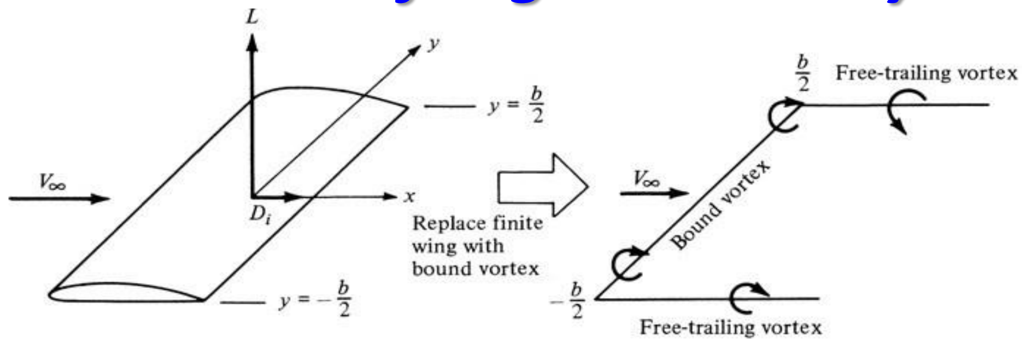
Lecture # 34: 3D Wing Aerodynamics: Lifting Line Theory – Part #3

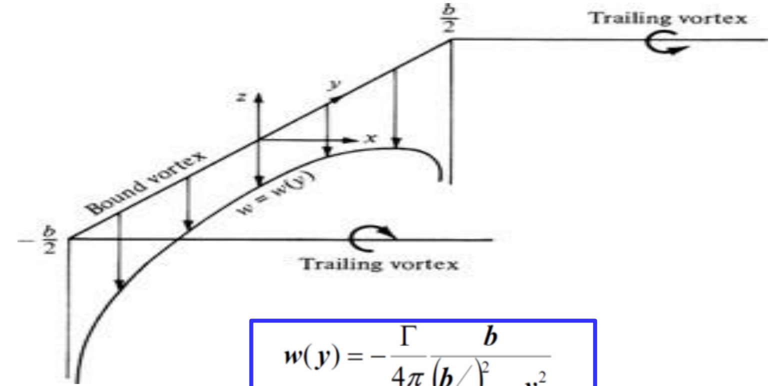
Dr. Hui HU
Department of Aerospace Engineering
Iowa State University, 2251 Howe Hall, Ames, IA 50011-2271
Tel: 515-294-0094 / Email: huhui@iastate.edu

3D Wing Aerodynamics

Prandtl's Lifting Line Theory

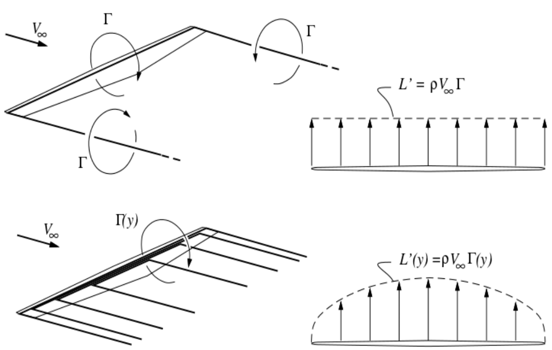


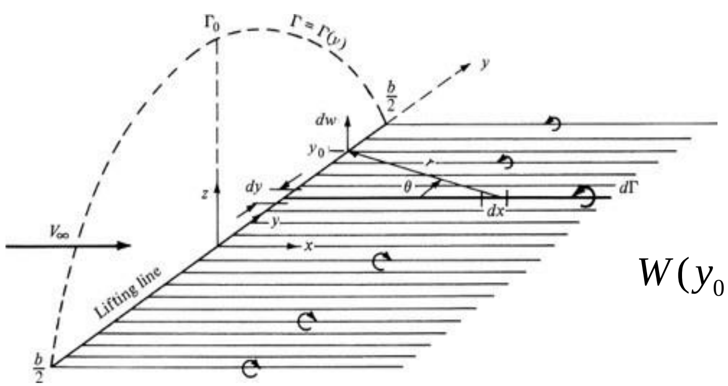
Finite wing Horseshoe vortex



$$w(y) = -\frac{\Gamma}{4\pi}\frac{b}{\left(b/2\right)^2 - y^2}$$

- ***However, the single vortex filament case is not sufficient to describe the physical conditions on the wing since the downwash at the wing tips is infinite, instead of zero!***





$$dw = -\frac{(d\Gamma/dy)\bullet dy}{4\pi(y_0 - y)}$$

$$W(y_0) = \int_{-b}^{b} dw = -\int_{-b}^{b} \frac{(d\Gamma/dy)}{4\pi(y_0 - y)} dy$$

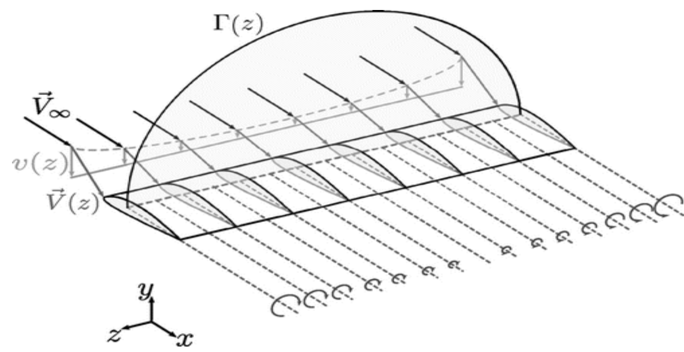
Fundamental equation for Prandtl's Lifting Line Theory:

$$\alpha(y_0) = \frac{\Gamma(y_0)}{\pi V_\infty c(y_0)} + \alpha_{L=0}(y_0) + \frac{1}{4\pi V_\infty}\int_{-b}^{b} \frac{(d\Gamma/dy)}{(y_0 - y)} dy$$

❑ Prandtl's Lifting Line Theory

❑ ***Fundamental equation for Prandtl's Lifting Line Theory can be expressed as:***

$$\alpha(y_0) = \frac{\Gamma(y_0)}{\pi V_\infty c(y_0)} + \alpha_{L=0}(y_0) + \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2} \frac{(d\Gamma/dy)}{(y_0 - y)} dy$$



- ***Once $\Gamma(y_0)$ is known:***

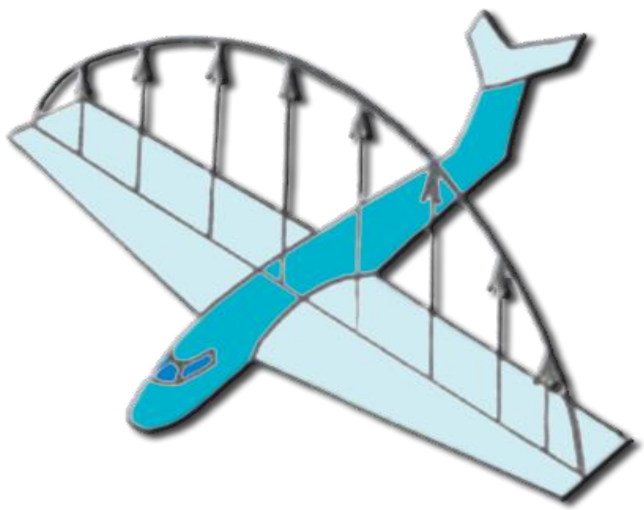
1. ***Lift distribution:*** $L'(y) = \rho V_\infty \Gamma(y)$

2. ***Total lift:*** $L = \int_{-b/2}^{b/2} L' dy = \rho V_\infty \int_{-b/2}^{b/2} \Gamma(y) dy$ $\quad C_L = \frac{L}{q_\infty S} = \frac{2}{V_\infty S}\int_{-b/2}^{b/2} \Gamma(y) dy$

3. ***Induced angle of attack:*** $\alpha_i(y_0) = \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2} \frac{(d\Gamma/dy)}{(y_0 - y)} dy$

4. ***Induced drag:*** $D_i = \int_{-b/2}^{b/2} D_i' dy = \int_{-b/2}^{b/2} L' \alpha_i \, dy = \rho V_\infty \int_{-b/2}^{b/2} \Gamma(y)\, \alpha_i(y) dy$

$$C_{D_i} = \frac{D_i}{q_\infty S} = \frac{2}{V_\infty S}\int_{-b/2}^{b/2} \Gamma(y)\, \alpha_i(y) dy$$

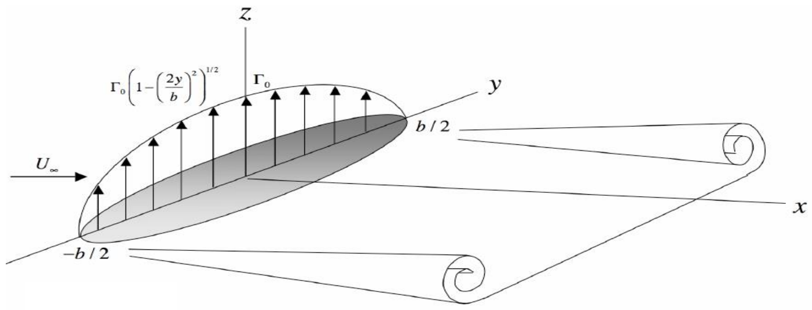


❑ PRANDTL'S LIFTING LINE THEORY

❑ *Consider a case with elliptical lift distribution*

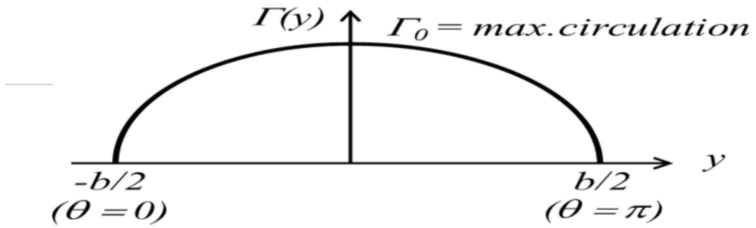
- ***Consider the following "elliptical" lift distribution:***

$$\Gamma(y) = \Gamma_0\sqrt{1-\left(\frac{y}{b/2}\right)^2}$$



- ***Compute the downwash velocity from:***

$$w(y_0) = -\frac{1}{4\pi}\int_{-b/2}^{b/2}\frac{d\Gamma/dy}{y_0-y}dy$$



- ***Induced downwash velocity:***

$$w(y) = -\frac{\Gamma_0}{2b}$$

- ***Induced AOA:***

$$\alpha_i = \frac{\Gamma_0}{2bV_\infty} = \frac{C_L}{\pi\,(b^2/S)} = \frac{C_L}{\pi\,AR}$$

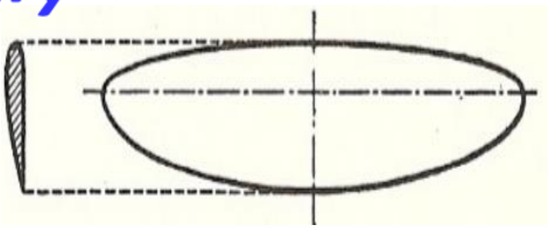
- ***Lift coefficient***

$$L = \rho V_\infty \Gamma_0 \frac{b\pi}{4} \quad\Rightarrow\quad C_L = \frac{L}{0.5\rho V_\infty^{\,2} S} = \frac{\pi\Gamma_0 b}{2V_\infty S}$$

- ***The Supermarine Spitfire***

❑ Prandtl's Lifting Line Theory

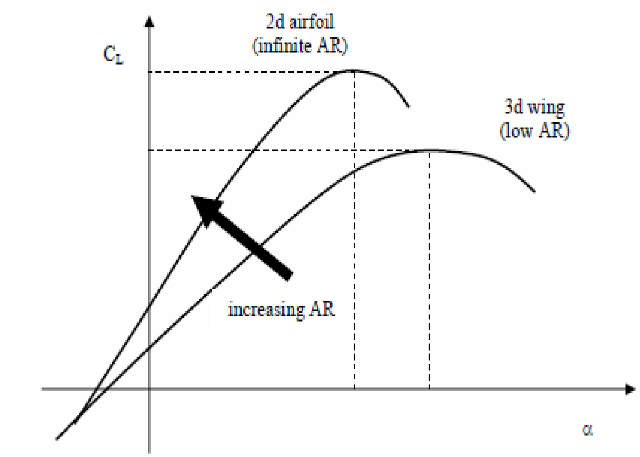
The elliptical lift distribution - summary



- ***Constant downwash along the span:*** $$\alpha_i = \frac{C_L}{\pi AR}$$

- ***Induced drag:*** $$C_{D_i} = \alpha_i C_L = \frac{C_L^2}{\pi AR}$$

- ***Lift slope:*** $$a = \frac{dC_L}{d\alpha} = \frac{a_0}{1 + a_0/\pi AR}$$



- ***Effect of increasing the wing aspect ratio:*** - *induced drag smaller*

  - *lift-slope larger* $(a \rightarrow a_0)$

- *Practical significance of the elliptical wing:*
  - *Optimum wing shape: minimal induced drag for given lift*
  - *Reference wing: reasonable approximation for real wings*

PRANDTL'S LIFTING LINE THEORY

Consider the case with elliptical lift distribution

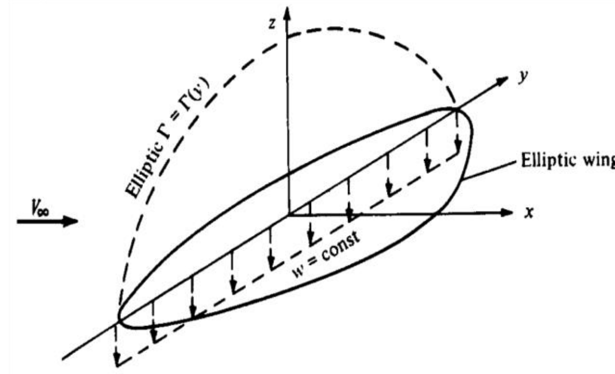
$$\Gamma(y) = \Gamma_o \sqrt{1 - \left(\frac{2y}{b}\right)^2}$$

$$y = \frac{b}{2}\cos\theta, \qquad dy = -\frac{b}{2}\sin\theta d\theta$$

When combined, Eq. (5.28) becomes

$$\Gamma(y) = \Gamma_o \sqrt{1 - \cos^2\theta}$$

$$\Gamma(\theta) = \Gamma_0 \sin\theta$$



Consider the case with a general lift distribution

$$\Gamma(\theta) = 2bV_\infty \sum_{n=1}^{N} A_n \sin(n\theta); \quad n = 1 \dots N$$

$$\Rightarrow \frac{d\Gamma}{dy} = \frac{d\Gamma}{d\theta}\frac{d\theta}{dy} = 2bV_\infty \sum_{n=1}^{N} nA_n \cos(n\theta)\frac{d\theta}{dy}$$

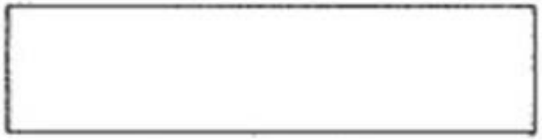
Fundamental equation for Prandtl's Lifting Line Theory:

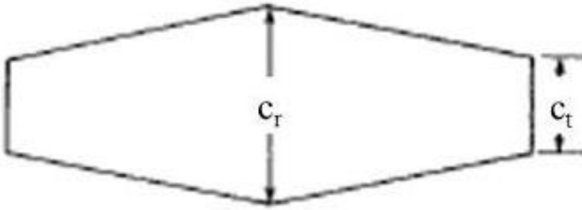
$$\alpha(y_0) = \frac{\Gamma(y_0)}{\pi V_\infty c(y_0)} + \alpha_{L=0}(y_0) + \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2} \frac{(d\Gamma/dy)}{(y_0 - y)} dy$$

$$\Rightarrow \alpha(\theta_0) = \frac{2b}{\pi c(\theta_0)} \sum_{n=1}^{N} A_n \sin(n\theta_0) + \alpha_{L=0}(\theta_0) + \frac{1}{\pi}\int_0^\pi \frac{\sum_{n=1}^{N} nA_n \cos(n\theta)}{\cos\theta - \cos\theta_0} d\theta$$

$$\Rightarrow \alpha(\theta_0) = \frac{2b}{\pi c(\theta_0)} \sum_{n=1}^{N} A_n \sin(n\theta_0) + \alpha_{L=0}(\theta_0) + \sum_{n=1}^{N} nA_n \frac{\sin(n\theta_0)}{\sin\theta}$$


Elliptic Wing


Rectangular Wing



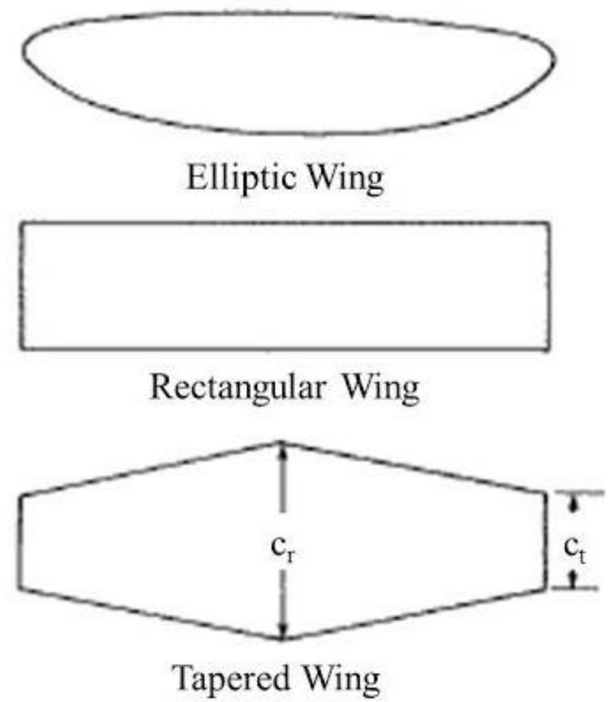
Tapered Wing

PRANDTL'S LIFTING LINE THEORY

Consider the case with a general lift distribution

$$\Gamma(\theta) = 2bV_\infty \sum_{n=1}^{N} A_n \sin(n\theta); \quad n = 1 \ldots N$$

$$\alpha(\theta_0) = \frac{2b}{\pi c(\theta_0)} \sum_{n=1}^{N} A_n \sin(n\theta_0) + \alpha_{L=0}(\theta_0) + \sum_{n=1}^{N} nA_n \frac{\sin(n\theta_0)}{\sin\theta_0}$$



Lift coefficient of the general wing:

$$C_L = \frac{L}{0.5\rho V_\infty{}^2 S} = \frac{\rho V_\infty}{0.5\rho V_\infty{}^2 S} \int_{-b/2}^{+b/2} \Gamma(y)dy$$

$$= \frac{2}{V_\infty S} \int_{-b/2}^{+b/2} \Gamma(y)dy\Gamma(\theta) = \frac{2b^2}{S} \sum_{n=1}^{N} A_n \int_0^{\pi} \sin(n\theta)\sin\theta d\theta$$

$$\because \int_0^{\pi} \sin(n\theta)\sin\theta d\theta = \begin{cases} \pi/2 & for\ n = 1 \\ 0 & for\ n \neq 1 \end{cases}$$

$$\therefore C_L = \frac{2b^2}{S}\frac{A_1\pi}{2} = A_1\pi AR$$

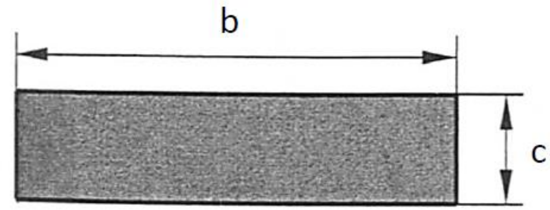
$$C_L = A_1 \pi AR$$

❑ Prandtl's Lifting Line Theory

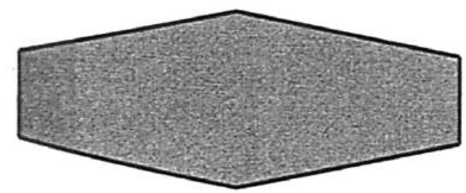
❑ Consider the case with a general lift distribution

$$\Gamma(\theta) = 2bV_\infty \sum_{n=1}^{N} A_n \sin(n\theta); \quad n = 1 \ldots N$$

$$\alpha(\theta_0) = \frac{2b}{\pi c(\theta_0)} \sum_{n=1}^{N} A_n \sin(n\theta_0) + \alpha_{L=0}(\theta_0) + \sum_{n=1}^{N} nA_n \frac{\sin(n\theta_0)}{\sin\theta_0}$$



rectangular wing



trapezoidal wing

❑ Induced drag coefficient of the general wing:

$$C_{D,i} = \frac{2}{V_\infty S} \int_{-b/2}^{+b/2} \Gamma(y)\alpha_i(y)dy = \frac{2b^2}{S} \int_0^\pi \left(\sum_{n=1}^{N} A_n \sin(n\theta) \right) \alpha_i(\theta) \sin\theta d\theta$$

$$\because \alpha_i(\theta_0) = \frac{1}{4\pi V_\infty} \int_{-b/2}^{+b/2} \frac{d\Gamma(y)/dy}{y_o - y} dy = \frac{1}{\pi} \sum_{n=1}^{N} A_n \int_0^\pi \frac{\cos(n\theta)}{\cos\theta - \cos\theta_0} d\theta = \sum_{n=1}^{N} nA_n \frac{\sin n\theta_0}{\sin\theta_0}$$

$$\therefore C_{D,i} = \frac{2b^2}{S} \int_0^\pi \left(\sum_{n=1}^{N} A_n \sin(n\theta) \right) \left(\sum_{n=1}^{N} A_n \sin(n\theta) \right) d\theta$$

$$\because \int_0^\pi \sin(m\theta)\sin(n\theta)d\theta = \begin{cases} 0 & for\ m = n \\ \frac{\pi}{2} & for\ m \neq n \end{cases}$$

$$\therefore C_{D,i} = \frac{2b^2}{S} \left(\sum_{n=1}^{N} nA_n^2 \right) \frac{\pi}{2} = \pi AR \left(\sum_{n=1}^{N} nA_n^2 \right)$$

$$= \pi AR \left(A_1^2 + \sum_{n=2}^{N} nA_n^2 \right) = \pi AR \cdot A_1^2 \left[1 + \sum_{n=2}^{N} n \left(\frac{A_n}{A_1} \right)^2 \right]$$

$$C_{D,i} = \pi AR \cdot A_1^2 \left[1 + \sum_{n=2}^{N} n \left(\frac{A_n}{A_1} \right)^2 \right]$$

$$By\ defining: \quad \delta = \sum_{n=2}^{N} n \left(\frac{A_n}{A_1} \right)^2 \ as\ the\ induced\ drag\ factor$$

$$\Rightarrow C_{D,i} = \pi AR \cdot A_1^2 [1 + \delta]$$

$$C_L = A_1 \pi AR \Rightarrow C_{D,i} = \frac{C_L^2}{\pi AR}[1 + \delta]$$

$$By\ defining: \quad e = \frac{1}{(1+\delta)}\ as\ the\ span\ effciency\ factor$$

$$\Rightarrow C_{D,i} = \frac{C_L^2}{\pi e AR}$$

PRANDTL'S LIFTING LINE THEORY

Consider the case with a general lift distribution

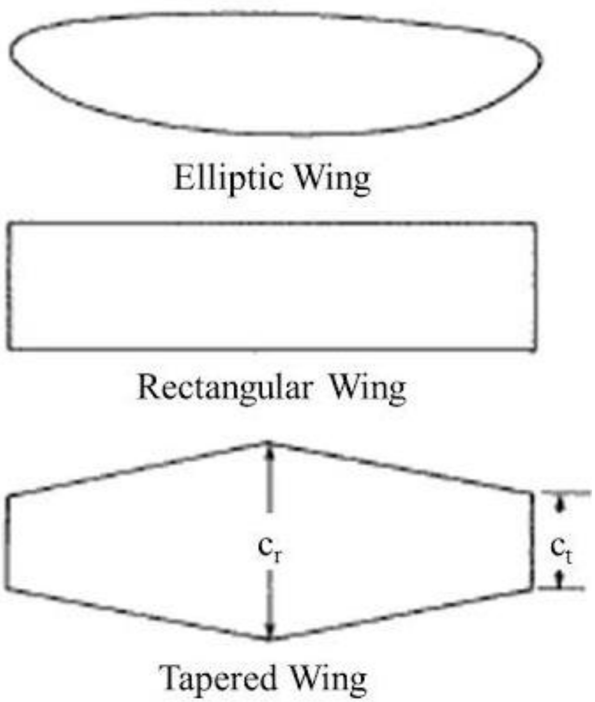
General lift distribution: summary and conclusions

$$\boxed{C_L = A_1 \pi \cdot AR}$$

$$C_{Di} = \pi AR \sum_{n=1}^{N} n A_n^{\,2} = \pi AR\, A_1^2 \left[1 + \sum_{n=2}^{N} n \left(\frac{A_n}{A_1} \right)^2 \right]$$

$$\boxed{C_{D_i} = \frac{C_L^{\,2}}{\pi AR}(1+\delta) \quad \text{where} \quad \delta = \sum_{n=2}^{N} n \left(\frac{A_n}{A_1} \right)^2 \geq 0}$$

$$\text{or: } \; C_{D_i} = \frac{C_L^2}{\pi \cdot eAR} \quad \text{where} \quad e = \frac{1}{(1+\delta)} \leq 1 \quad \text{the "span efficiency factor"}$$



Conclusion:

- *The elliptic wing (δ = 0, e = 1) gives the lowest possible induced drag (for given lift and aspect ratio).*

PRANDTL'S LIFTING LINE THEORY

Why aren't all wings designed to be elliptic?

1. At high angles of attack a wing with uniform cross section along the span and no twist will stall simultaneously all along the span causing sudden loss of aileron control. Increased chord at the wing tips helps maintain control authority.

2. Stall near the wing root is preferred and the wing can be twisted to reduce the angle of attack near the tips. This is called washout.

3. The induced drag penalty is relatively small even for relatively large deviations from an elliptical shape.

4. The compound curves involved in constructing an elliptic wing increase cost and complexity of manufacture.

❑ PRANDTL'S LIFTING LINE THEORY

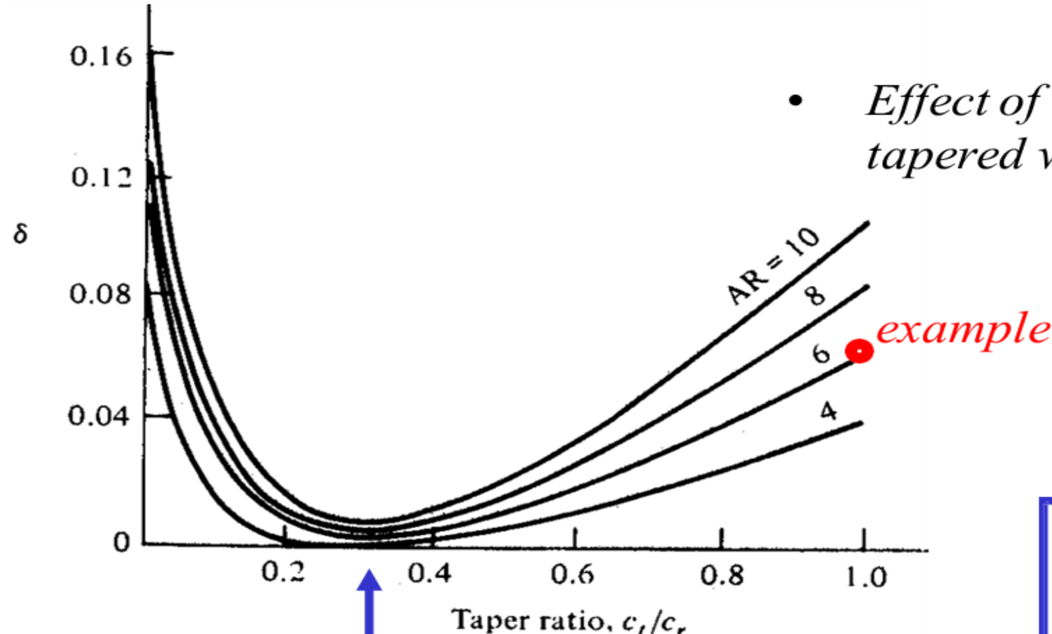
❑ Consider the case with a general lift distribution

Effect of wing planform and aspect ratio

$$C_{D_i} = \frac{C_L^{\ 2}}{\pi A}(1+\delta)$$

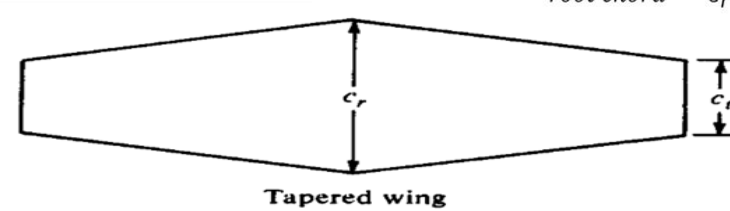
$$a = \frac{a_0}{1+(a_0/\pi A)(1+\delta)}$$

- *Values of δ depend on **planform** and **aspect ratio** of the wing*



- *Effect of wing **planform** on δ for a tapered wing*

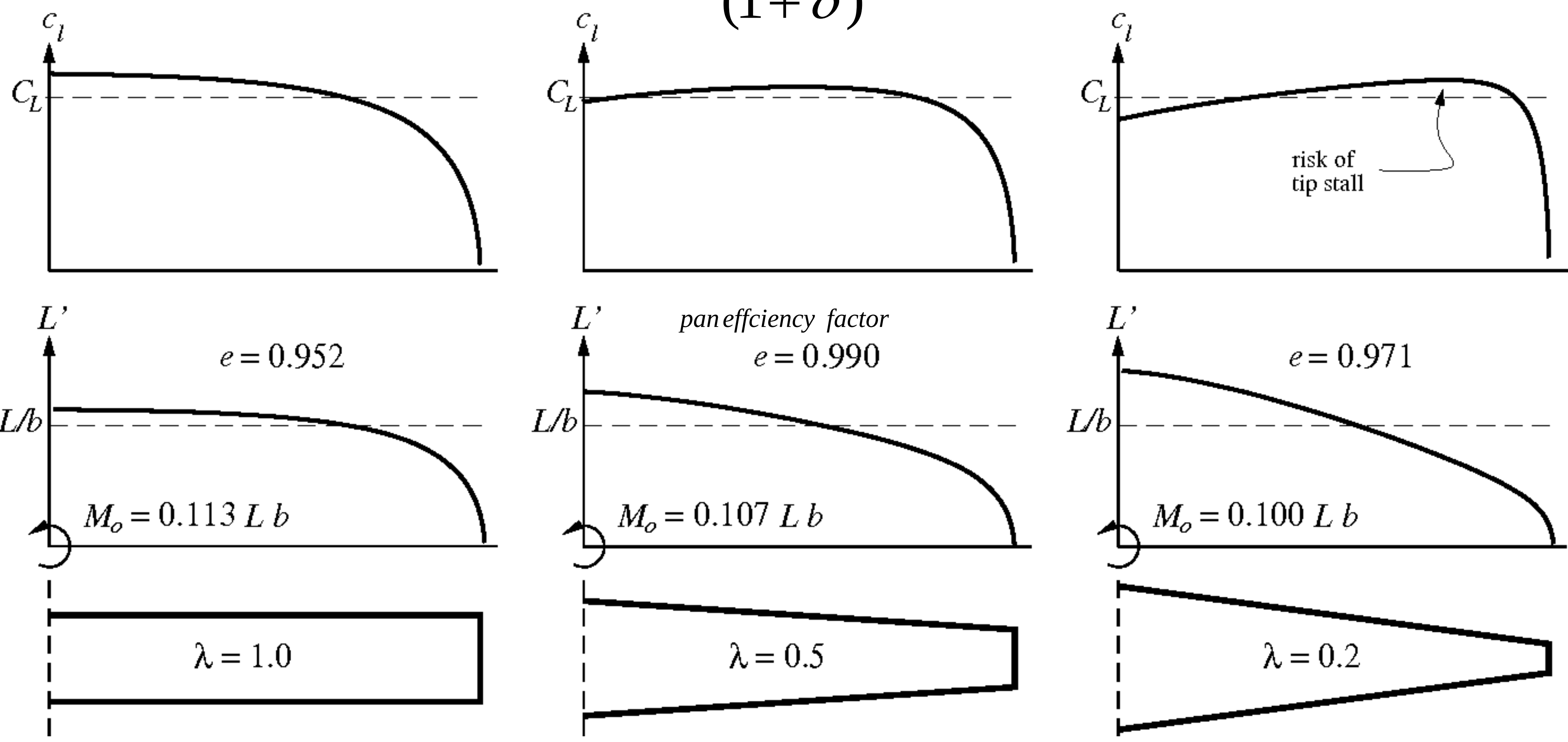
Taper Ratio $\lambda = \frac{\text{tip chord}}{\text{root chord}} = \frac{C_t}{C_r}$



A tapered wing with taper ratio $c_t/c_r = 0.3$ is almost as good as an elliptical wing!

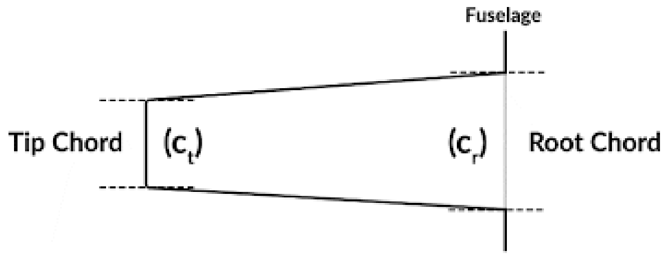
Prandtl's Lifting Line Theory

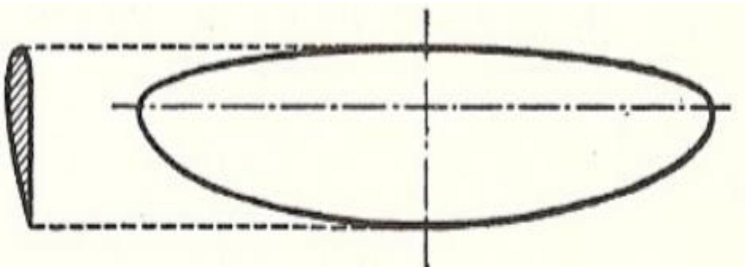
- *Span efficient factor:* $$e = \frac{1}{(1+\delta)}$$



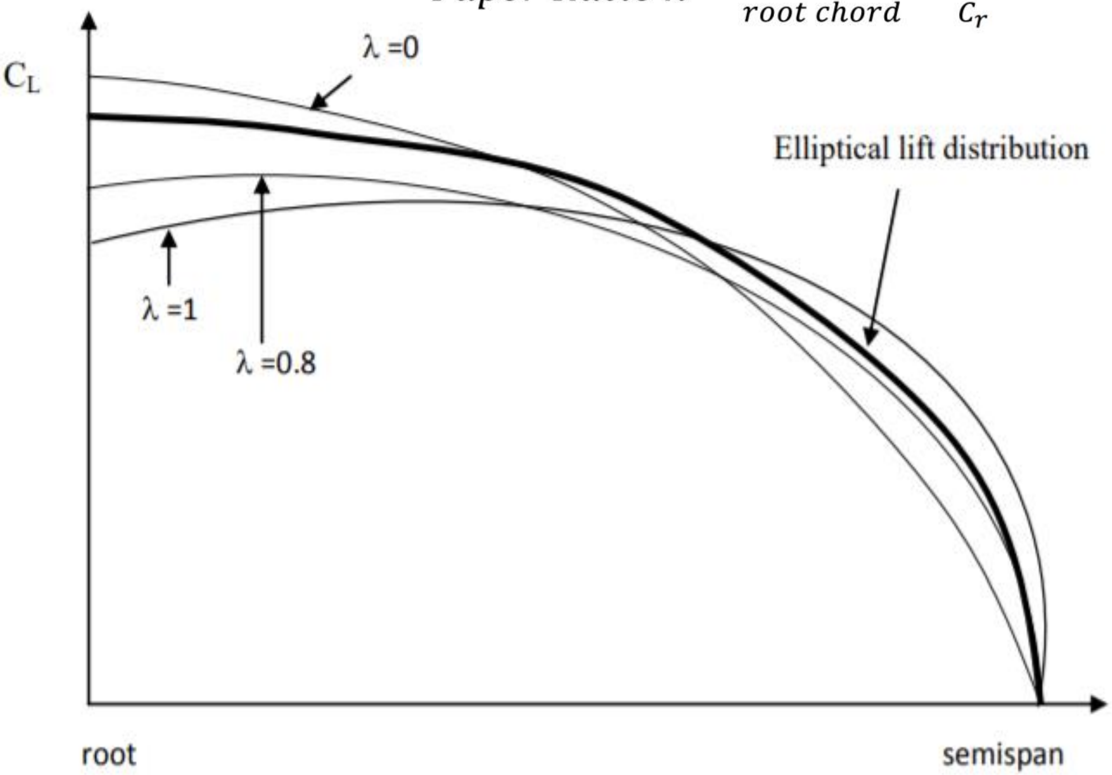
- *Load distributions, root bending moment, and span efficiency for three taper ratios. All three cases have AR= 10, and no wing twist.*

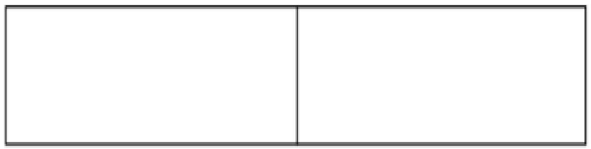
❑ Prandtl's Lifting Line Theory



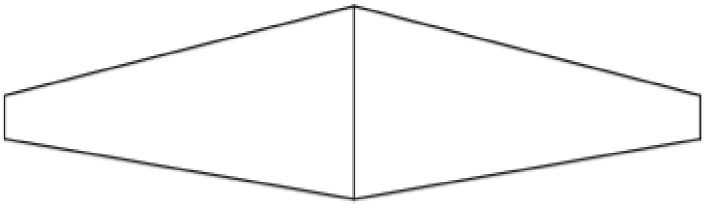


$$Taper\ Ratio\ \lambda = \frac{tip\ chord}{root\ chord} = \frac{C_t}{C_r}$$

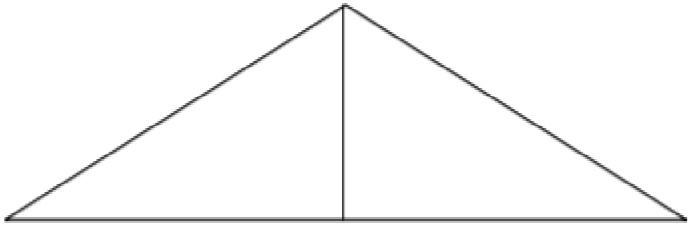




a. Rectangle (λ =1)



b. Trapezoid $0 < \lambda < 1$ (straight tapered)



c. Triangle (delta) $\lambda = 0$

- *The typical effect of taper ratio on the lift distribution*
- *Wings with various taper ratio*

❑ PRANDTL'S LIFTING LINE THEORY

❑ *Consider the case with a general lift distribution*

Final conclusions
the effect of wing planform on the induced drag

$$C_{D_i} = \frac{C_L{}^2}{\pi AR}(1+\delta)$$

- *In order to reduce the induced drag it is more important to increase the aspect ratio AR than trying to approach the elliptic lift distribution accurately*
- *A tapered wing with taper ratio of $c_t/c_r = 0.3$ is almost as good as an elliptical wing and is much easier to manufacture*
- *Note that the induced drag factor δ is a constant (i.e., independent of α) only for a wing without twist!*
- *Remember: total drag = induced drag + profile drag (~ viscosity)*

❑ PRANDTL'S LIFTING LINE THEORY

❑ **Consider the case with elliptical lift distribution**

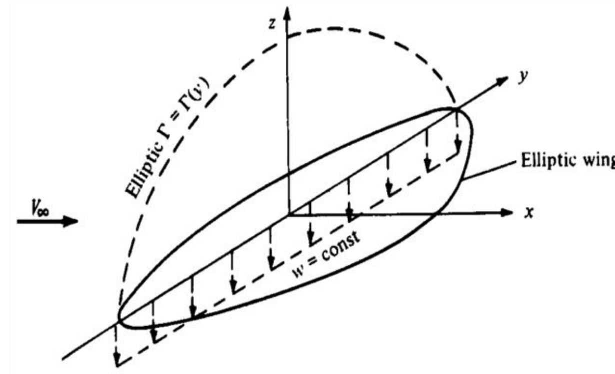
$$\Gamma(y) = \Gamma_o \sqrt{1 - \left(\frac{2y}{b}\right)^2}$$

$$y = \frac{b}{2}\cos\theta, \qquad dy = -\frac{b}{2}\sin\theta d\theta$$

When combined, Eq. (5.28) becomes

$$\Gamma(y) = \Gamma_o \sqrt{1 - \cos^2\theta}$$

$$\Gamma(\theta) = \Gamma_0 \sin\theta$$



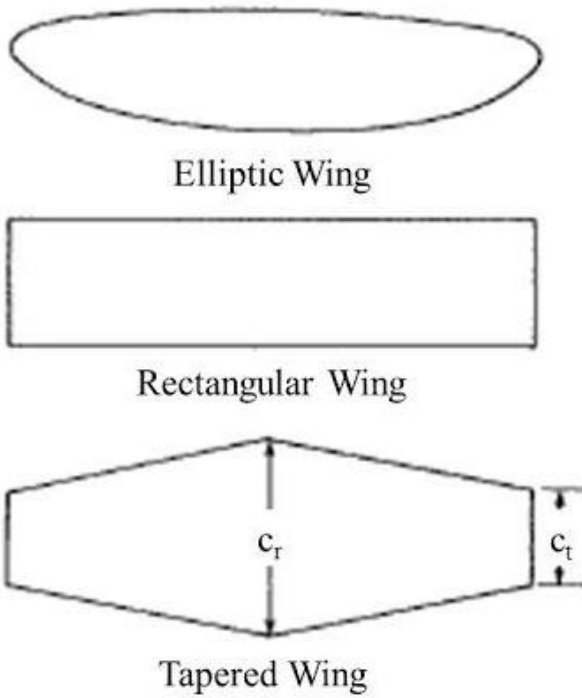
❑ **Consider the case with a general lift distribution**

$$\Gamma(\theta) = 2bV_\infty \sum_{n=1}^{N} A_n \sin(n\theta); \quad n = 1 \ldots N$$

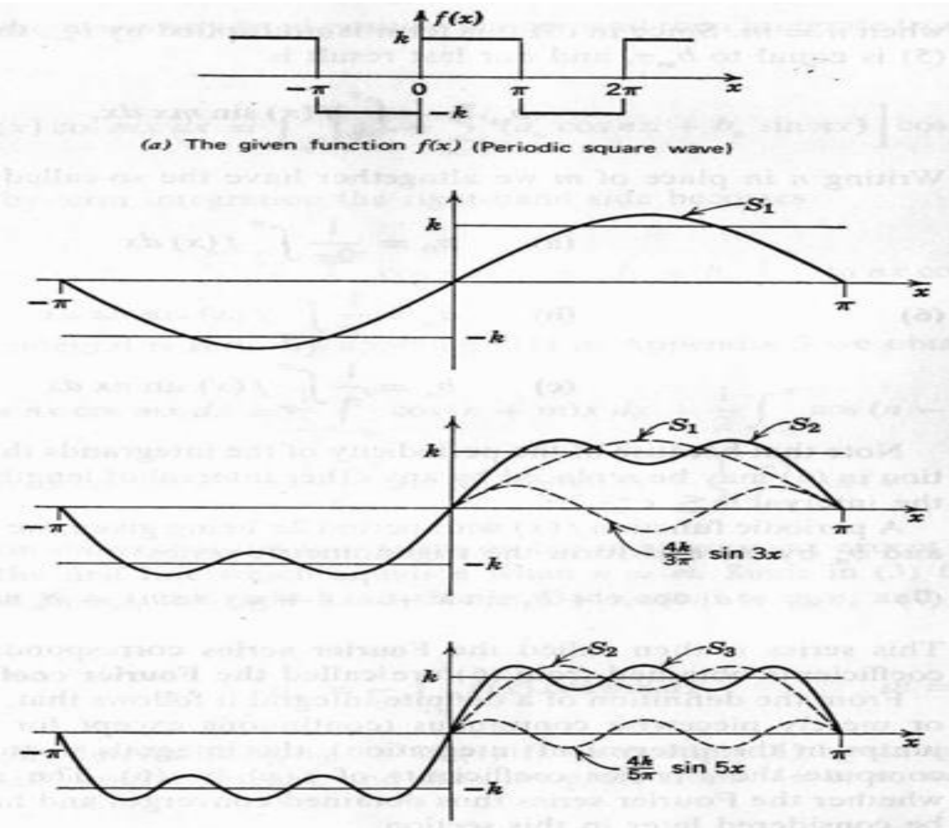
$$\alpha_i(\theta_0) = \sum_{n=1}^{N} nA_n \frac{\sin(n\theta_0)}{\sin\theta_0}$$

$$C_L = A_1 \pi AR$$

$$C_{D,i} = \pi AR \cdot A_1^2 \left[1 + \sum_{n=2}^{N} n\left(\frac{A_n}{A_1}\right)^2\right]$$



❑ Fourier Sine Series to represent Rectangular & Triangle functions



(a) The given function f(x) (Periodic square wave)

(b) The first three partial sums of the corresponding Fourier series

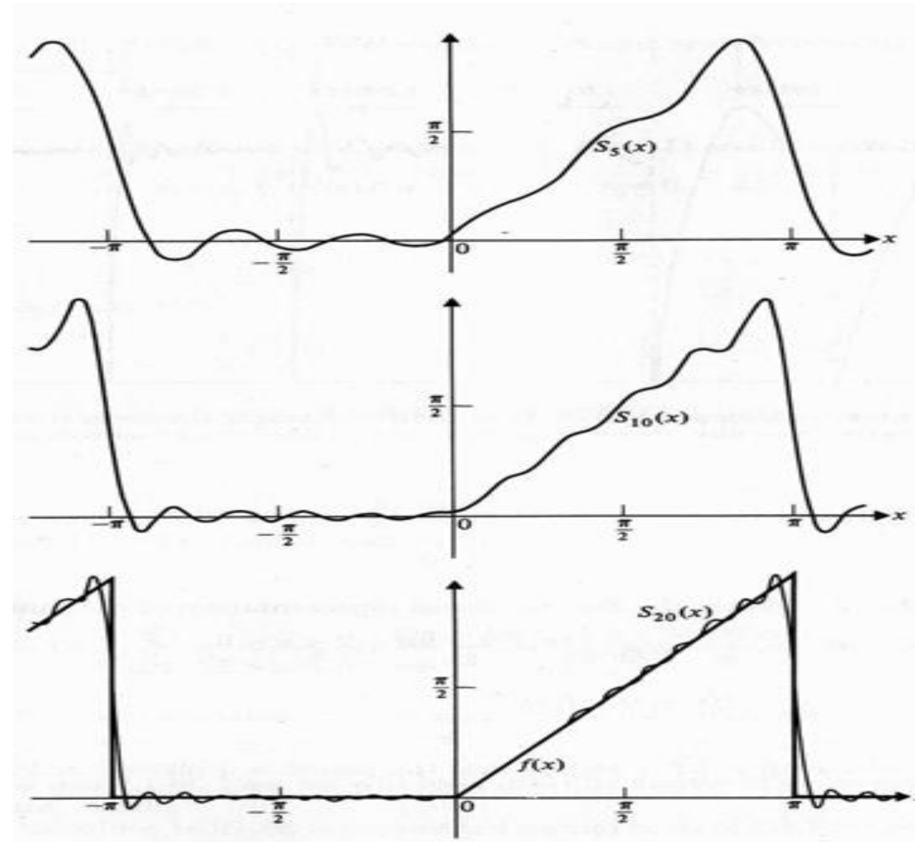


FIGURE 7.1–8 The graph of Eq. (7.1–20) for $N = 5, 10, 20$.

- *It was found that only 4 terms were needed to get about 95% of the total energy to represent the wakes behind a turbine stator*
- *The approximation functions are generally good representations of the actual functions if the actual function is smooth.*
- *Fortunately, wing circulation distributions are usually quite smooth and require relatively few terms.*

❑ Prandtl's Lifting Line Theory

❑ Consider the case with a general lift distribution

The relation between the A_n and the wing geometry

Solve Prandtl's wing equation:

$$\frac{c_l}{a_0} = \frac{2\Gamma}{a_0 V_\infty c} = \alpha - \alpha_{L=0} - \alpha_i$$

- substitute:

$$\Gamma(\theta) = 2bV_\infty \sum_{n=1}^{N} A_n \sin n\theta \qquad \alpha_i(\theta) = \sum_{n=1}^{N} nA_n \frac{\sin n\theta}{\sin\theta}$$

$$\boxed{\frac{4b}{a_0 c}\sum_{n=1}^{N} A_n \sin n\theta + \sum_{n=1}^{N} nA_n \frac{\sin n\theta}{\sin\theta} = \alpha - \alpha_{L=0}}$$

Numerical solution method:

- *Take a truncated series with N unknown coefficients: $A_1, A_2, \ldots A_N$*
- *Take N different spanwise locations on the wing where the equation is to be satisfied: θ_1, θ_2, .. θ_N; (but not at the tips, so: $0 < \theta_1 < \pi$)*
- *System of N equations with N unknowns (Solve $N \times N$ matix)*
- *Note: it is not possible to solve for only one coefficient, as for the thin airfoil theory of 2D airfoils.*

❑ PRANDTL'S LIFTING LINE THEORY

❑ *Consider the case with a general lift distribution*

Numerical example of the wing equation (1)

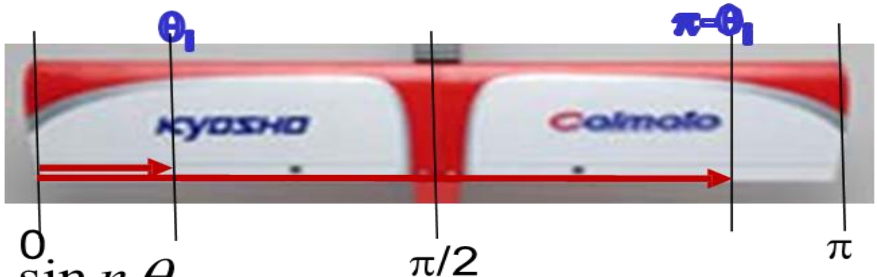
- ***Consider:*** rectangular wing: $c = \text{constant}$; $\text{span} = b$; $b/c = AR$;
  without twist: $\alpha = \text{constant}$; $\alpha_{L=0} = 0$
- ***Evaluate the wing equation at the N control points at θ_i:***

$$\sum_{n=1}^{N}\left(\frac{4A}{a_0} + \frac{n}{\sin\theta_i}\right)A_n \sin n\theta_i = \alpha \qquad i = 1, 2, \dots N$$

- ***The wing is symmetrical*** $\rightarrow$ $A_2, A_4, \dots$ are zero

$$\sin\theta_i = \sin(\pi - \theta_i)$$

$$A_n \sin n\theta_i = A_n \sin n(\pi - \theta_i)$$



If n is even :

$$A_n \sin n\theta_i = A_n \sin(n\pi - n\theta_i) = -A_n \sin n\,\theta_i$$

$$A_n = 0 \text{ for n is even number}$$

If n is odd :

$$A_n \sin n\theta_i = A_n \sin(n\pi - n\theta_i) = A_n \sin(\pi - n\theta_i) = A_n \sin n\theta_i$$

$$A_n \neq 0 \text{ for n is odd number}$$

Prandtl's Lifting Line Theory

Consider the case with a general lift distribution

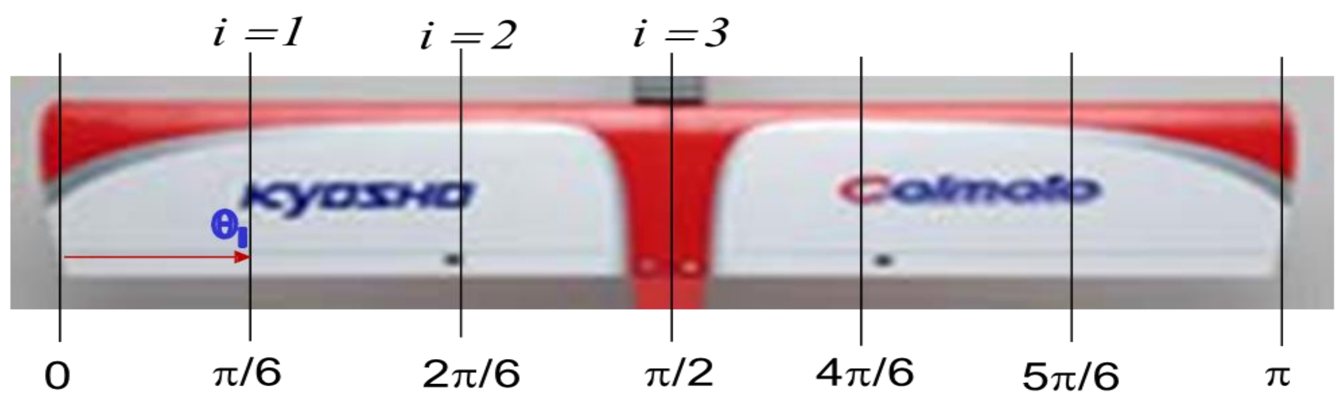
Numerical example of the wing equation (2)

- *Evaluate the wing equation at the N control points at θ_i:*

$$\sum_{n=1}^{N}\left(\frac{4A}{a_0}+\frac{n}{\sin\theta_i}\right)A_n \sin n\theta_i \quad = \quad \alpha \qquad i = 1, 2, \dots N$$

- *The wing is symmetrical* $\rightarrow$ *A_2, A_4, … are zero*
  - *take only A_1, A_3, … as unknowns*
  - *take only control points on half of the wing: $0 < \theta_i \leq \pi/2$*

- <u>*Example for N=3:*</u>
  - *take A_1, A_3, A_5 as unknowns*
  - *take control points (equidistant in θ): $\theta_1 = \pi/6$, $\theta_2 = \pi/3$, $\theta_3 = \pi/2$*
  - *take lift-slope of the airfoils $a_0 = 2\pi$, and wing aspect ratio $AR = 2\pi$*



PRANDTL'S LIFTING LINE THEORY

Consider the case with a general lift distribution

Numerical example of the wing equation (3)

$$\sum_{n=1}^{N}\left(\frac{4A}{a_0}+\frac{n}{\sin\theta_i}\right)A_n\sin n\theta_i = \alpha$$

$i = 1, 2, \ldots N$

- *A=AR in the equation*

– $i=1,\ \theta_1=\pi/6,$

$$\left(\frac{4A}{a_0}+\frac{1}{\sin(\pi/6)}\right)A_1\sin(\pi/6)+\left(\frac{4A}{a_0}+\frac{3}{\sin(\pi/6)}\right)A_3\sin(3\pi/6)+\left(\frac{4A}{a_0}+\frac{5}{\sin(\pi/6)}\right)A_5\sin(5\pi/6)=\alpha$$

– $i=2,\ \theta_2=\pi/3,$

$$\left(\frac{4A}{a_0}+\frac{1}{\sin(\pi/3)}\right)A_1\sin(\pi/3)+\left(\frac{4A}{a_0}+\frac{3}{\sin(\pi/3)}\right)A_3\sin(3\pi/3)+\left(\frac{4A}{a_0}+\frac{5}{\sin(\pi/3)}\right)A_5\sin(5\pi/3)=\alpha$$

– $i=3,\ \theta_3=\pi/2$

$$\left(\frac{4A}{a_0}+\frac{1}{\sin(\pi/2)}\right)A_1\sin(\pi/2)+\left(\frac{4A}{a_0}+\frac{3}{\sin(\pi/2)}\right)A_3\sin(3\pi/2)+\left(\frac{4A}{a_0}+\frac{5}{\sin(\pi/2)}\right)A_5\sin(5\pi/2)=\alpha$$

Prandtl's Lifting Line Theory

Consider the case with a general lift distribution

Numerical example of the wing equation (3)

$$\sum_{n=1}^{N}\left(\frac{4A}{a_0}+\frac{n}{\sin\theta_i}\right)A_n \sin n\theta_i \quad = \quad \alpha \qquad\qquad i=1,2,\dots N$$

- A=AR in the equation

– $i=1,\ \theta_1=\pi/6,$

$$(4+2)A_1(0.5)+(4+6)A_3(1)+(4+10)A_5(0.5)=\alpha$$

$$3A_1+10A_3+7A_5=\alpha$$

– $i=2,\ \theta_2=\pi/3,$

$$\left(4+\frac{2}{\sqrt{3}}\right)A_1\frac{\sqrt{3}}{2}+\left(4+3(\frac{2}{\sqrt{3}})\right)A_3(0)+\left(4+5(\frac{2}{\sqrt{3}})\right)A_5(-0.866)=\alpha$$

$$4.464A_1-8.464A_5=\alpha$$

– $i=3,\ \theta_3=\pi/2$

$$(4+1)A_1-(4+3)A_3+(4+5)A_5=\alpha$$

$$5A_1-7A_3+9A_5=\alpha$$

Prandtl's Lifting Line Theory

Consider the case with a general lift distribution

Numerical example: the rectangular wing (*N*=3)

- *The set of equations becomes:* *with solution*:

$$\begin{pmatrix} 3 & 10 & 7 \\ 4.464 & 0 & -8.464 \\ 5 & -7 & 9 \end{pmatrix}\begin{pmatrix} A_1 \\ A_3 \\ A_5 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} A_1 \\ A_3 \\ A_5 \end{pmatrix} = \alpha \begin{pmatrix} 0.2316 \\ 0.0277 \\ 0.0040 \end{pmatrix}$$

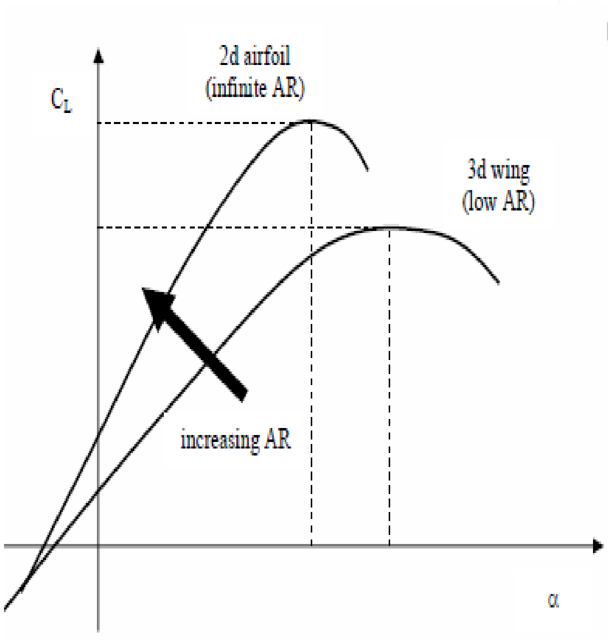
- *Evaluation of the properties of the rectangular wing (with $AR = a_0 = 2\pi$):*

		N=3	*N*=20
$C_L = \pi AR\, A_1 = 4.572\alpha \longrightarrow$	$a = \dfrac{dC_L}{d\alpha}$	$= 4.572$	(4.583)
	τ	$= 0.176$	(0.166)
$\delta = \sum_{n=2}^{N} n\left(\dfrac{A_n}{A_1}\right)^2 \geq 0 \longrightarrow$	δ	$= 0.044$	(0.051)
	e	$= 0.957$	(0.951)

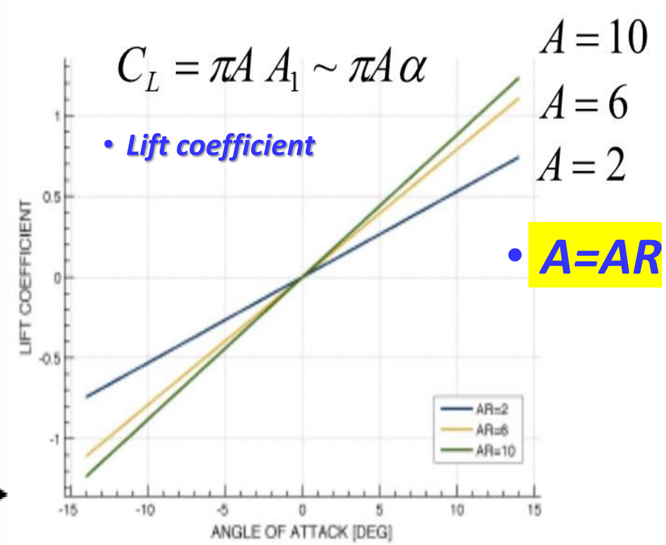
- ***Note: with $\delta \approx 0.05$: only 5% more induced drag than elliptical wing!***

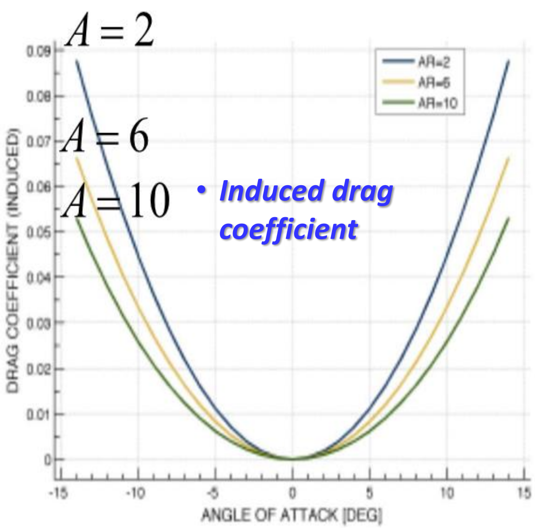
❑ PRANDTL'S LIFTING LINE THEORY

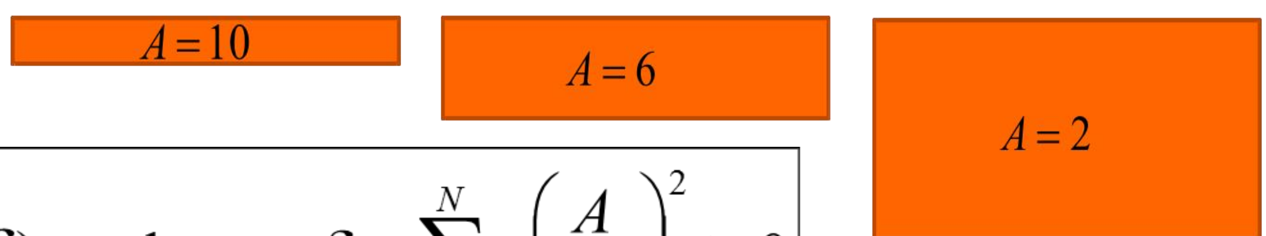
❑ ***<u>Consider the case with a general lift distribution</u>***



Sample Problem : Effect of Aspect Ratio Results





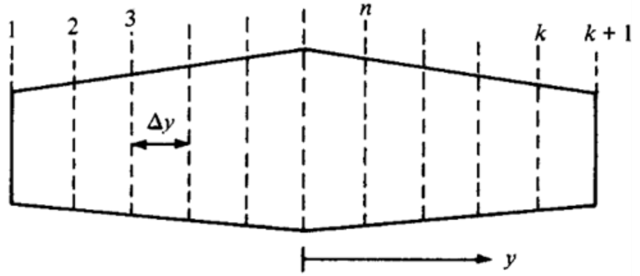


$$C_{D_i} = \frac{C_L^{\,2}}{\pi A}(1+\delta) \quad \text{where} \quad \delta = \sum_{n=2}^{N} n\left(\frac{A_n}{A_1}\right)^2 \geq 0$$

Prandtl's Lifting Line Theory

A numerical nonlinear lifting-line method

Given the wing shape and the angle of attack α:



1. *Divide the wing in spanwise positions:* y_n
2. *Assume an initial circulation distribution*

   $\Gamma_n=\Gamma(y_n)$, *e.g. elliptical*

3. *Calculate the induced angle of attack:*

$$\alpha_i(y_n) = \frac{1}{4\pi V_\infty} \int_{-b/2}^{b/2} \frac{(d\Gamma/dy)}{(y_n - y)} dy$$

(evaluate the integral numerically)

4. *Calculate:* $\alpha_{eff}(y_n) = \alpha - \alpha_i(y_n)$
5. *Calculate lift coefficient:* $c_l(y_n) = c_l(\alpha_{eff}(y_n))$
6. *Update circulation:* $\Gamma(y_n) = \frac{V_\infty c(y_n)}{2} \cdot c_l(y_n)$

iterate until convergence

(under relaxation

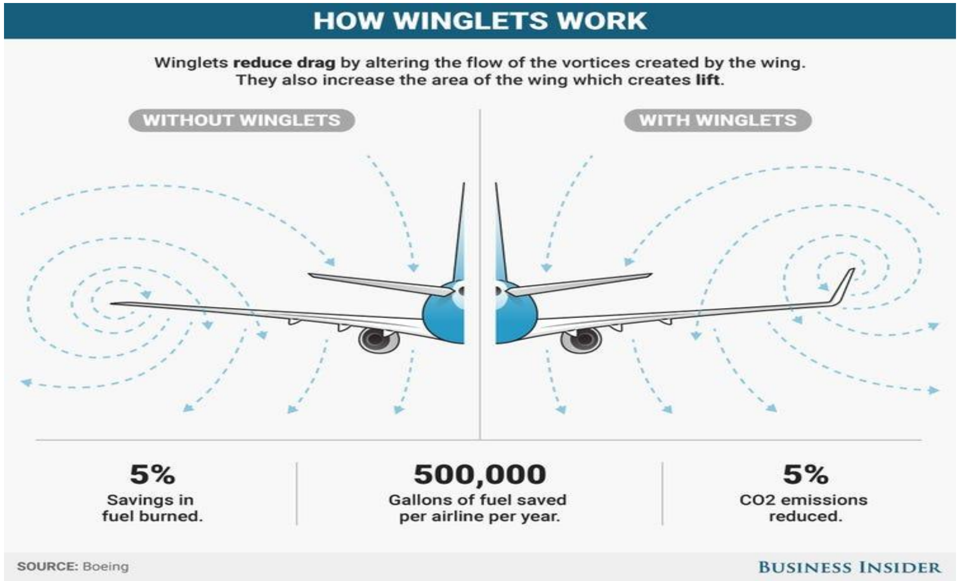
3D Wing theory - a summary

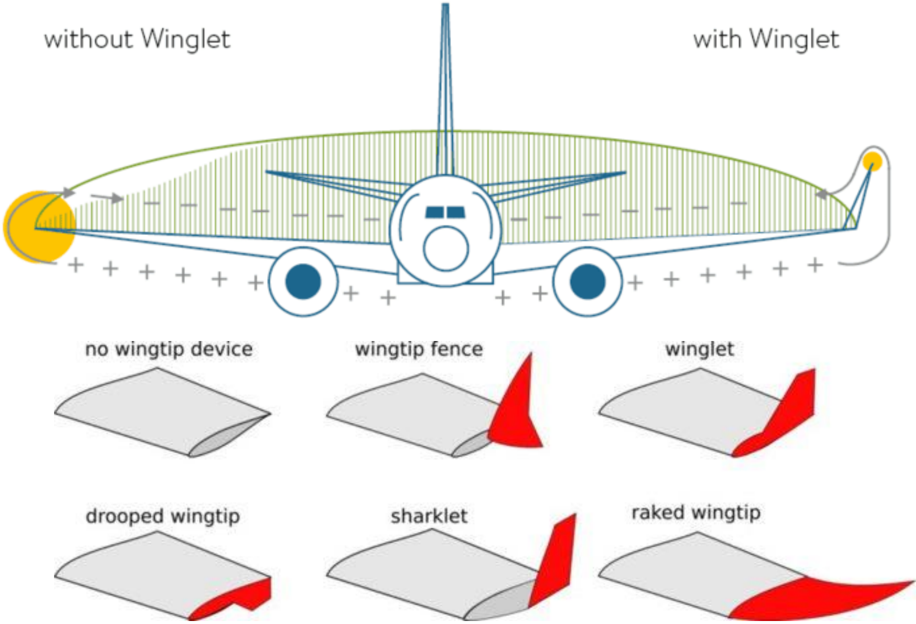
- ***Lifting-line theory:***
  - *The wing is replaced by a bound vortex at the 1/4-chord line of the wing with varying circulation $\Gamma(y)$: the* ***lifting line***
  - *The trailing vortices form a flat sheet of distributed vorticity: the* ***vortex wake***
- ***Limitations of the classical theory:***
  - *slender wings (large aspect ratio, or: span>>chord)*
  - *straight wings (no wing sweep)*
  - *moderate aerodynamic loading (no deformation of the vortex wake)*
  - *linear relation* $c_l \sim \alpha_{eff}$
- ***Extensions:***

Chapter 5.4 – Anderson Textbook: non-linear lifting-line theory: $c_l(\alpha_{eff})$

Chapter 5.5 – Anderson Textbook: methods where the wing is represented by a ***vortex-sheet*** *(instead of a line): lifting-surface / vortex-lattice methods*

Wingflex of B787



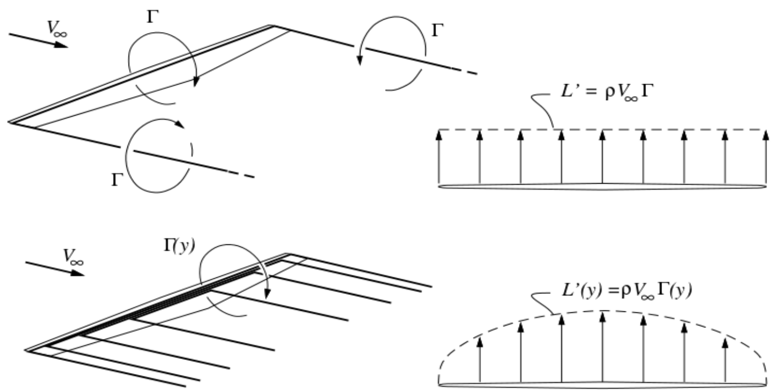


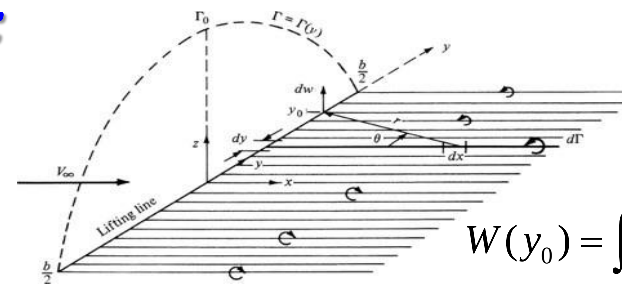
- *Boeing 787 and 737 MAX in flight*

- *Why the Wings of Boeing 787 are curved?*

3D Wing Theory

Prandtl's Lifting Line Theory:



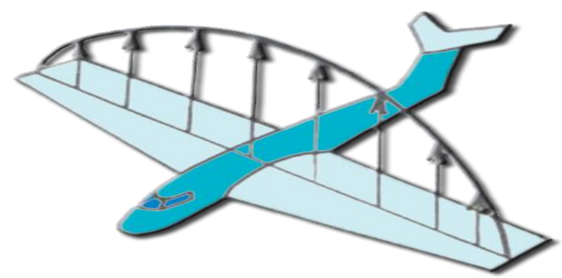


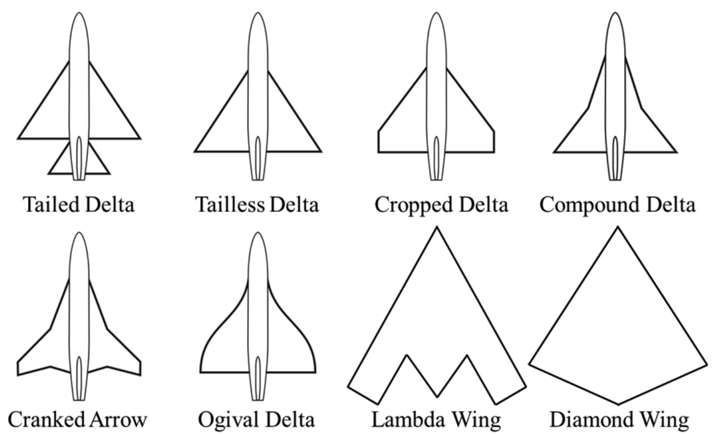
$$dw = -\frac{(d\Gamma / dy)\cdot dy}{4\pi(y_0 - y)}$$

$$W(y_0) = \int_{-b}^{b} dw = -\int_{-b}^{b} \frac{(d\Gamma / dy)}{4\pi(y_0 - y)} dy$$

$$\alpha(y_0) = \frac{\Gamma(y_0)}{\pi V_\infty c(y_0)} + \alpha_{L=0}(y_0) + \frac{1}{4\pi V_\infty} \int_{-b/2}^{b/2} \frac{(d\Gamma / dy)}{(y_0 - y)} dy$$

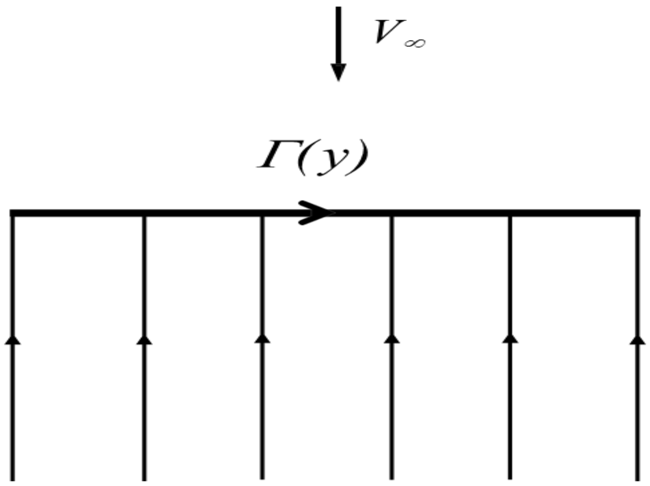
- *A disadvantage of the lifting line theory is that all of the action associated with the bound vortex occurs at the quarter chord point, such that only the lift and drag coefficients are computed but not the moment coefficient.*
- *Unfortunately, the moment coefficient is essential to the performance calculations.*
- *An answer is found in the **vortex lattice method** which not only provides the pressure distribution but also anchors the results to the actual geometry rather than implicitly through the $\alpha_{L=0}$. This is essential not only for calculating moments but for many practical wing planforms like delta wings.*





3D Wing Theory

Prandtl's Lifting Line Theory:



wing

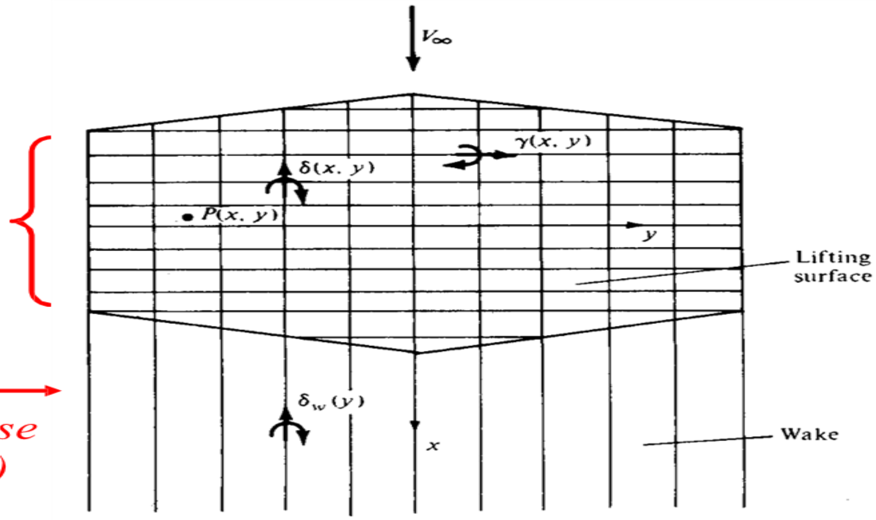
wake (streamwise vorticity)

Lifting line:

wing represented by a vortex filament (only spanwise vorticity)

valid only for slender wings

Vortex Lattice Method:



Lifting surface:

wing represented by a vortex sheet with distributed spanwise and chordwise vorticity

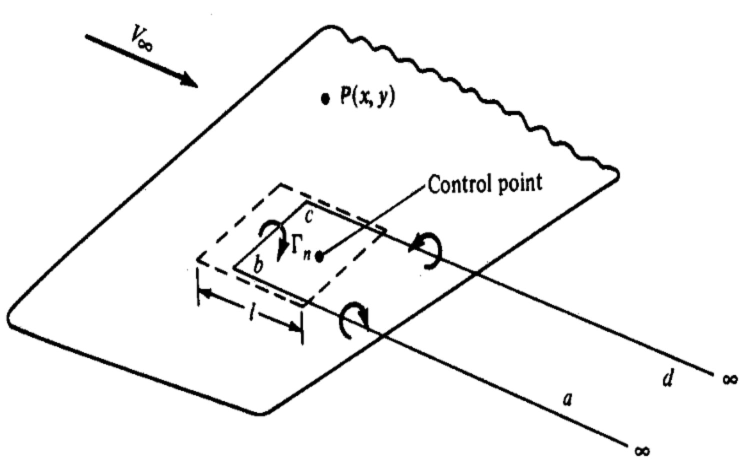
- F-22 Raptor

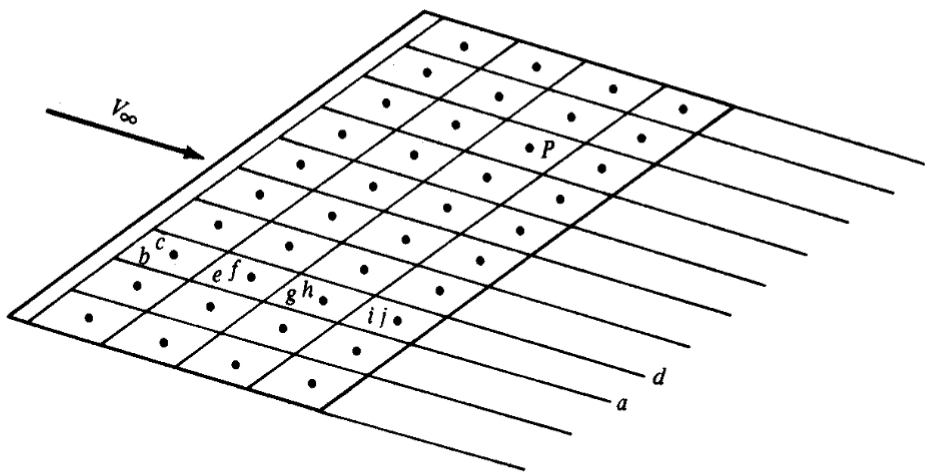
- Rafale flighter

❑ PRANDTL'S LIFTING LINE THEORY

Lifting-surface theory - numerical implementation

- ***3D vortex-panel methods:***
  - *the wing is represented by panels with distributed vorticity (three-dimensional extension of the vortex-panel method in section 4.9)*
- ***Vortex-Lattice methods:***
  - *distributed vorticity is concentrated into a lattice of horseshoe vortices*





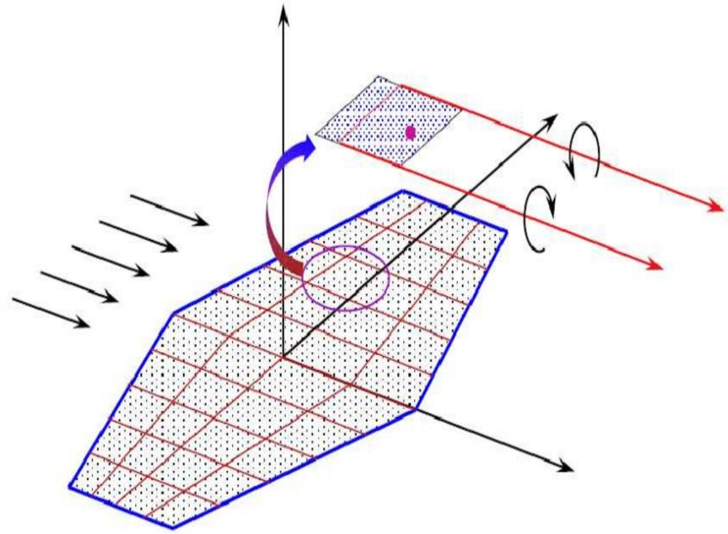
- *A single horseshoe vortex*
- *The vortex-lattice system on a finite wing*

3D WING THEORY

Vortex Lattice Method:

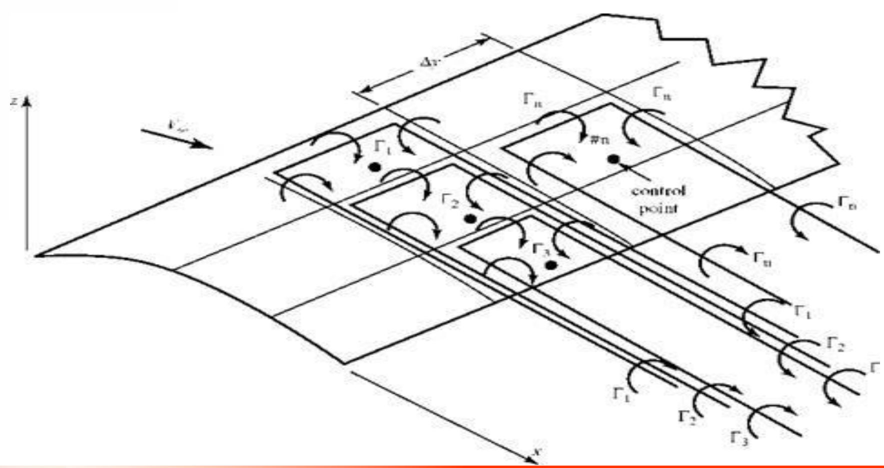
Essential ideas:

- Panel the wing with discrete spanwise, γ, and streamwise, δ, distribution of vorticies.
- Set a "control point" somewhere on this panel to apply the flow tangency condition.
- Biot-Savart to determine the induced velocity from all points.
- Solution of a system of equations determines the discrete vorticity distributions via the downwash equation:



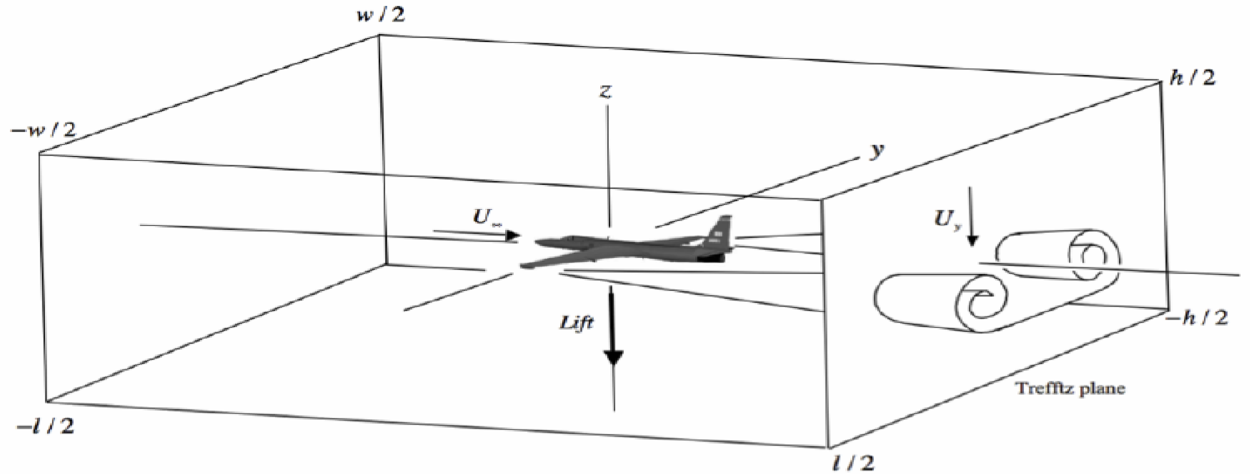
$$w(x,y) = -\frac{1}{4\pi}\iint_S \frac{(x-\xi)\gamma(\xi,\eta)+(y-\eta)\delta(\xi,\eta)}{\left[(x-\xi)^2+(y-\eta)^2\right]^{3/2}}d\xi d\eta$$

$$-\frac{1}{4\pi}\iint_W \frac{(y-\eta)\delta_W(\xi,\eta)}{\left[(x-\xi)^2+(y-\eta)^2\right]^{3/2}}d\xi d\eta$$



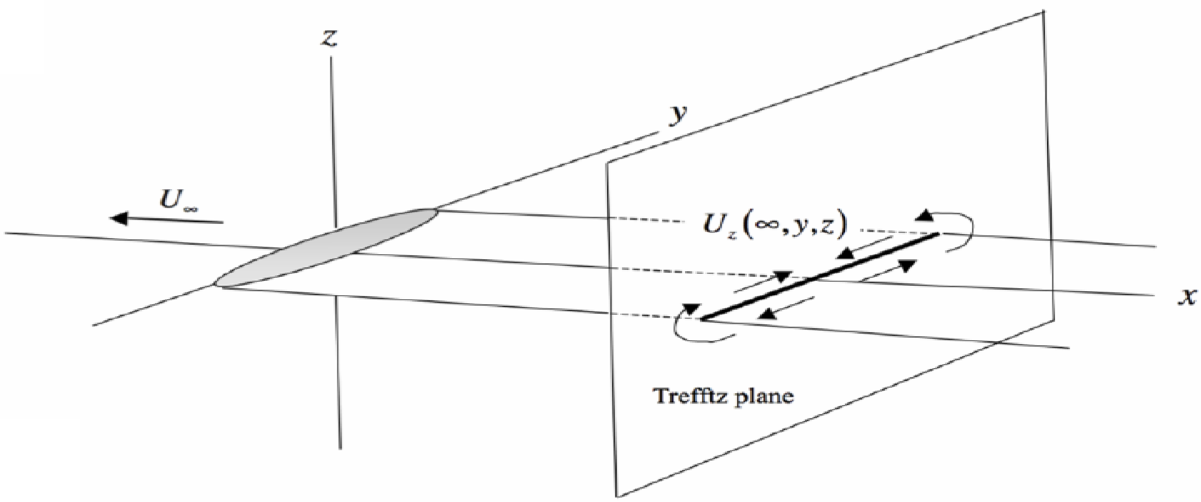
❑ 3D Wing Theory

Problem we would like to treat



Trefftz plane intersecting the rolled up wake far behind an aircraft.

Problem we actually treat



Trefftz plane intersecting the flat, straight vortex sheet from a wing

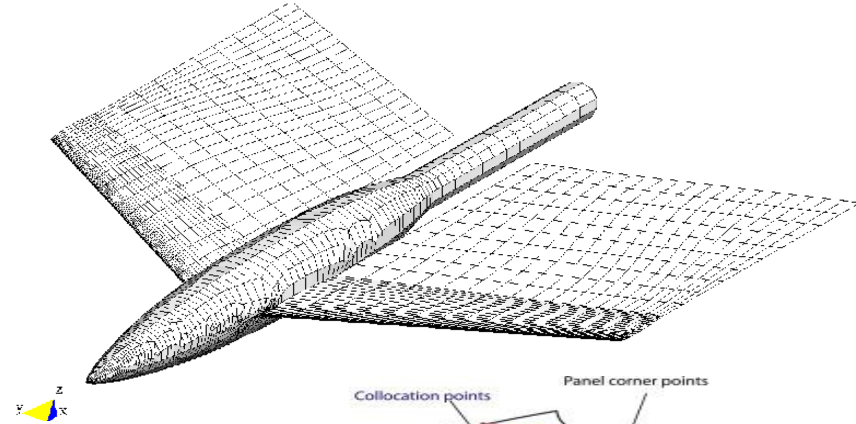
❑ 3D WING THEORY

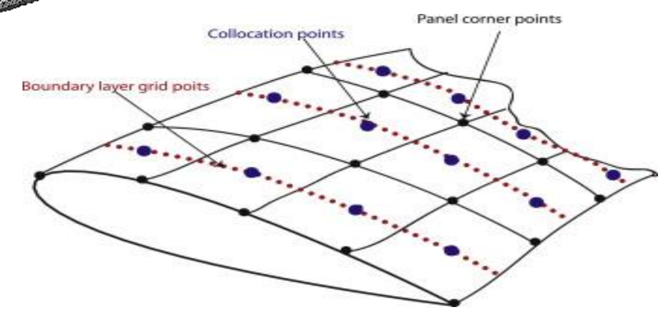
❑ 3D Panel Method:

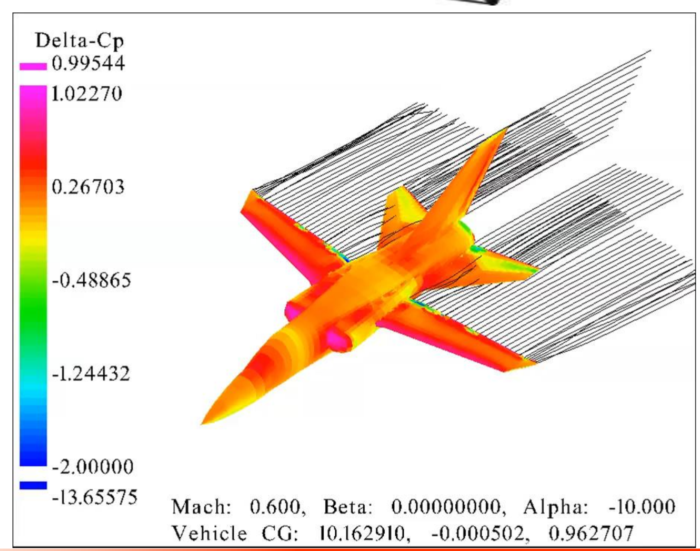
<u>Basic Idea:</u>

- *Distribute sources, doublets or vortices on the surface of a body.*
- *Apply the flow tangency condition.*
- *Solve for the unknown source, doublet and vortex strengths.*

This approach is widely used in the industry for preliminary design considerations and allows us to apply the surface tangency conditions to all points on the wing. A large code is written for this purpose and generally takes a good deal of effort to define the geometry and apply the method.



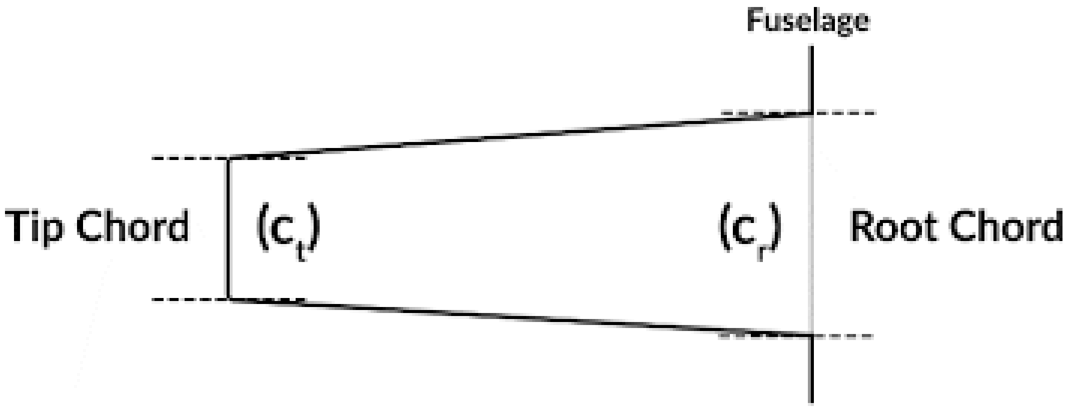




Prandtl's Lifting Line Theory

Example #1

- *For a finite wing with an aspect ratio of 8.0 and taper ration of 0.8. The airfoil section is thin and symmetrical.*
  - *Please calculate the lift and induced drag coefficient for the wing when it is at an angle of attack α=5.0°*



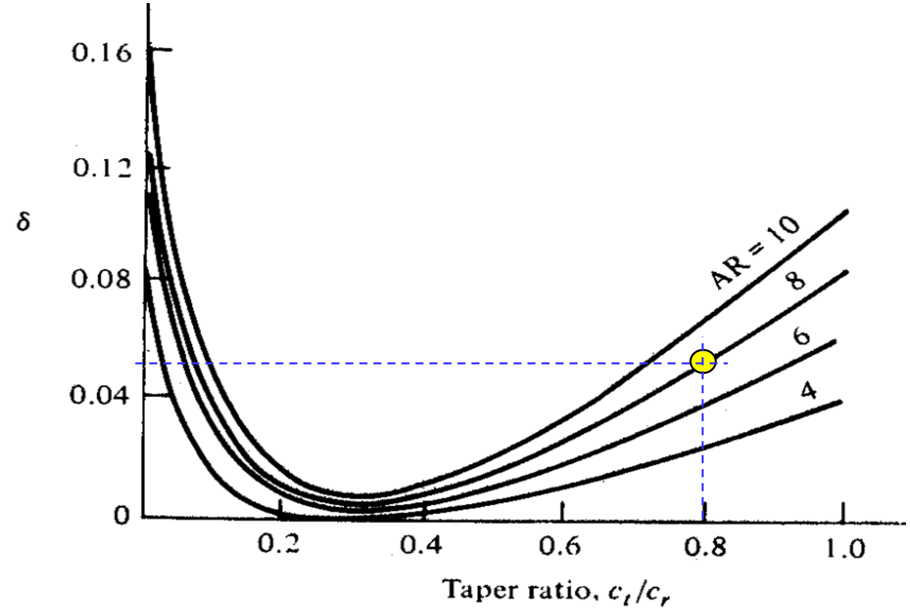
$$Taper\ Ratio\ \lambda = \frac{tip\ chord}{root\ chord} = \frac{C_t}{C_r}$$

❑ Prandtl's Lifting Line Theory

❑ Example #1

$$C_{D_i} = \frac{C_L^{\ 2}}{\pi AR}(1+\delta)$$

$$a = \frac{a_0}{1+(a_0/\pi AR)(1+\delta)}$$



$$a = \frac{a_0}{1 + a_0/\pi\,\text{AR}(1+\delta)} = \frac{2\pi}{1 + 2\pi(1.055)/8\pi} = 4.97\ \text{rad}^{-1}$$

$$= 0.0867\ \text{degree}^{-1}$$

Since the airfoil is symmetric, $\alpha_{L=0} = 0°$. Thus,

$$c_L = a\alpha = (0.0867\ \text{degree}^{-1})(5°) = \boxed{0.4335}$$

From Equation (5.61),

$$c_{D,i} = \frac{C_L^2}{\pi\,\text{AR}}(1+\delta) = \frac{(0.4335)^2(1+0.055)}{8\pi} = \boxed{0.00789}$$

Prandtl's Lifting Line Theory

Example #2

- *Consider a rectangular wing with an aspect ratio of 6, an induced drag factor $\delta = 0.055$, and a zero-lift angle of attack of $-2°$. At an angle of attack of $3.4°$, the induced drag coefficient for this wing is 0.01.*
  - *Calculate the induced drag coefficient for a similar wing (a rectangular wing with the same airfoil section) at the same angle of attack, but with an aspect ratio of 10.*
  - *Assume that the induced factors for drag and the lift slope, δ and τ, respectively, are equal to each other (i.e., $\delta = \tau$). Also, for $AR = 10$, $\delta = 0.105$.*

Prandtl's Lifting Line Theory

Solution

We must recall that although the angle of attack is the same for the two cases compared here (AR = 6 and 10), the value of C_L is different because of the aspect-ratio effect on the lift slope. First, let us calculate C_L for the wing with aspect ratio 6. F

$$C_{D,i} = \frac{C_L^2}{\pi AR}(1+\delta)$$

$$C_L^2 = \frac{\pi AR C_{D,i}}{1+\delta} = \frac{\pi(6)(0.01)}{1+0.055} = 0.1787$$

Hence, $C_L = 0.423$

The lift slope of this wing is therefore

$$\frac{dC_L}{d\alpha} = \frac{0.423}{3.4^\circ-(-2^\circ)} = 0.078/\text{ degree } = 4.485/rad$$

❑ Prandtl's Lifting Line Theory

❑ Solution

The lift slope for the airfoil (the infinite wing) can be obtained from Equation

$$a=\frac{a_0}{1+(a_0/\pi AR)(1+\tau)}$$

$$\frac{dC_L}{d\alpha}=a=\frac{a_0}{1+(a_0/\pi AR)(1+\tau)}$$

$$4.485=\frac{a_0}{1+[(1.055)a_0/\pi(6)]}=\frac{a_0}{1+0.056a_0}$$

Solving for a_0, we find that this yields $a_0=5.989$/rad. Since the second wing (withAR = 10) has the same airfoil section, then a_0 is the same. The lift slope of the second

wing is given by

$$a=\frac{a_0}{1+(a_0/\pi AR)(1+\tau)}=\frac{5.989}{1+[(5.989)(1.105)/\pi(10)]}=4.95/rad$$

$$=0.086/\text{ degree}$$

The lift coefficient for the second wing is therefore

$$C_L=a\left(\alpha-\alpha_{L=0}\right)=0.086\left[3.4^{\circ}-\left(-2^{\circ}\right)\right]=0.464$$

❑ *Solution*

In turn, the induced drag coefficient is

$$C_{D,i} = \frac{C_L^2}{\pi AR}(1+\delta) = \frac{(0.464)^2(1.105)}{\pi(10)} = 0.0076$$

Note: This problem would have been more straightforward if the lift coefficients had been stipulated to be the same between the two wings rather than the angle of attack. Then Equation **(5.61)** would have yielded the induced drag coefficient directly. A purpose of this example is to reinforce the rationale behind Equation **(5.65)**, which readily allows the scaling of drag coefficients from one aspect ratio to another, as long as the lift coefficient is the same. This allows the scaled drag-coefficient data to be plotted versus C_L (not the angle of attack) as in Figure 5.22. However, in the present example where the angle of attack is the same between both cases, the effect of aspect ratio on the lift slope must be explicitly considered, as we have done above.

$$C_{D,1} = C_{D,2} + \frac{C_L^2}{\pi e}\left(\frac{1}{AR_1} - \frac{1}{AR_2}\right) \quad (5.65)$$

❑ PRANDTL'S LIFTING LINE THEORY

❑ *Solution*

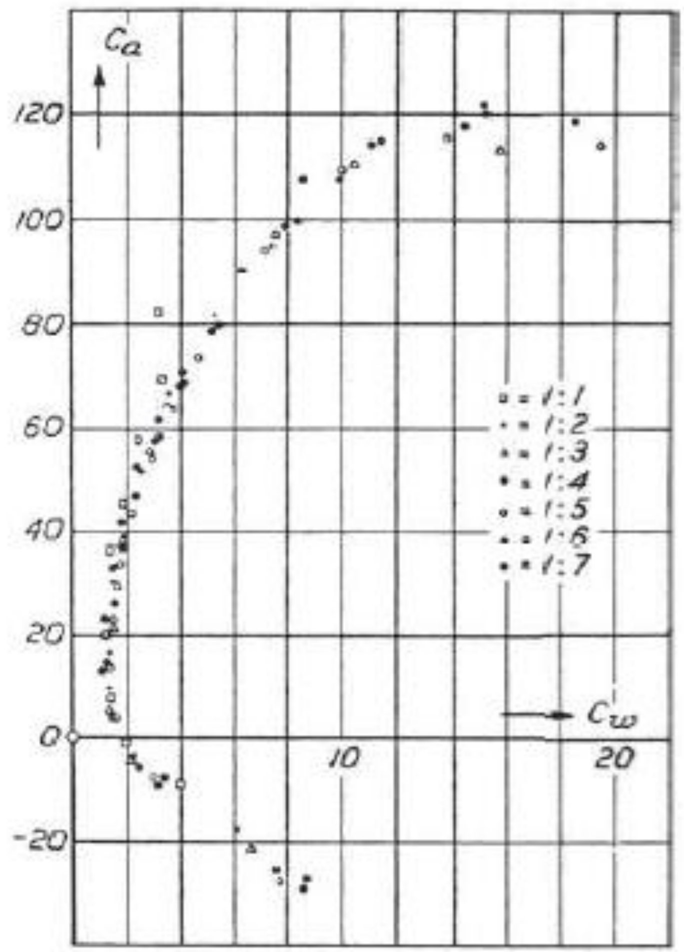


Figure 5.22 Data of Figure 5.21 scaled by Prandtl to an aspect ratio of 5.