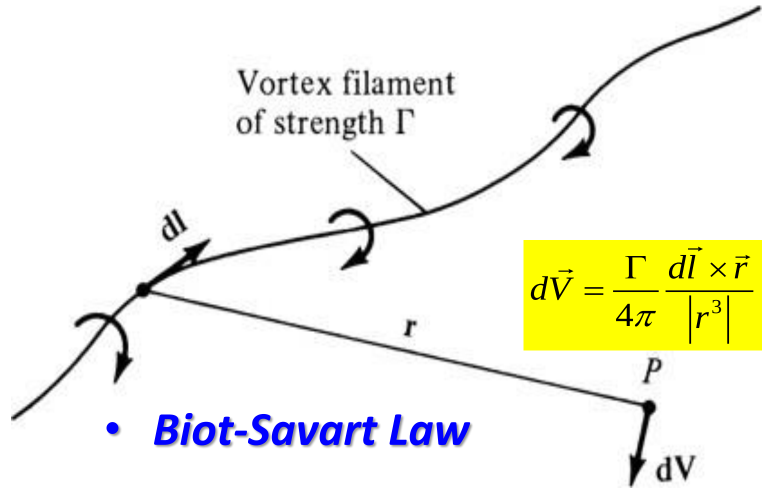


- *The finite wing is replaced with a bound vortex.*
- *Due to Helmholtz's theorem, a vortex filament cannot end in the fluid.*
  - *Therefore, assume the vortex filament continues as two free vortices trailing downstream from the wing tips to infinity.*
- *Bound vortex + Trailing vortices —> Horseshoe Vortex*

❑ 3D Wing Aerodynamics

❑ *Prandtl's Lifting Line Theory*

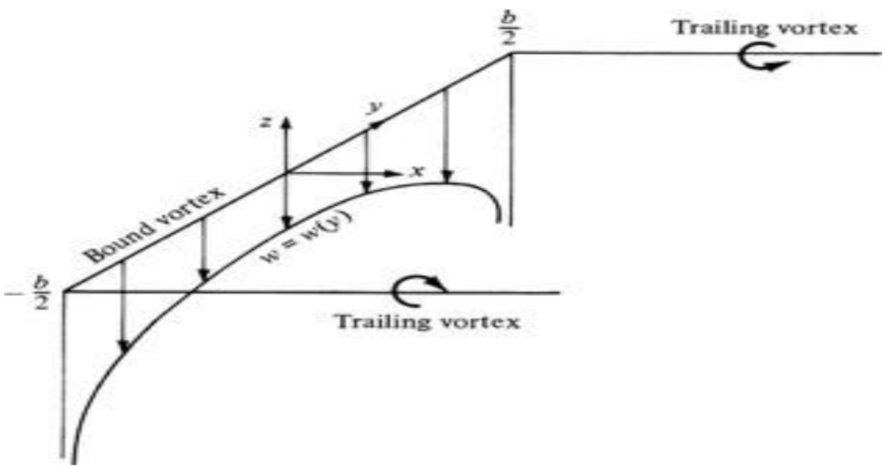




$$d\vec{V} = \frac{\Gamma}{4\pi}\frac{d\vec{l} \times \vec{r}}{\left|r^3\right|}$$

- ***Biot-Savart Law***

- *The Biot-Savart law allows us to determine the downwash along the wing and results in:*



$$w(y) = -\frac{\Gamma}{4\pi\left(b/2 + y\right)} - \frac{\Gamma}{4\pi\left(b/2 - y\right)}$$

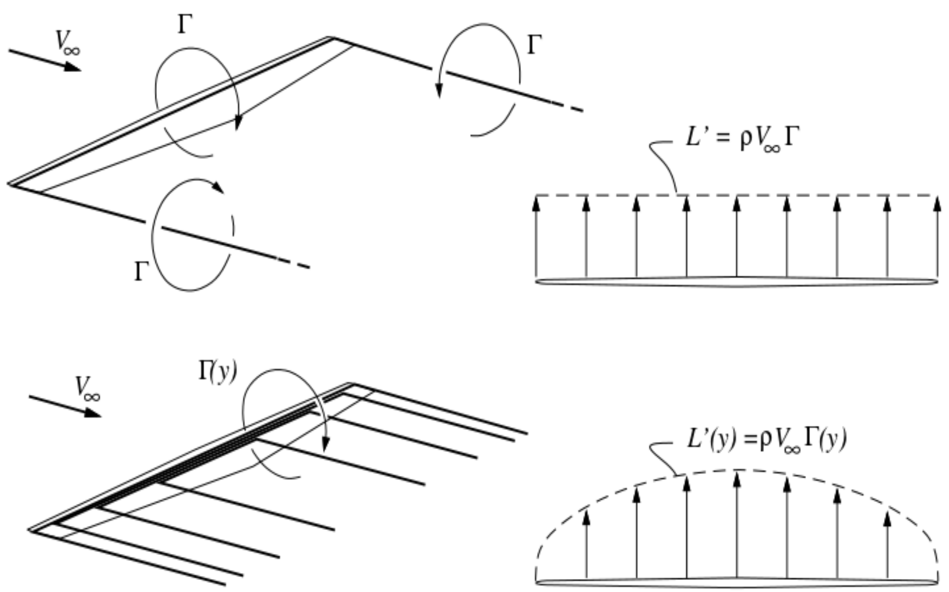
or

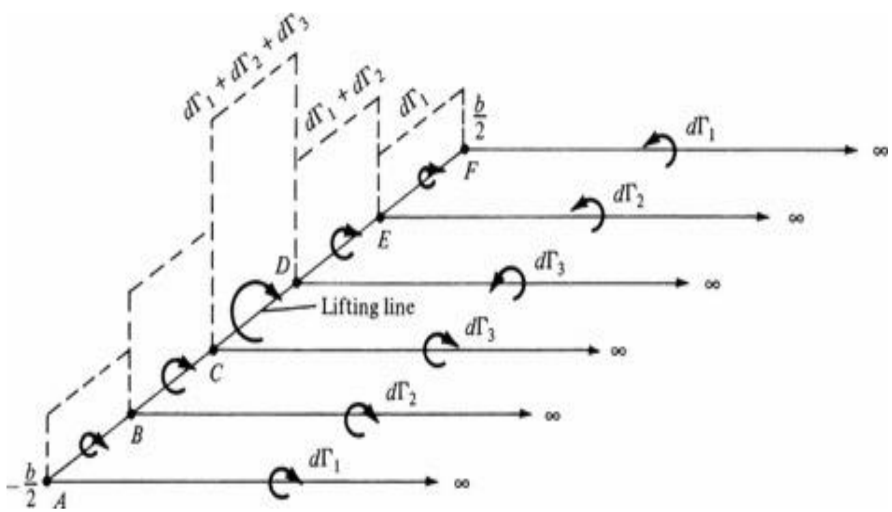
$$w(y) = -\frac{\Gamma}{4\pi}\frac{b}{\left(b/2\right)^2 - y^2}$$

- ***However, the single vortex filament case is not sufficient to describe the physical conditions on the wing since** the downwash at the wing tips is infinite, instead of zero!*

❑ PRANDTL'S LIFTING LINE THEORY

- *Instead of representing the wing by a single horseshoe vortex, superimpose using a large number of horseshoe vortices.*
  - *Each horseshoe with a different length of the bound vortex.*
  - *All bound vortices coincident along a single line - Lifting Line.*
- *The series of trailing vortices represents pairs of vortices.*
  - *Each pair is associated with a given horseshoe vortex.*
- *The strength of each trailing vortex is equal to the change in circulation along the lifting line.*





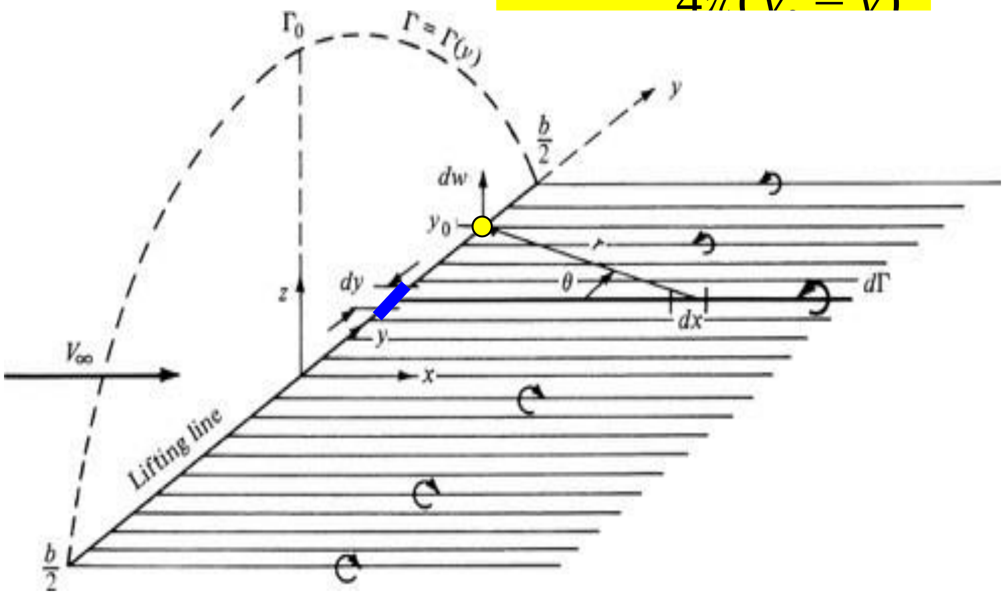
❑ PRANDTL'S LIFTING LINE THEORY

❑ *Strength of the trailing vortex at position, y, along the wingspan:*

- *Take a small segment of the lifting line, dy, at position y.*

$$dw = -\frac{(d\Gamma / dy)\bullet dy}{4\pi(y_0 - y)}$$

- *The change in circulation of the lifting line over the segment is:*
  $d\Gamma = (d\Gamma/dy)dy$.
- *This is equal to the strength of the trailing vortex.*
- *The contribution dw to the induced velocity at position.*



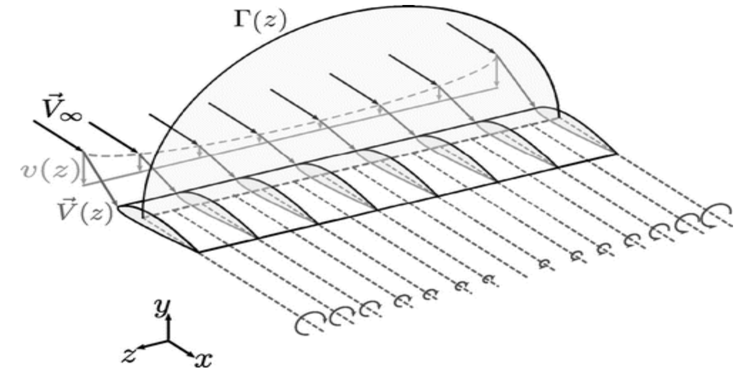
❑ *Total velocity at position y_0 induced by the entire vortex wake*

$$W(y_0) = \int_{-b/2}^{b/2} dw = -\int_{-b/2}^{b/2} \frac{(d\Gamma / dy)}{4\pi(y_0 - y)} dy$$

❑ PRANDTL'S LIFTING LINE THEORY

❑ *Determining the induced angle of attack of the lifting line.*

- ***Total velocity at position y_0 induced by the entire vortex wake***

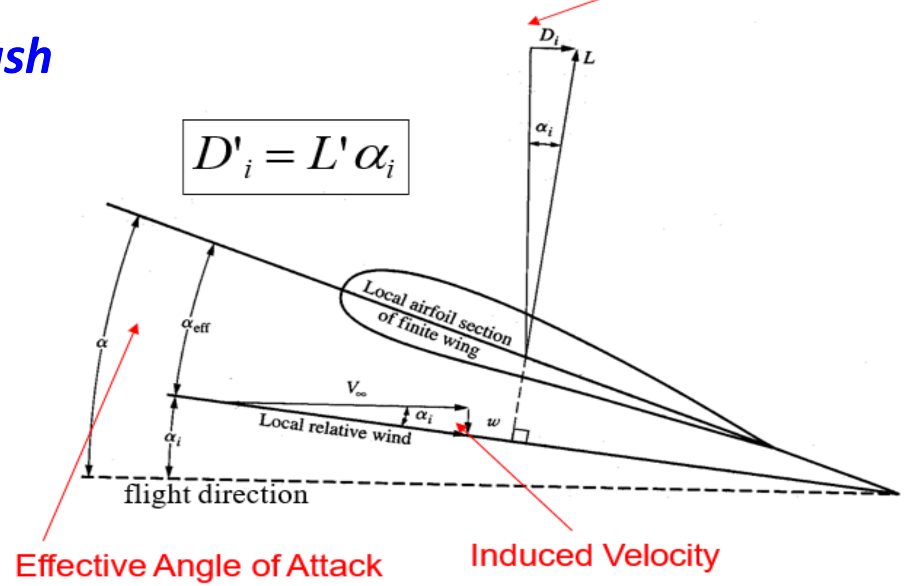


$$W(y_0) = \int_{-b/2}^{b/2} dw = -\int_{-b/2}^{b/2} \frac{(d\Gamma / dy)}{4\pi(y_0 - y)} dy$$

$$\boxed{\alpha_{\mathrm{eff}} = \alpha - \alpha_i}$$

- ***Induced angle of attack due to downwash***

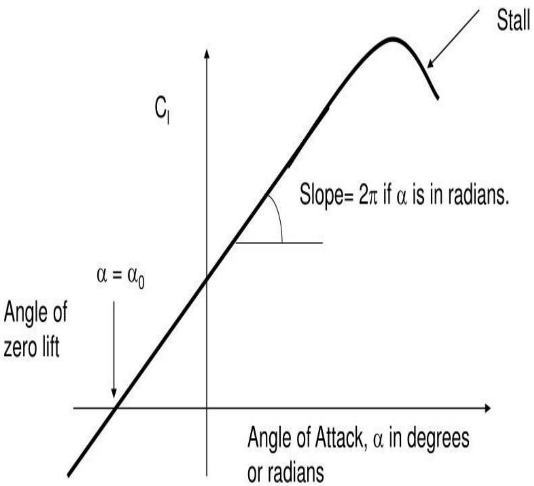
$$\boxed{D'_i = L'\alpha_i}$$

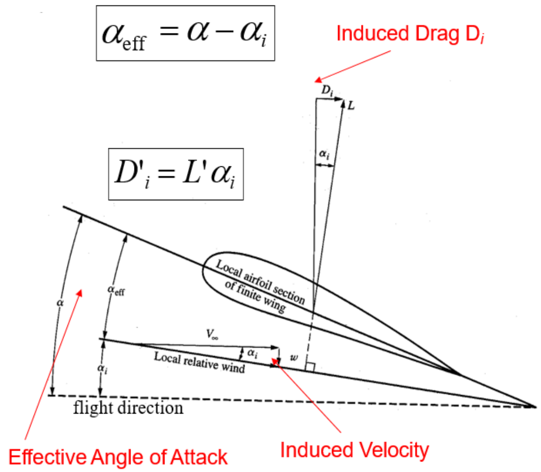


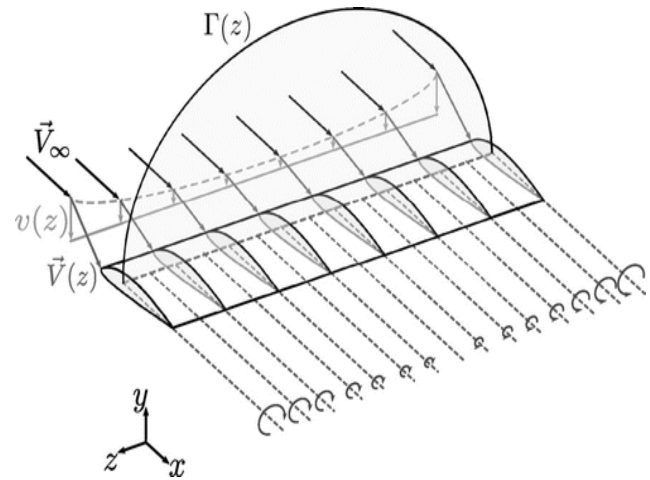
$$\alpha_i(y_0) = \tan^{-1}[-\frac{w(y_0)}{V_\infty}] \approx -\frac{w(y_0)}{V_\infty}$$

$$\Rightarrow \alpha_i(y_0) = \frac{1}{4\pi V_\infty} \int_{-b}^{b} \frac{(d\Gamma / dy)}{(y_0 - y)} dy$$

❑ Prandtl's Lifting Line Theory







- Use '2D' airfoil theory, but modified by the effective flow direction: $\left(a_0 = \dfrac{dc_l}{d\alpha}\right)$

$$c_l = c_l(\alpha_{\text{eff}}) = a_0[\alpha_{\text{eff}} - \alpha_{L=0}] = a_0[\alpha - \alpha_i - \alpha_{L=0}]$$

- From the relation between lift and circulation:

$$c_l = \frac{L'}{\frac{1}{2}\rho V_\infty{}^2 c} = \frac{\rho V_\infty \Gamma}{\frac{1}{2}\rho V_\infty{}^2 c} = \frac{2\Gamma(y_0)}{V_\infty c(y_0)}$$

- combination:

$$\alpha = \frac{c_l}{a_0} + \alpha_{L=0} + \alpha_i$$

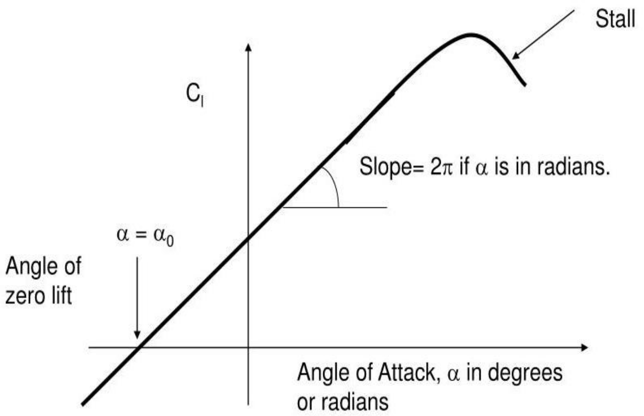
$$\alpha_i(y_0) = -\frac{w(y_0)}{V_\infty} = \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2}\frac{(d\Gamma/dy)}{(y_0 - y)}dy$$

$$\boxed{\alpha(y_0) = \frac{2\Gamma(y_0)}{a_0 V_\infty c(y_0)} + \alpha_{L=0}(y_0) + \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2}\frac{(d\Gamma/dy)}{(y_0 - y)}dy}$$

PRANDTL'S LIFTING LINE THEORY

- ***According thin airfoil theory for 2D airfoil***

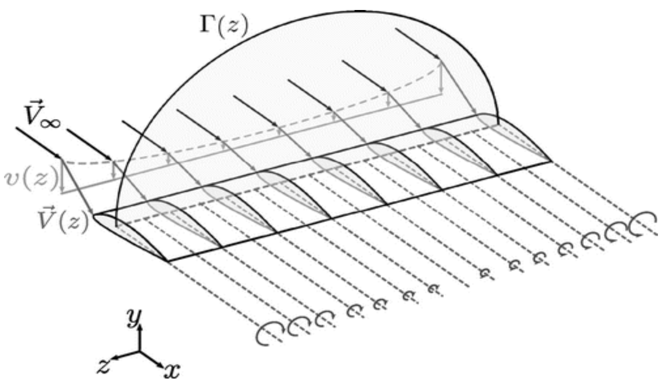
$$a_o = \frac{dc_l}{d\alpha} = 2\pi$$



Fundamental equation for Prandtl's Lifting Line Theory can be expressed as:

$$\boxed{\alpha(y_0) = \frac{\Gamma(y_0)}{\pi V_\infty c(y_0)} + \alpha_{L=0}(y_0) + \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2} \frac{(d\Gamma/dy)}{(y_0 - y)} dy}$$

- ***All the other quantities are known for a finite wing of given design at a given geometric angle of attack in a freestream with given velocity.***
- ***The question is an integro-differential equation, in which the only unknow is Γ.***
- ***A solution to the above equation would yield $\Gamma = \Gamma(y_0)$, where y_0 ranges along the span from $-b/2$ to $b/2$.***



❑ PRANDTL'S LIFTING LINE THEORY

❑ ***Fundamental equation for Prandtl's Lifting Line Theory can be expressed as:***

$$\alpha(y_0) = \frac{\Gamma(y_0)}{\pi V_\infty c(y_0)} + \alpha_{L=0}(y_0) + \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2} \frac{(d\Gamma/dy)}{(y_0 - y)}\,dy$$

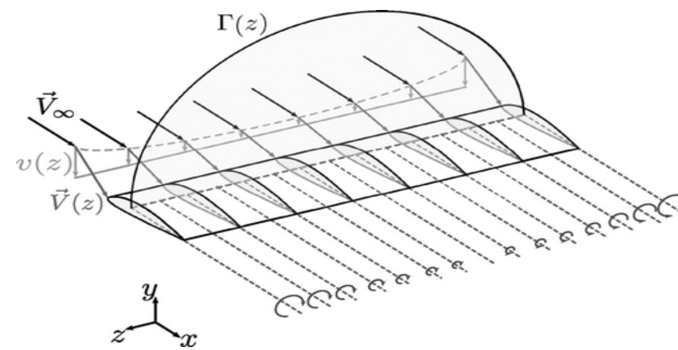
Some remarks:

1. ***This equation describes the relation between circulation and wing properties.***
2. ***It is linear in Γ.***
3. ***The circulation Γ is proportional to V_∞ (Lift ~ $\rho V_\infty \Gamma$ ~ ρV_∞^2).***
4. ***For a wing without twist (α and $\alpha_{L=0}$ are constant):***
   - ***Circulation Γ is proportional to $\alpha - \alpha_{L=0}$ for every value of α .***
   - ***The lift distribution has the same form as the wing shape (which depends on $a_0(y)$, $c(y)$ and b, therefore, on the wing shape).***
   - ***The total lift is zero when $\alpha = \alpha_{L=0}$ and then: $\Gamma \equiv 0$ along the spanwise y_o direction***
5. ***For a wing with twist (α and $\alpha_{L=0}$ are not constant): THIS IS NOT SO***
   - ***in particular: total zero lift is in general not accompanied by: $\Gamma \equiv 0$ along the spanwise y_o direction.***

Prandtl's Lifting Line Theory

- **Fundamental equation for Prandtl's Lifting Line Theory can be expressed as:**

$$\alpha(y_0) = \frac{\Gamma(y_0)}{\pi V_\infty c(y_0)} + \alpha_{L=0}(y_0) + \frac{1}{4\pi V_\infty}\int_{-b}^{b} \frac{(d\Gamma/dy)}{(y_0 - y)} dy$$



- **Once $\Gamma(y_0)$ is known:**

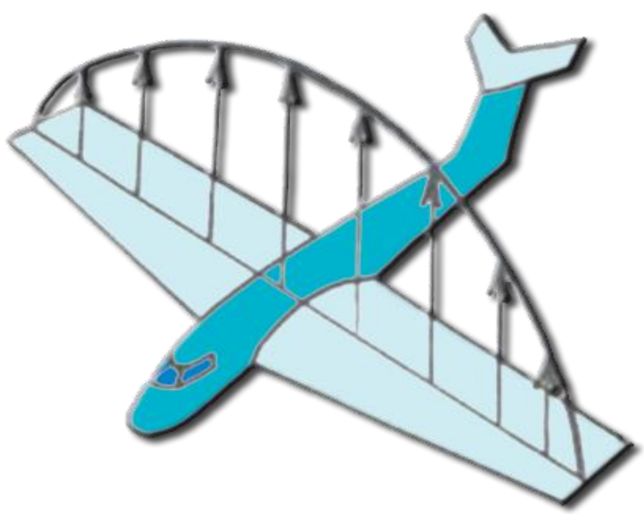
1. **Lift distribution:** $L'(y) = \rho V_\infty \Gamma(y)$

2. **Total lift:** $L = \int_{-b/2}^{b/2} L' dy = \rho V_\infty \int_{-b/2}^{b/2} \Gamma(y) dy$ $\quad \boxed{C_L = \frac{L}{q_\infty S} = \frac{2}{V_\infty S}\int_{-b/2}^{b/2} \Gamma(y) dy}$

3. **Induced angle of attack:** $\boxed{\alpha_i(y_0) = \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2} \frac{(d\Gamma/dy)}{(y_0 - y)} dy}$

4. **Induced drag:** $D_i = \int_{-b/2}^{b/2} D_i' dy = \int_{-b/2}^{b/2} L' \alpha_i \, dy = \rho V_\infty \int_{-b/2}^{b/2} \Gamma(y)\, \alpha_i(y) dy$

$$\boxed{C_{D_i} = \frac{D_i}{q_\infty S} = \frac{2}{V_\infty S}\int_{-b/2}^{b/2} \Gamma(y)\, \alpha_i(y) dy}$$



❑ Prandtl's Lifting Line Theory

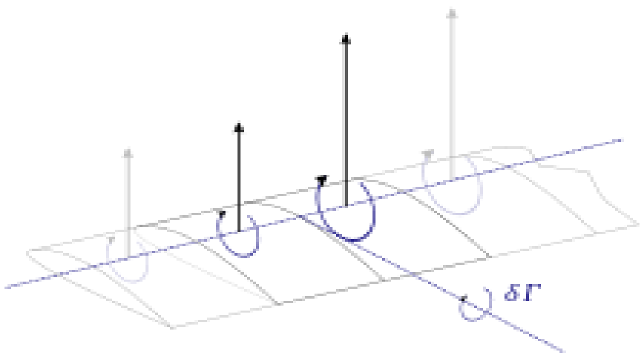
❑ ***Fundamental equation for Prandtl's Lifting Line Theory can be expressed as:***

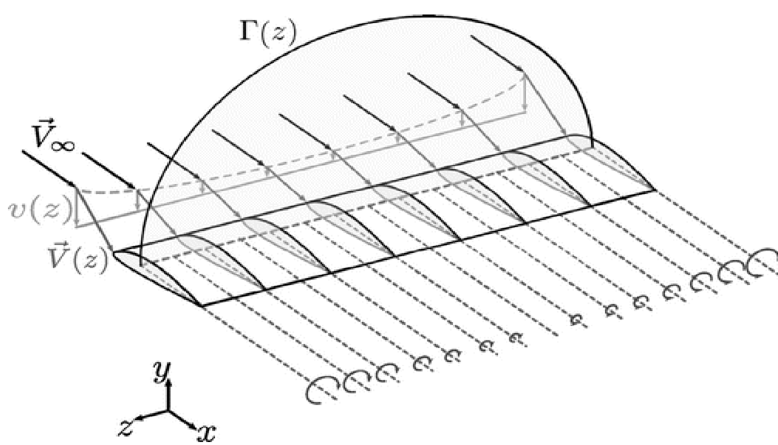
$$\alpha(y_0) = \frac{\Gamma(y_0)}{\pi V_\infty c(y_0)} + \alpha_{L=0}(y_0) + \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2} \frac{(d\Gamma/dy)}{(y_0 - y)} dy$$

Two approaches can be taken:

- ***Direct approach** – A wing planform is given with a distribution of aerodynamic twist, the equation is solved and lift and drag information extracted.*

- ***Inverse approach** – A lift distribution is proposed and the corresponding planform distribution developed.*



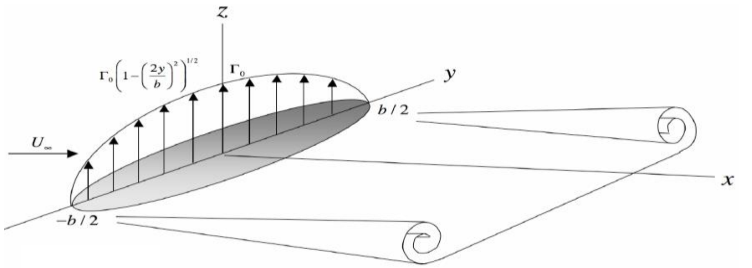


Prandtl's Lifting Line Theory

Consider a case with elliptical lift distribution

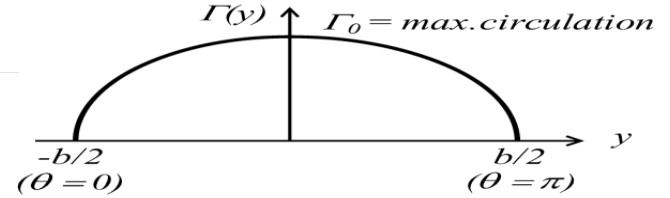
- *Consider the following "elliptical" lift distribution:*

$$\Gamma(y) = \Gamma_0 \sqrt{1-\left(\frac{y}{b/2}\right)^2}$$



- *Compute the downwash velocity from:*

$$w(y_0) = -\frac{1}{4\pi}\int_{-b/2}^{b/2} \frac{d\Gamma/dy}{y_0 - y}\,dy$$



- *With coordinate transformation:*

$$y = -\frac{b}{2}\cos\theta \qquad dy = \frac{b}{2}\sin\theta\, d\theta \qquad \boxed{\Gamma(\theta) = \Gamma_0 \sin\theta}$$

$$w(\theta_0) = -\frac{1}{2\pi b}\int_0^{\pi} \frac{d\Gamma/d\theta}{\cos\theta - \cos\theta_0}\,d\theta = -\frac{\Gamma_0}{2\pi b}\underbrace{\int_0^{\pi}\frac{\cos\theta}{\cos\theta - \cos\theta_0}\,d\theta}_{=\pi} = -\frac{\Gamma_0}{2b}$$

$$\alpha_i = -\frac{w}{V_\infty} = \frac{\Gamma_0}{2bV_\infty}$$

Downwash and induced angle of attack are constant over the span of the wing!

Prandtl's Lifting Line Theory

Consider a case with elliptical lift distribution

- **Calculation of the total lift:** $dy = \frac{b}{2}\sin\theta\, d\theta$ $\qquad \Gamma(\theta) = \Gamma_0 \sin\theta$

$$L = \rho V_\infty \int_{-b/2}^{b/2} \Gamma(y)\, dy = \rho V_\infty \frac{b}{2}\int_0^{\pi} \Gamma(\theta)\sin\theta\, d\theta = \rho V_\infty \Gamma_0 \frac{b}{2}\int_0^{\pi} \sin^2\theta\, d\theta = \rho V_\infty \Gamma_0 \frac{b}{4}\pi$$

- **Relation between Γ_0 and C_L:** $$\Gamma_0 = \frac{4L}{\rho V_\infty b\pi} = \frac{4\cdot(C_L \frac{1}{2}\rho V_\infty^2 S)}{\rho V_\infty b\pi} = \frac{2V_\infty S}{b\pi} C_L$$

- **The induced angle of attack:** $$\boxed{\alpha_i = \frac{\Gamma_0}{2bV_\infty} = \frac{C_L}{\pi\,(b^2/S)} = \frac{C_L}{\pi\, AR}}$$

- *$AR = b^2/S$: is called the 'aspect ratio'(AR) of the wing ("slankheid")*
- *Typical values:*
  - *$AR = 6 \sim 8$ for subsonic aircraft*
  - *$AR = 10 \sim 22$ for glider aircraft*

❑ PRANDTL'S LIFTING LINE THEORY

❑ *Consider a case with elliptical lift distribution*

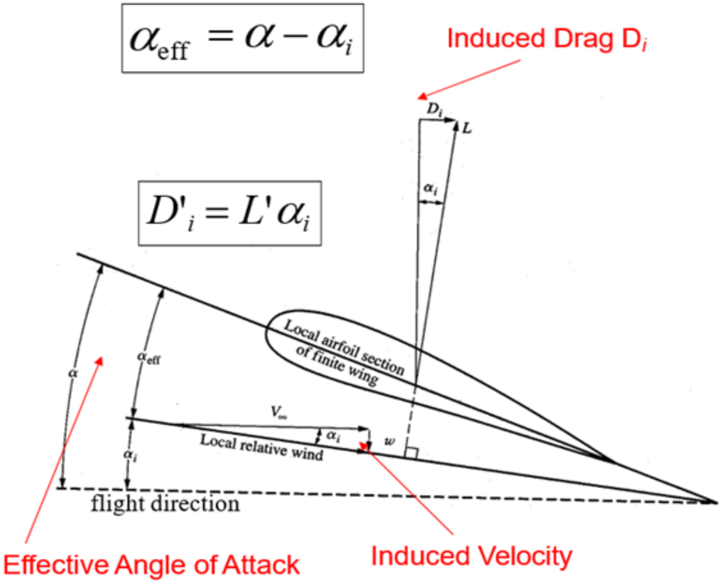
- ***Calculation of the induced drag:***

$$D_i = \int_{-b/2}^{b/2} L'\,\alpha_i\,dy = \alpha_i \int_{-b/2}^{b/2} L'\,dy = \alpha_i\,L$$

Note that $\alpha_i = \dfrac{C_L}{\pi AR}$ *is constant here*

$$\alpha_{\text{eff}} = \alpha - \alpha_i$$

$$D'_i = L'\alpha_i$$



$$\boxed{C_{D_i} = \alpha_i C_L = \frac{C_L^{\,2}}{\pi AR}}$$

Conclusions:

- *The induced drag is the "drag due to lift"*
- *Remember : total drag* $C_D = c_d + C_{D_i}$
- $C_{D_i} \sim C_L^{\,2}$ *: quadratic dependence*
- $C_{D_i} \sim \dfrac{1}{A}$ *: large AR decreases induced drag*

Prandtl's Lifting Line Theory

Consider a case with elliptical lift distribution

The geometry can be found by going back to the lift coefficient and Eqs. (5.23), (5.21) and (5.2)

$$C_l = \frac{2\Gamma(y_o)}{V_\infty c(y_o)} \qquad (5.23)$$

$$C_l = 2\pi\left[\alpha_{eff} - \alpha_{L=0}\right] \qquad (5.21)$$

$$\alpha_{eff} = \alpha - \alpha_i \qquad (5.2)$$

Then using the idea that the induced a.o.a. is constant and if there is no aerodynamic twist we see from Eqs. (5.21) and (5.2) that

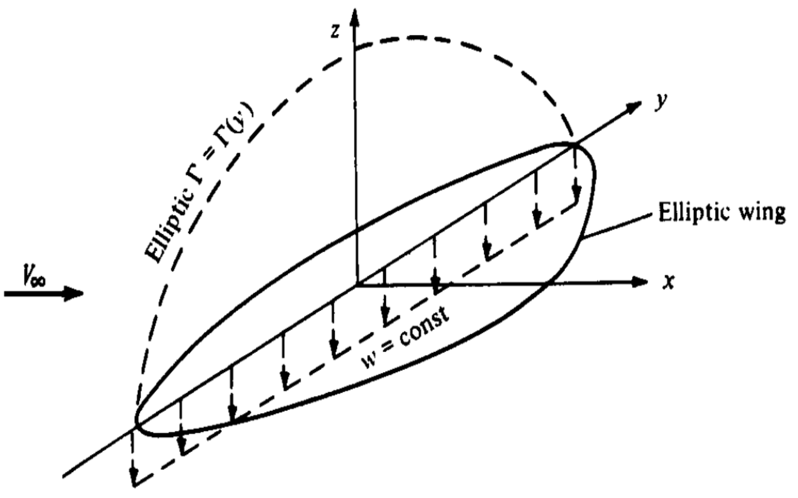
$$C_l = const. \qquad (5.47)$$

Elliptic Lift Distribution

Combining Eqs. (5.47) and (5.23) gives

$$const. = \frac{2\Gamma(y_o)}{V_\infty c(y_o)} \Rightarrow c(y_o) = const._2 \Gamma(y_o)$$

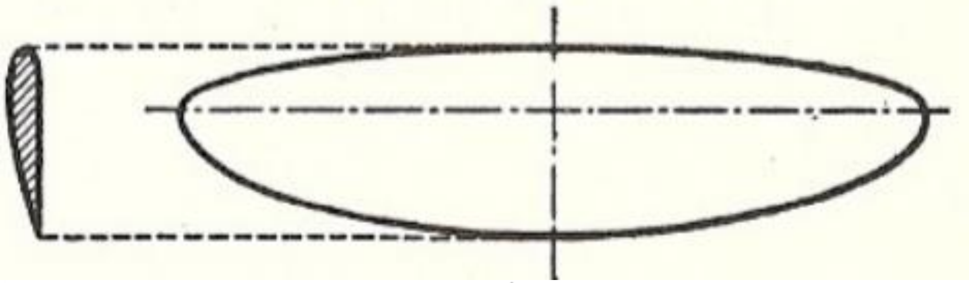
- ***The wing must have an elliptical planform in order to have an elliptical lift distribution!***



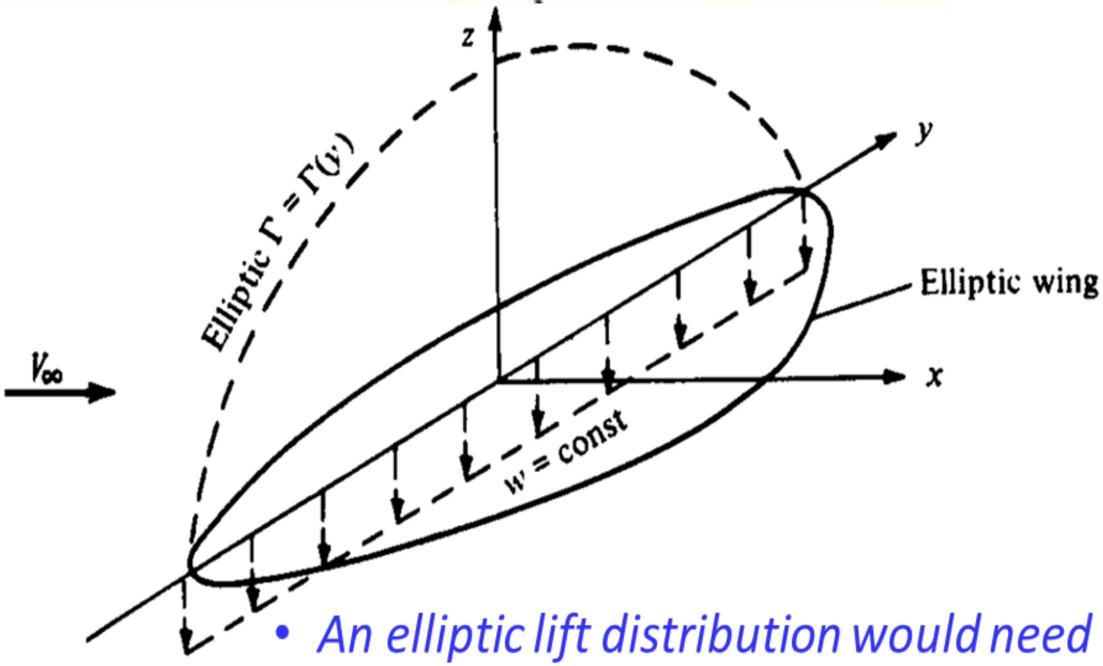
❑ PRANDTL'S LIFTING LINE THEORY

❑ Consider a case with elliptical lift distribution

An elliptical wing planform: (note straight 1/4-chord line)



1/4-chord line

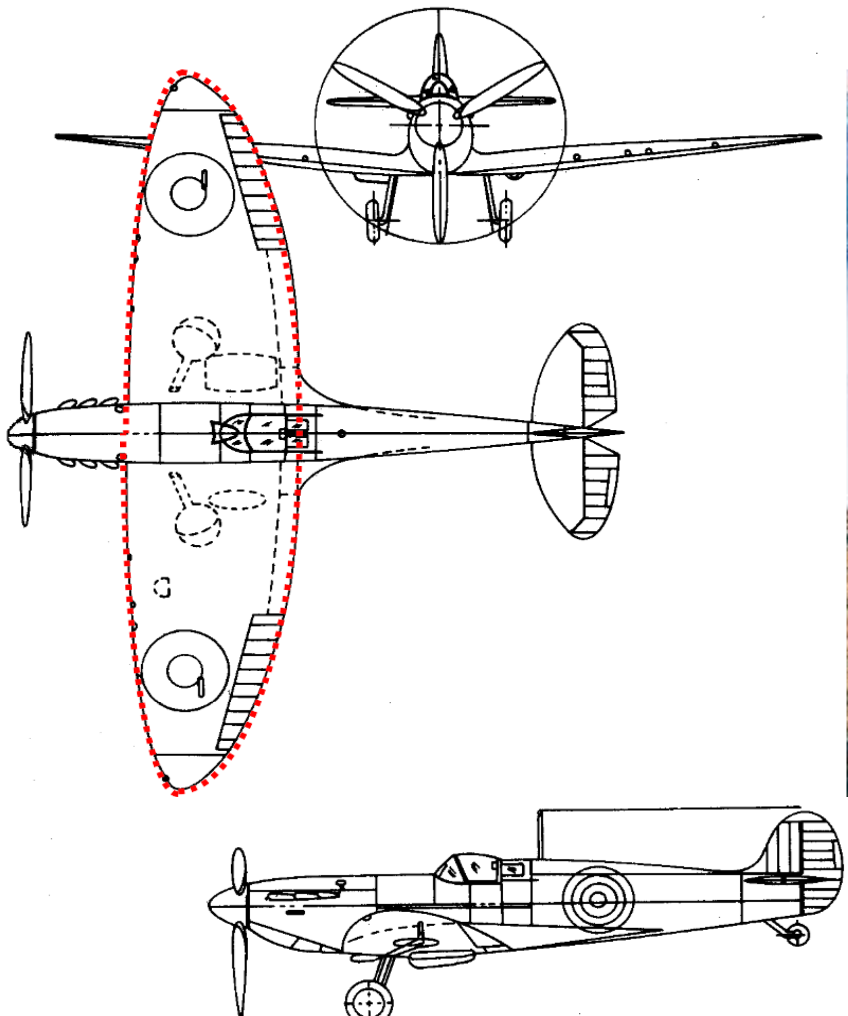


- *The SPITFIRE with elliptical wing*

- An elliptic lift distribution would need an elliptic wing planform and a constant downwash.

❑ Prandtl's Lifting Line Theory

- *The Supermarine Spitfire*



- ***How The Spitfire Became An Aviation Masterpiece | The Birth Of A Legend | Timeline:***
- [***https://www.youtube.com/watch?v=fR03Mmv2bUs***](https://www.youtube.com/watch?v=fR03Mmv2bUs)

Prandtl's Lifting Line Theory

Consider a case with elliptical lift distribution

- *We found that:*
  - $c_l = C_L \quad (= constant)$
  - $\alpha_i = \dfrac{C_L}{\pi AR} \quad (= constant)$

  $\Bigg\}$ *for an elliptic wing*

  - $c_l = a_0[\alpha - \alpha_i - \alpha_{L=0}]$

    where: $a_0 = \dfrac{dc_l}{d\alpha}$

  $\Bigg\}$ *for a general wing*

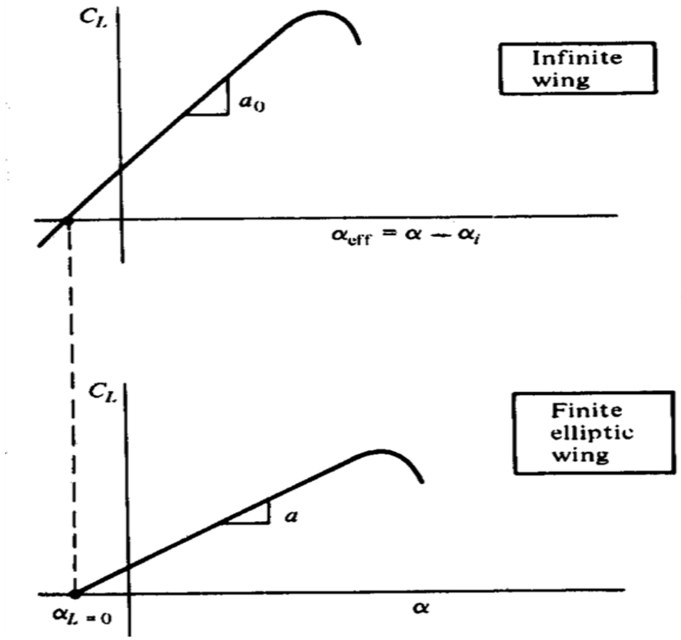
- *Combining:* $C_L = c_l = a_0[\alpha - \alpha_i - \alpha_{L=0}] = a_0\left[\alpha - \alpha_{L=0} - \dfrac{C_L}{\pi AR}\right]$

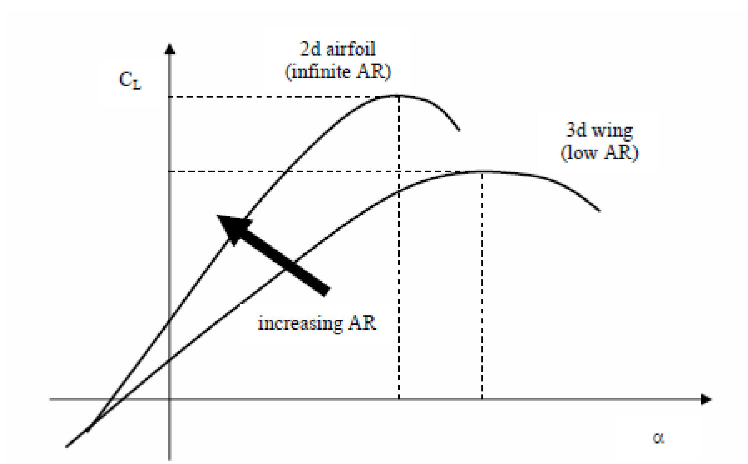
- *Solve for C_L* : $\boxed{C_L\left(1 + \dfrac{a_0}{\pi AR}\right) = a_0\,(\alpha - \alpha_{L=0})}$

note: $C_L = 0$ *when* $\alpha = \alpha_{L=0}$ *and:* $\boxed{\dfrac{dC_L}{d\alpha} = \dfrac{a_0}{1 + a_0/\pi AR}}$

Prandtl's Lifting Line Theory

Consider a case with elliptical lift distribution





- *For an elliptic wing:*

$$\boxed{\frac{dC_L}{d\alpha} = \frac{a_0}{1 + a_0 / \pi AR}}$$

- *The lift slope is reduced.*
- *physical explanation: the downwash reduces the effective angle of attack:*

$$\frac{dC_L}{d\alpha} = \frac{dc_l}{d\alpha} = \frac{dc_l}{d\alpha_{\text{eff}}} \cdot \frac{d\alpha_{\text{eff}}}{d\alpha}$$

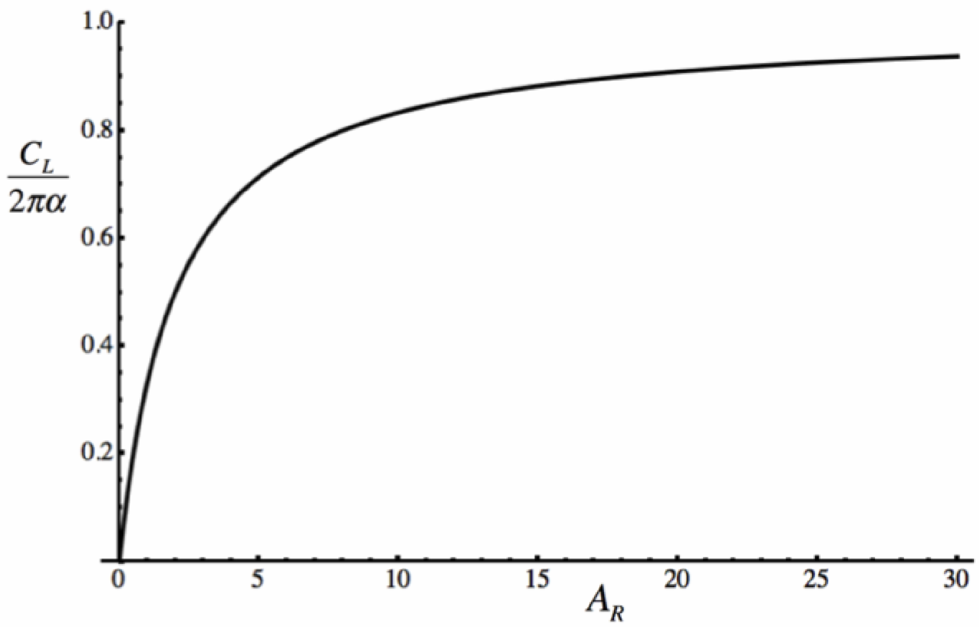
$$\frac{dc_l}{d\alpha_{\text{eff}}} = a_0 \qquad \frac{d\alpha_{\text{eff}}}{d\alpha} = 1 - \frac{d\alpha_i}{d\alpha} < 1$$

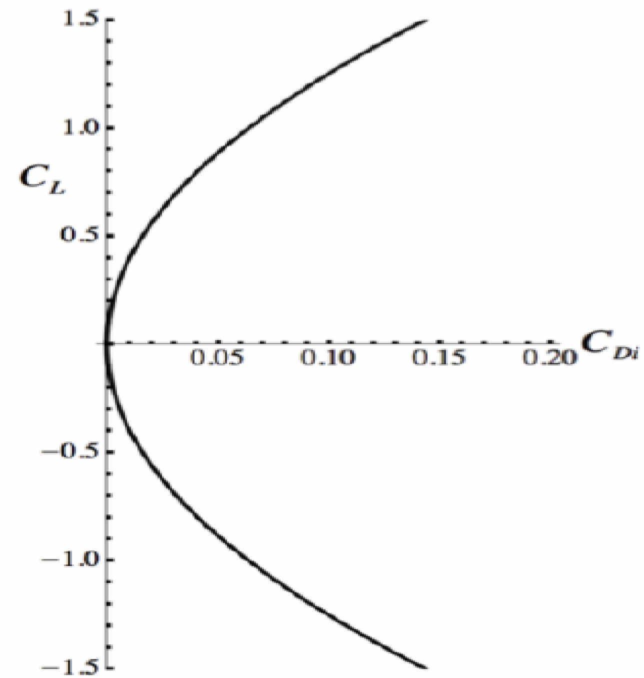
Prandtl's Lifting Line Theory

- *Consider a case with elliptical lift distribution*

$$\frac{C_L}{2\pi\alpha} = \left(\frac{A_R}{2 + A_R}\right)$$

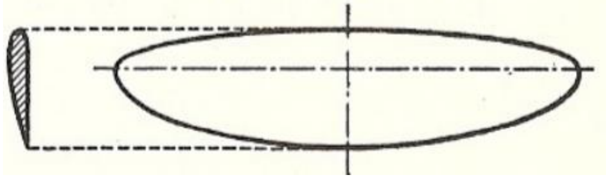
$$C_{D_i} = \frac{1}{\pi} C_L^2 \left(\frac{S}{b^2}\right) = \frac{1}{\pi} \frac{C_L^2}{A_R}$$





❑ Prandtl's Lifting Line Theory

The elliptical lift distribution - summary



- ***Constant downwash along the span:*** $$\alpha_i = \frac{C_L}{\pi AR}$$

- ***Induced drag:*** $$C_{D_i} = \alpha_i C_L = \frac{C_L^{\,2}}{\pi AR}$$

- ***Lift slope:*** $$a = \frac{dC_L}{d\alpha} = \frac{a_0}{1 + a_0/\pi AR}$$

- ***Effect of increasing the wing aspect ratio:*** *- induced drag smaller*

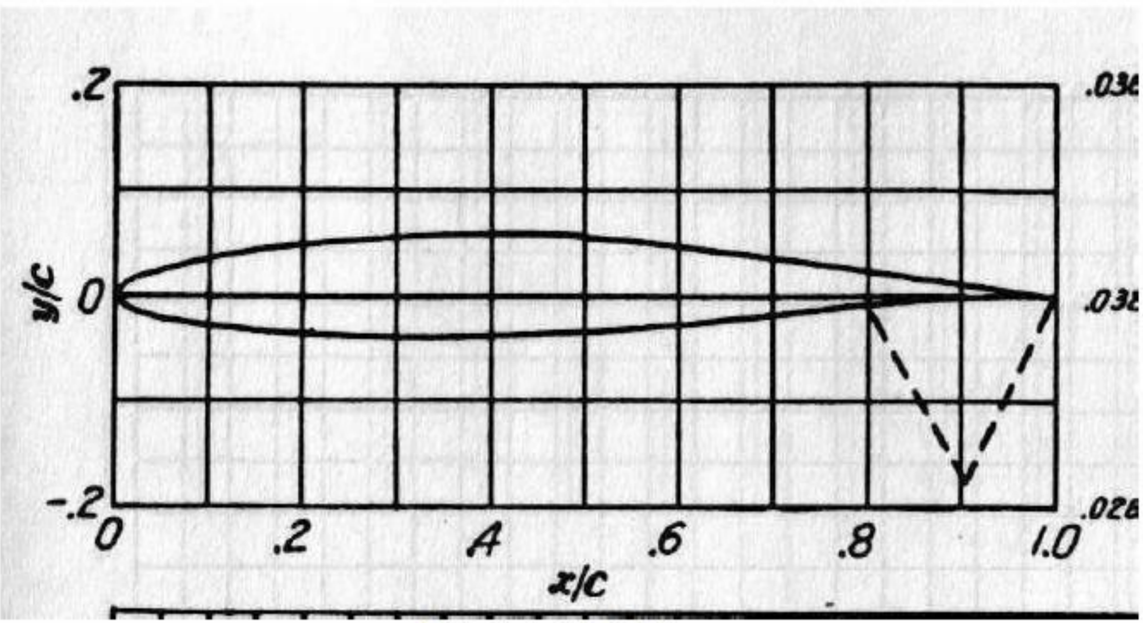
  - lift-slope larger $(a \rightarrow a_0)$

- *Practical significance of the elliptical wing:*
  - *<u>Optimum wing shape</u>: minimal induced drag for given lift*
  - *<u>Reference wing</u>: reasonable approximation for real wings*

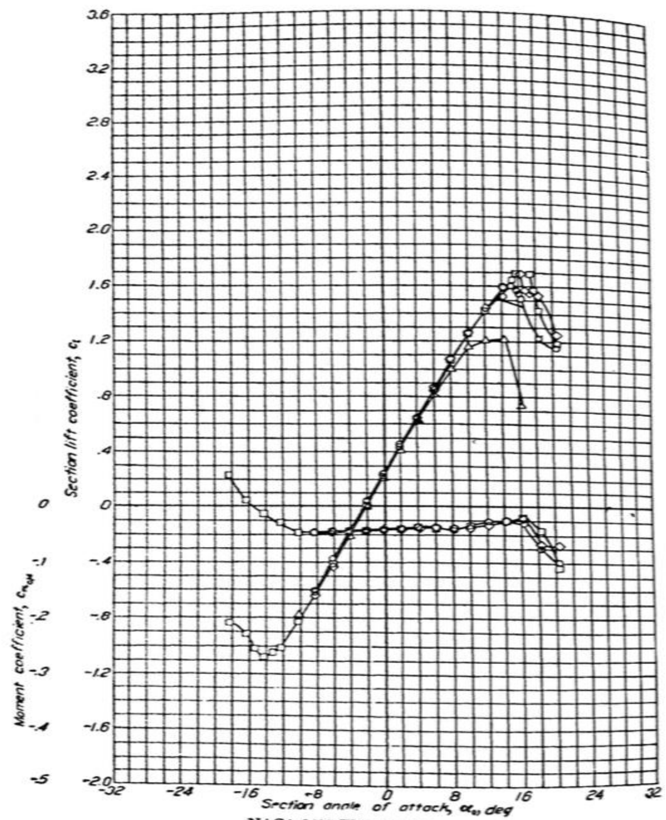
❑ PRANDTL'S LIFTING LINE THEORY

Example Problem: Consider an elliptic wing with 10m span and 2.5m root chord. If the wing is made up of NACA 64-210 wing sections and is flying 50m/s at a geometric angle of attack of 8 degrees, compute

1. C_L and C_{D_i}
2. L and D_i
3. The acceleration of this wing if it has a mass of 1 Mg at sea level.



NACA 64-210



NACA 2412 Wing Section

- *The chart shows that $\alpha_{L=0} \approx -1.8°$*

Prandtl's Lifting Line Theory

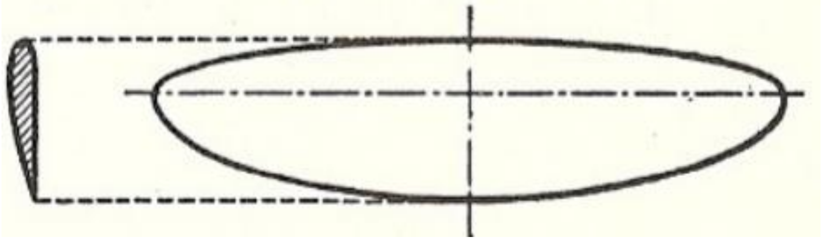
$$\Gamma_o = \frac{2\pi(50\,m/s)\left[8^o - (-1.8^o)\right]}{\left[\dfrac{2}{2.5m} + \dfrac{\pi}{10m}\right]} = 48.23\frac{m^2 \cdot rad}{\sec}$$

$$S = \pi\frac{b}{2}\frac{c_r}{2} = \frac{\pi(10m)(2.5m)}{4} = 19.63m^2$$

$$AR = b^2/S = (10m)^2/19.63m^2 = 5.09$$

$$C_L = \frac{b\pi\Gamma_o}{2V_\infty S} = \frac{\pi(10m)\left(48.23\dfrac{m^2 \cdot rad}{\sec}\right)}{2(50\,m/s)(19.63m^2)}$$

$$\boxed{\begin{gathered} C_L = 0.77 \\ C_{D_i} = \frac{C_L^2}{\pi AR} = \frac{(0.77)^2}{\pi(5.09)} = 0.037 \end{gathered}}$$



PRANDTL'S LIFTING LINE THEORY

<u>Lift and Drag calculations</u>

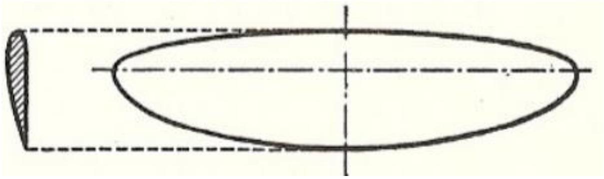
$$L = \frac{1}{2}\rho_\infty V_\infty^2 S C_L$$

$$L = \frac{1}{2}\left(1.225\,kg/m^3\right)\left(50\,m/s\right)^2\left(19.63m^2\right)(0.77) = 23.1kN$$

$$D_i = \frac{1}{2}\rho_\infty V_\infty^2 S C_{D_i}$$

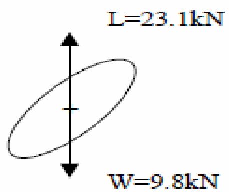
$$D_i = \frac{1}{2}\left(1.225\,kg/m^3\right)\left(50\,m/s\right)^2\left(19.63m^2\right)(0.037) = 1.1kN$$

- ***Since the generated lift is 23.1kN, the total weight of the airplane can be up to 2357 Kg***



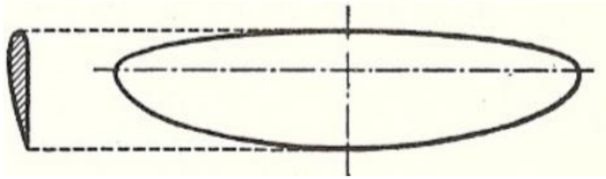
PRANDTL'S LIFTING LINE THEORY

Acceleration calculations



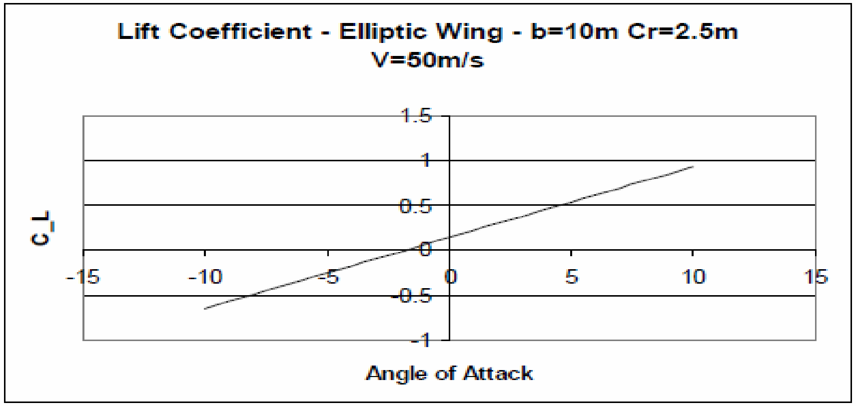
$$F_{net} = 23.1kN - 9.8kN = 13.3kN$$

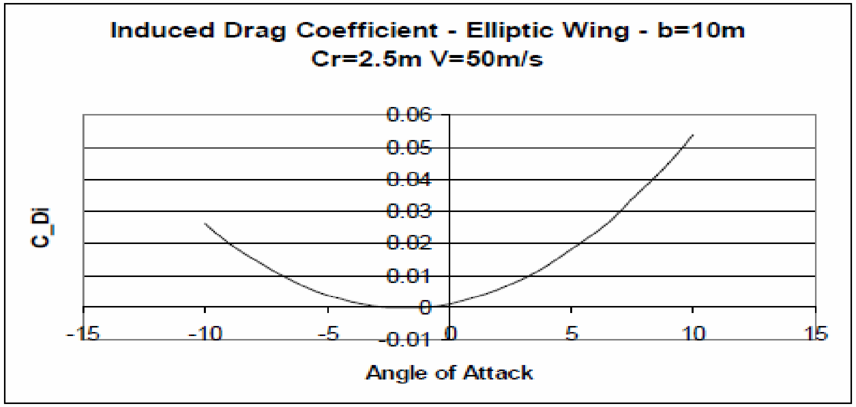
$$\vec{a} \approx 1.36g$$



Exercise: Develop C_L and C_{D_i} over the range of a.o.a from

$-10^o \Rightarrow +10^o$





Prandtl's Lifting Line Theory

Why aren't all wings designed to be elliptic?

1. At high angles of attack a wing with uniform cross section along the span and no twist will stall simultaneously all along the span causing sudden loss of aileron control. Increased chord at the wing tips helps maintain control authority.

2. Stall near the wing root is preferred and the wing can be twisted to reduce the angle of attack near the tips. This is called washout.

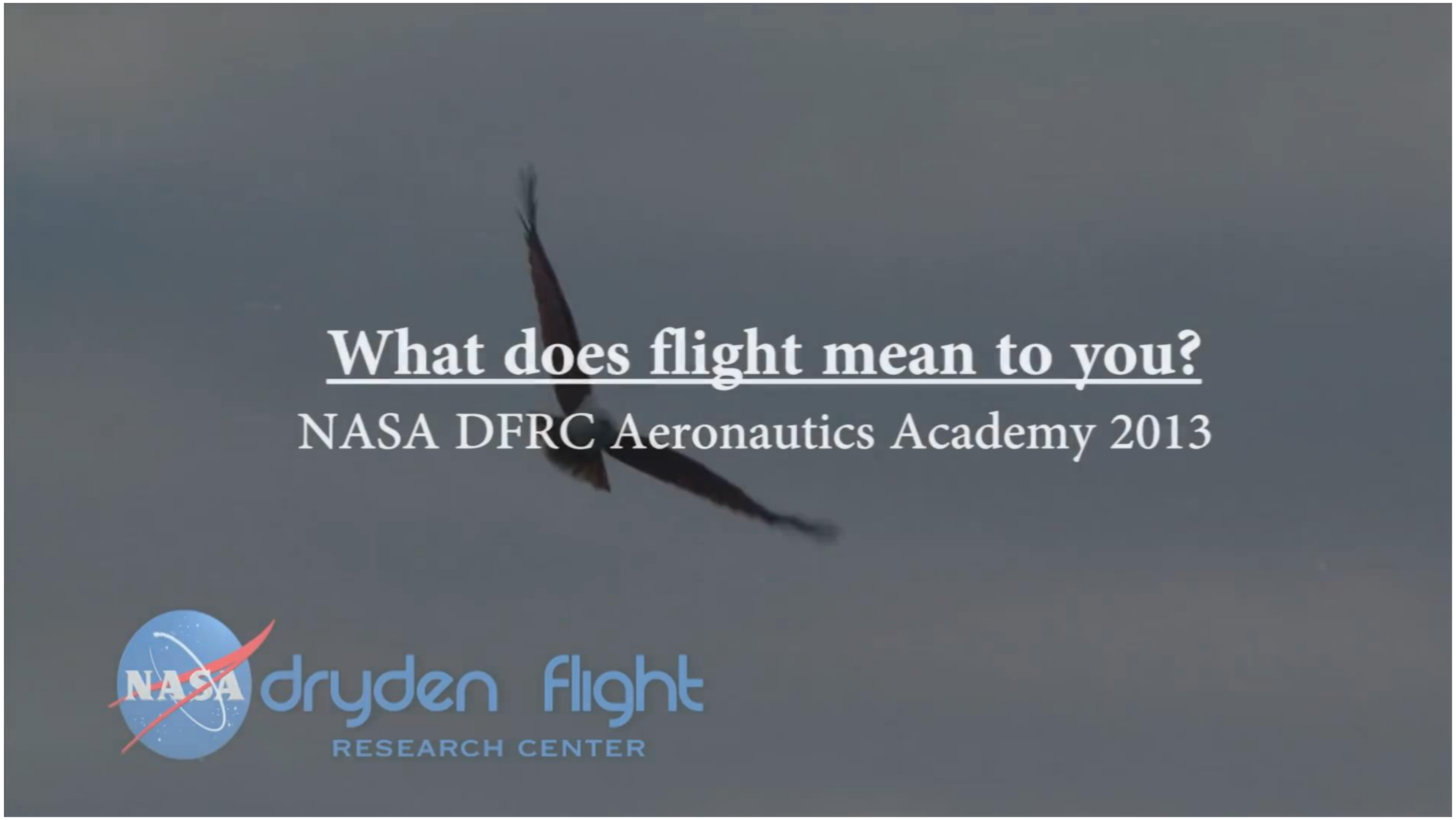
3. The induced drag penalty is relatively small even for relatively large deviations from an elliptical shape.

4. The compound curves involved in constructing an elliptic wing increase cost and complexity of manufacture.

Prandtl's Lifting Line Theory

Proving Prandtl- With A Twist!

- *A group of college aerospace engineering students in the 2012-2013 Aeronautics Academy at NASA's Dryden Flight Research Center have proven German aerodynamicist Ludwig Prandtl's theory on how to overcome one of the thorny problems of flight.*



https://www.youtube.com/watch?v=Hr0I6wBFGpY&t=135s