Lecture # 38: Boundary Layer Flows

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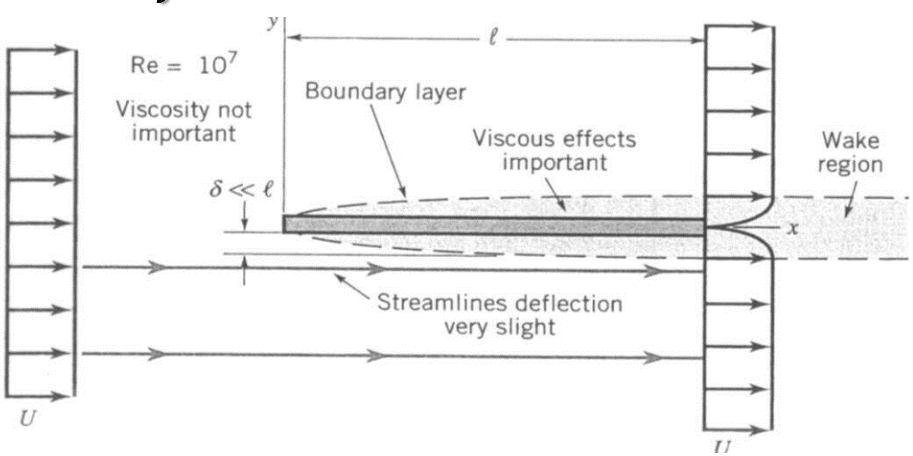
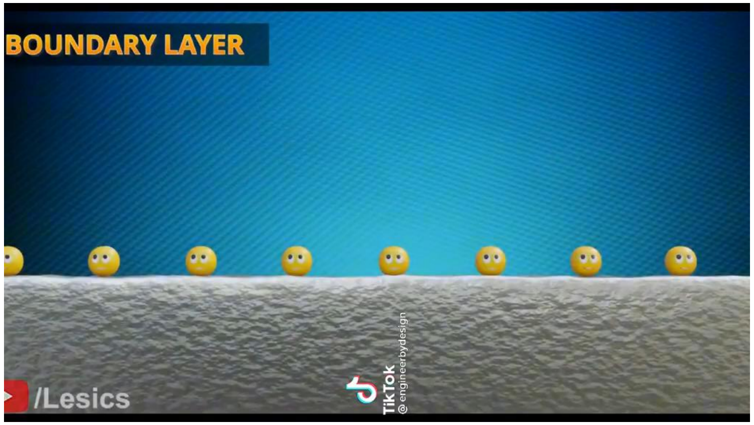
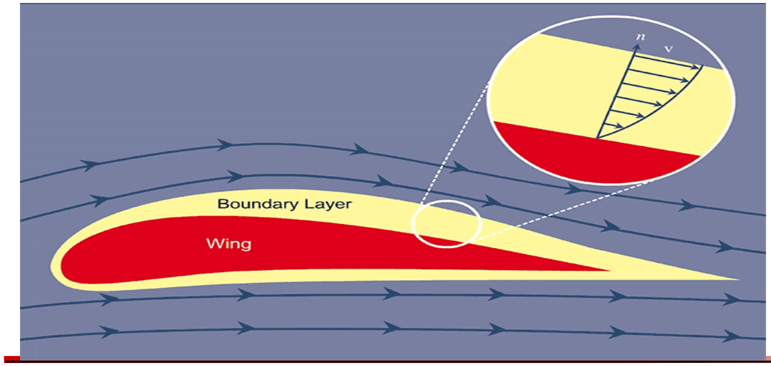
❑ Boundary Layer Flows

- *The boundary layer is a very thin layer of air flowing over the surface of an object (like a wing).*
- *As air moves past the wing, the molecules right next to the wing stick to the surface. Each layer of molecules in the boundary layer moves faster than the layer closer to the surface.*
- *At the outer edge of the boundary layer, the molecules move at the same velocity (free stream velocity) as the molecules outside the boundary layer.*
- ***Ludwig Prandtl** revolutionized fluid dynamics when he introduced the boundary layer concept in the early 1900s.*



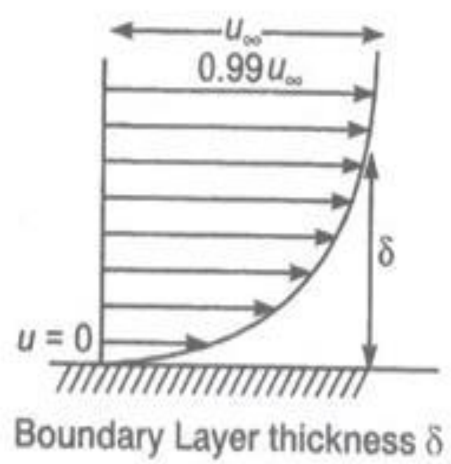


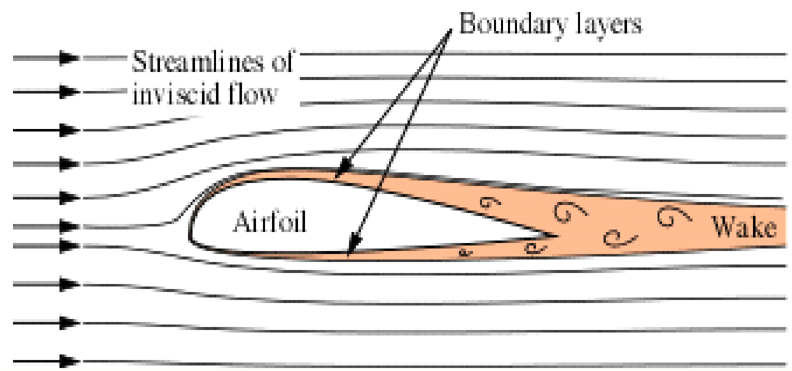


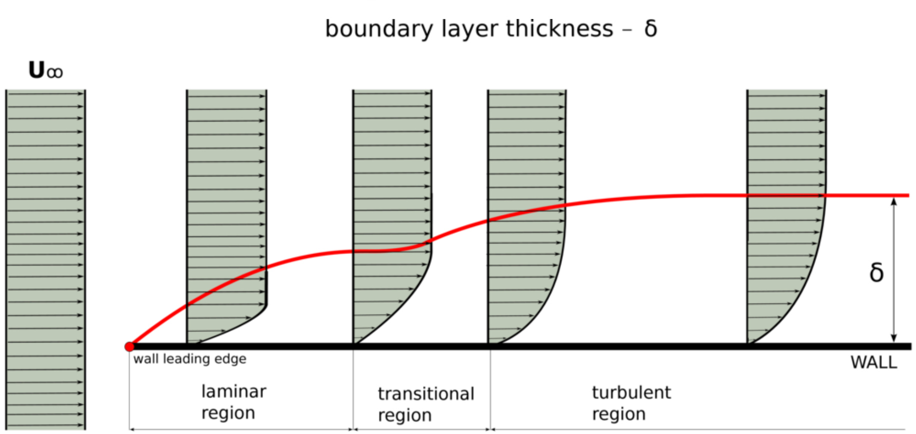
- ***What is a Boundary Layer?***
- ***https://www.youtube.com/watch?v=GqVCTNCwfQk***

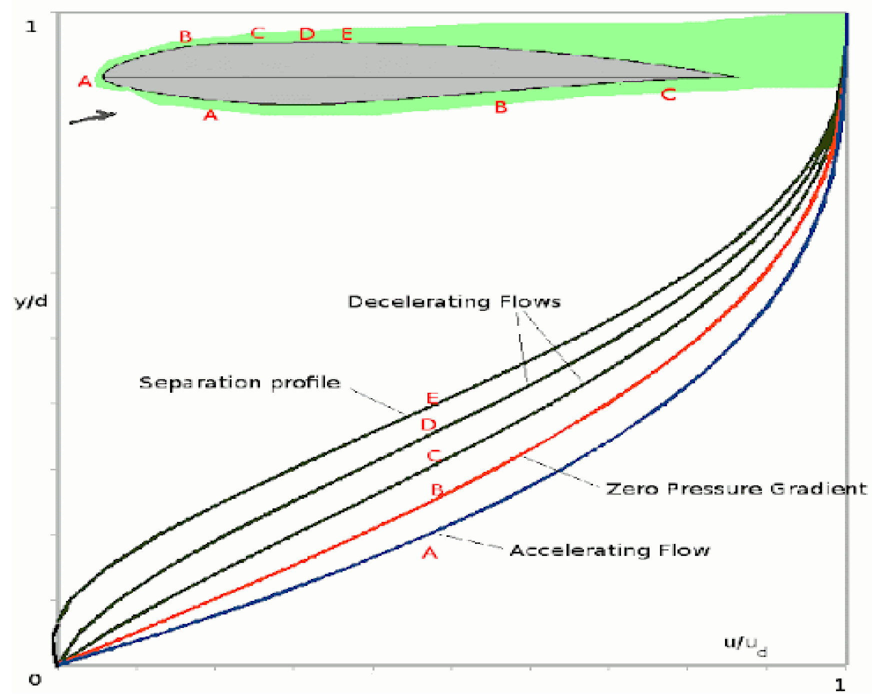
❑ Boundary Layer Flows

- *Viscous effects are limited to small region (thickness~δ) around the surface.*
- *Boundary layer thickness is defined at distance above the surface where velocity has reached 99% of the external flow.*



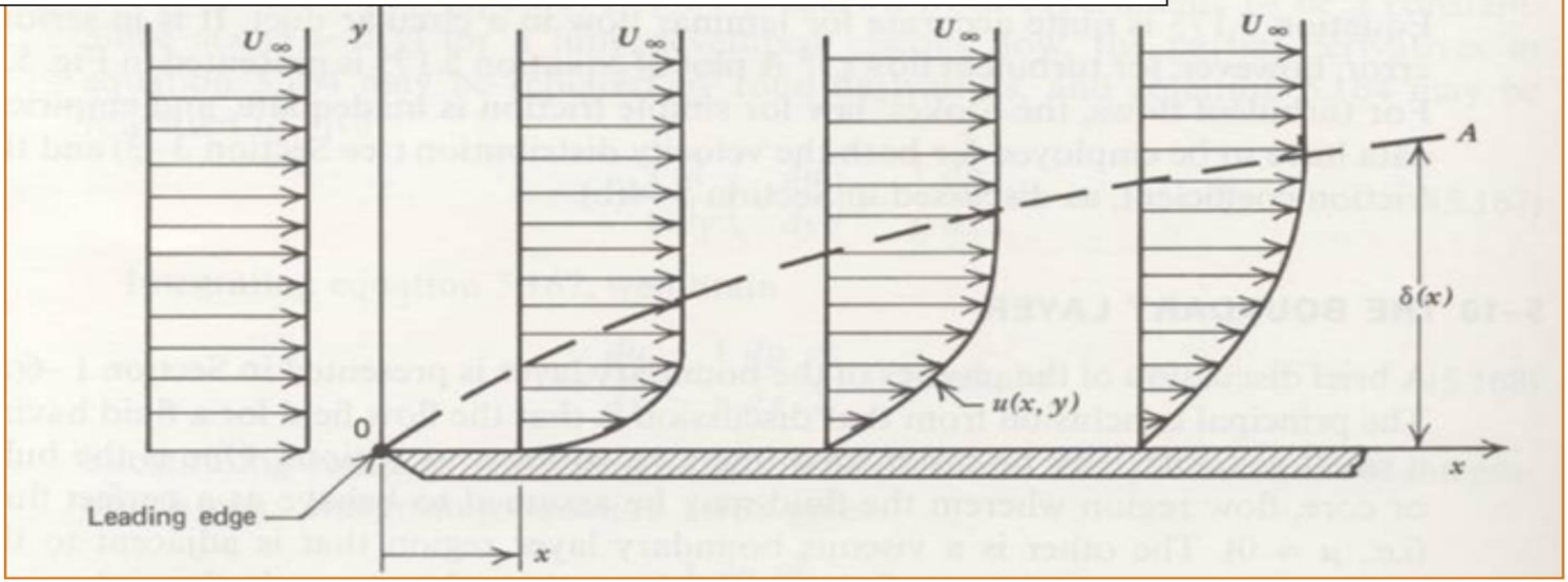






Boundary Layer Flows

BOUNDARY LAYER THICKNESS:
δ is y where $u(x,y) = 0.99\, U_\infty$



This definition for δ is completely arbitrary, why not 98%, 95%, etc.

Blasius showed theoretically that $\delta/x = 5/Re_x$

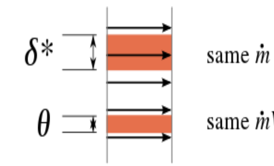
$(Re_x = \rho U_\infty x/\mu)$

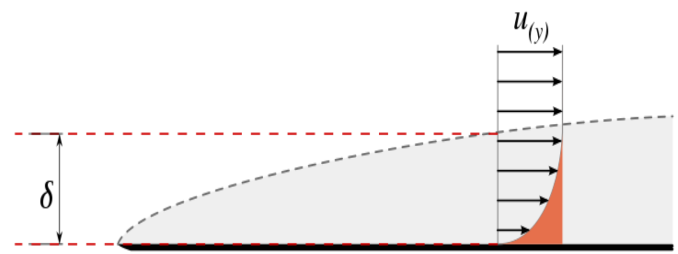
❑ Boundary Layer Flows

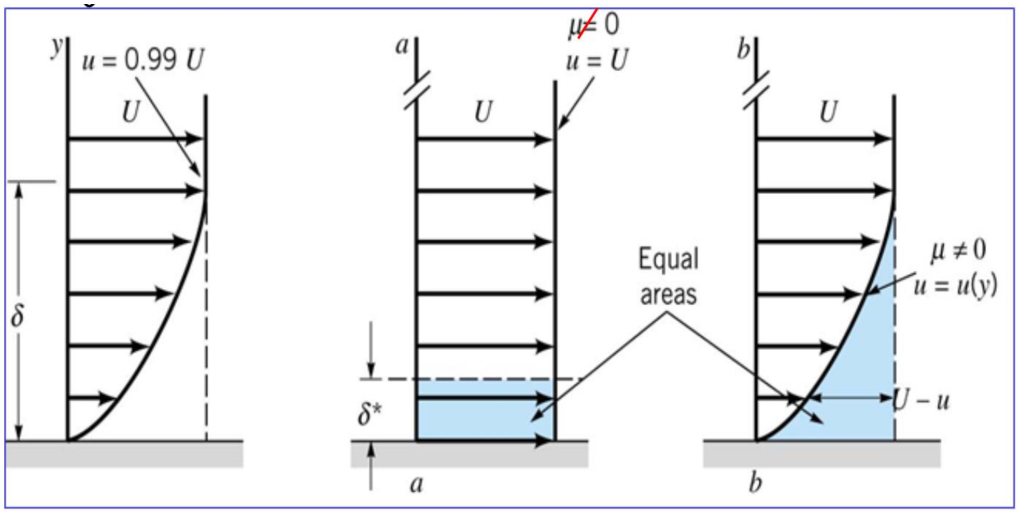
- ***Displacement thickness,*** δ^*

- Displacement thickness is the amount the streamlines outside the boundary layer appear to be ‘displaced’.

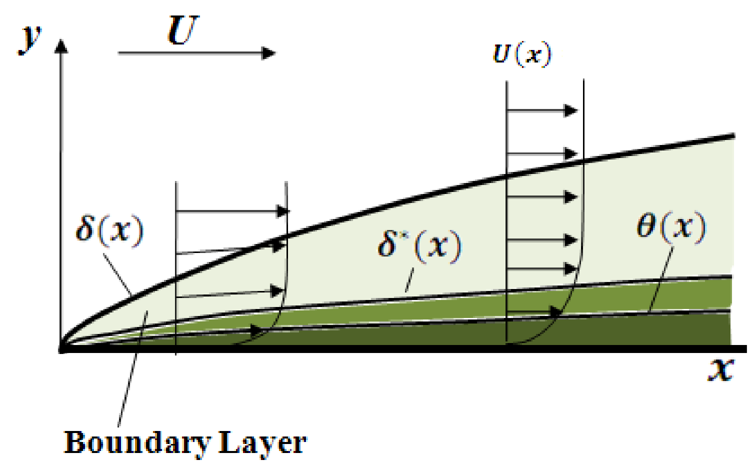
$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U_e}\right) dy$$







Because of the velocity deficit, U-u, within the bdy layer, the mass flux through b-b is less than a-a. However if we displace the plate a distance δ*, the mass flux along each section will be identical.

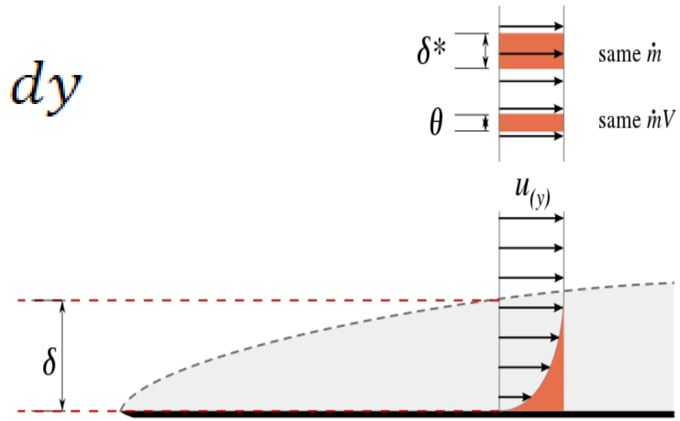


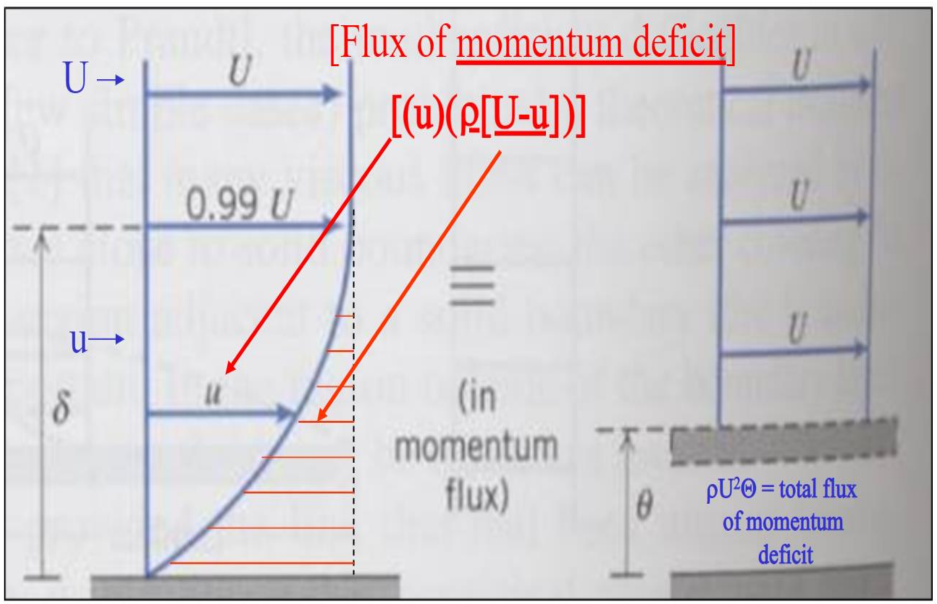
❑ Boundary Layer Flows

- ***Momentum thickness, θ***

Momentum thickness represent the momentum deficit caused by the presence of the solid wall (boundary layer)

$$\theta = \int_0^\infty \frac{u}{U_e}\left(1 - \frac{u}{U_e}\right) dy$$





$\rho U_e^2 \Theta = {}_0\!\int^\infty \rho\, u(U_e - u)dy$ $\Theta = {}_0\!\int^\infty [u/U_e]\,(1 - u/U_e)dy$

- *H is the shape factor defined as the ratio between the displacement and momentum thicknesses*

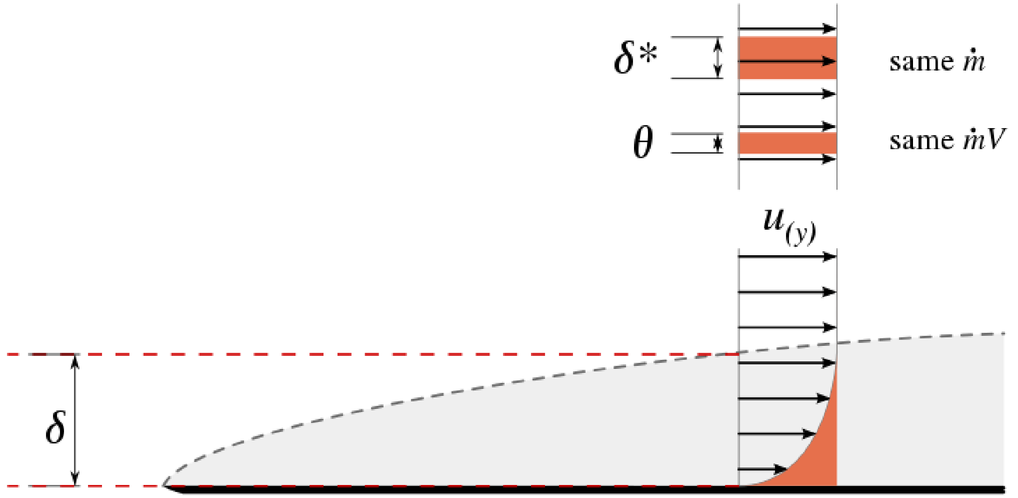
$$H = \frac{\delta^*}{\theta}$$

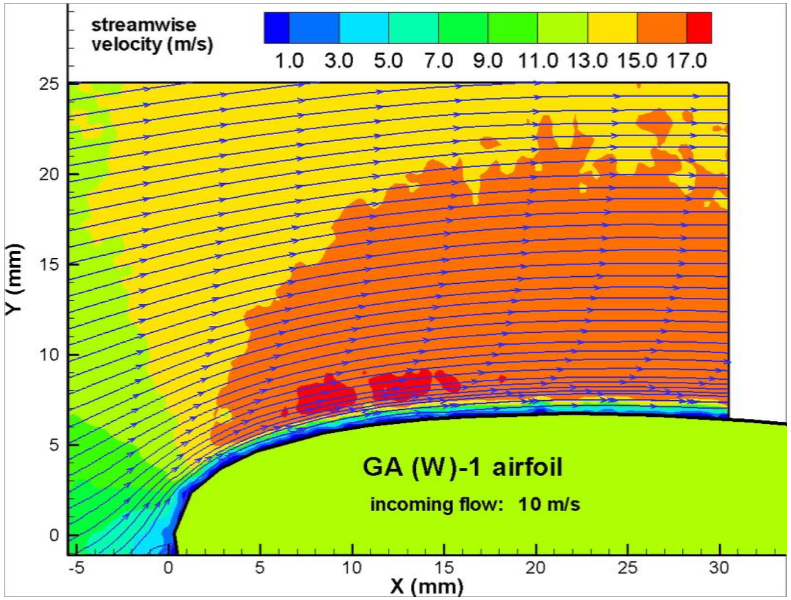
Boundary Layer Flows

- **_Example #1_**

- _Problem 01 : Calculate displacement and momentum thickness for the local velocity profile given by the formula_

$$u(y) = U_0(1 - e^{-\alpha y})$$





❑ Boundary Layer Flows

- ***Example #1***

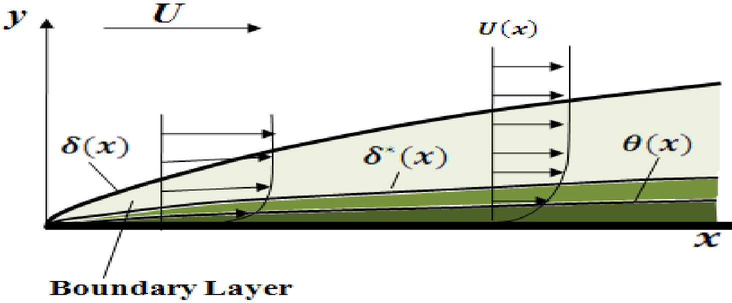
- ***<u>Problem 01</u> : Calculate displacement and momentum thickness for the local velocity profile given by the formula***

Solution:

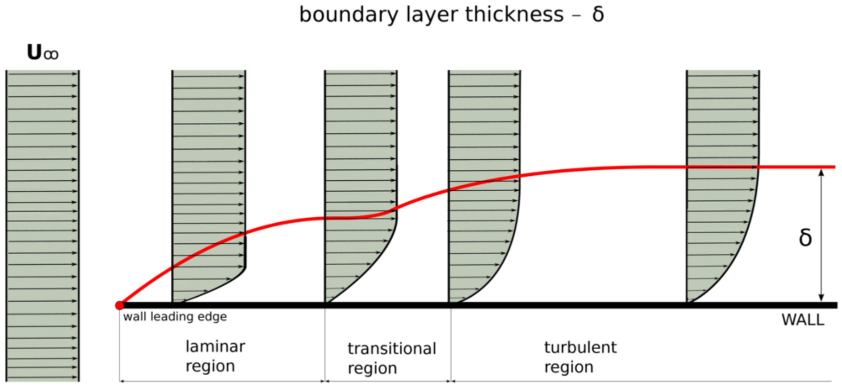
$$\delta_* = \lim_{\delta\to\infty}\int_0^{\delta}(1-1+e^{-\alpha y})dy = -\tfrac{1}{\alpha}\lim_{\delta\to\infty}[e^{-\alpha\delta}-e^{0}] = \tfrac{1}{\alpha}$$

$$\theta = \lim_{\delta\to\infty}\int_0^{\delta}(1-e^{-\alpha y})e^{-\alpha y}dy = \lim_{\delta\to\infty}\int_0^{\delta}e^{-\alpha y}dy - \lim_{\delta\to\infty}\int_0^{\delta}e^{-2\alpha y}dy = \tfrac{1}{\alpha}-\tfrac{1}{2\alpha} = \tfrac{1}{2\alpha}$$

The shape factor is $H = \dfrac{\delta_*}{\theta} = 2$



❑ Navier-Stoke Equation for 2D Viscous Flows



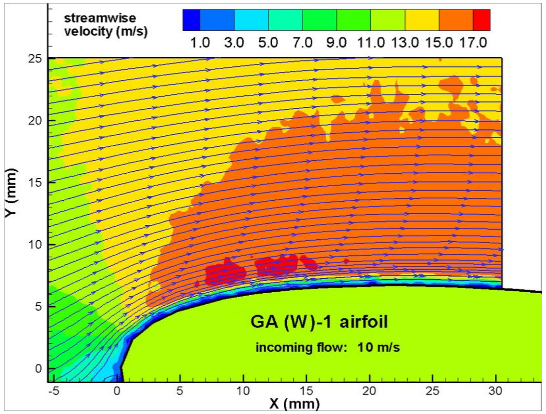
Navier-Stokes equations

x-momentum

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu \nabla^2 u$$

y-momentum

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \mu \nabla^2 v$$



These are 2D incompressible Navier-Stokes equations.

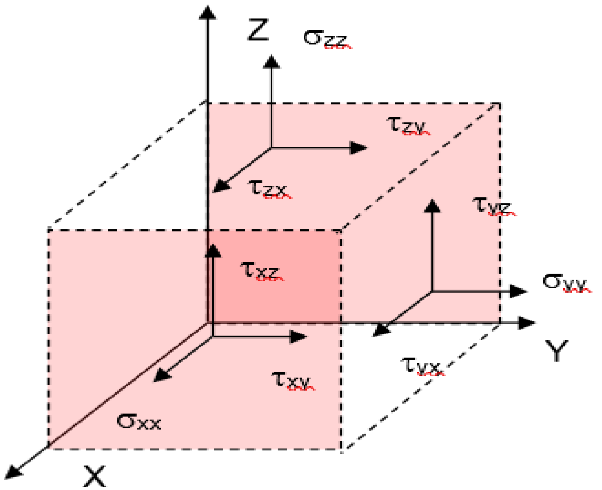
Continuity equation remains the same:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

There are three equations with three unknowns (u, v, p).

❑ Navier-Stoke Equation for 2D Viscous Flows

Conservation of momentum



For Newtonian fluid

$$\tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{xx} = -p + 2\mu\frac{\partial u}{\partial x}$$

Therefore, the x-momentum becomes

$$\rho\frac{Du}{Dt} = \frac{\partial}{\partial x}\left(-p + 2\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right)$$

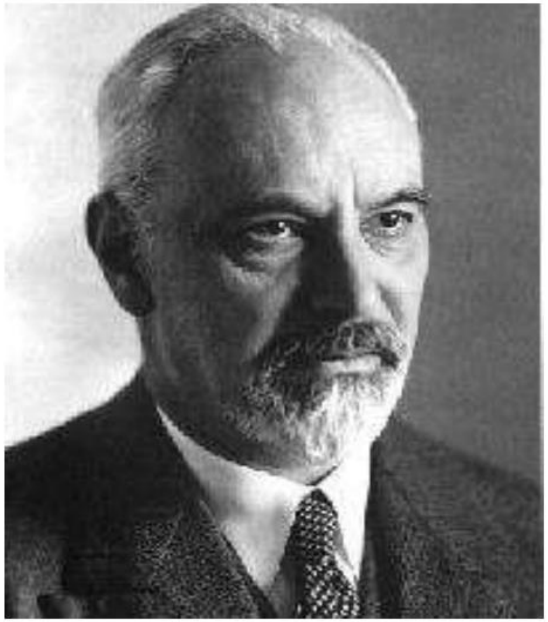
$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \mu\frac{\partial^2 u}{\partial y^2} + \mu\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)$$

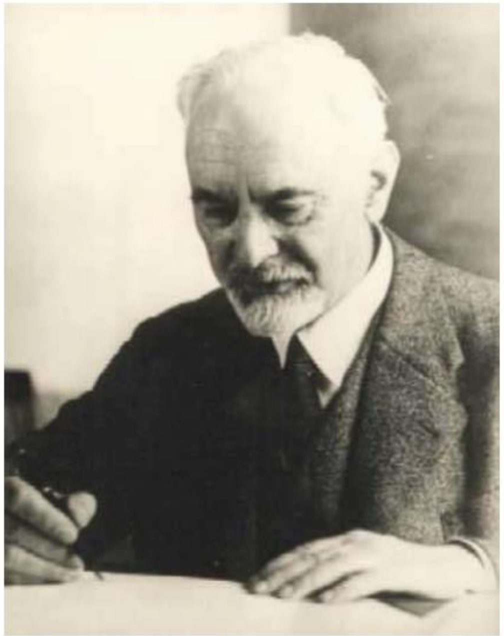
$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu\underbrace{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}_{\nabla^2 u} + \mu\frac{\partial}{\partial x}\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}_{\vec{\nabla}.\vec{V} = 0}$$

Boundary Layer Equations

Boundary Layer Equations

- Boundary Layer concept was founded by Ludwig Prandtl and it revolutionized the concept of solving Navier Stokes questions.
- Boundary Layer Equations are partial differential equations that apply inside the boundary layer.





Boundary Layer Flows

Non-dimensional variables

- Making the equations of motion non-dimensional helps to gauge the importance of various terms.
- Use a reference length, L and a reference velocity V_∞ to normalize all the terms
- Let's note the dimensional variables with superscript *

$$u = \frac{u^*}{V_\infty}, x = \frac{x^*}{L}, y = \frac{y^*}{L}, t = \frac{t^*}{L/V_\infty}, p = \frac{p^* - p_\infty}{\rho V_\infty^2}$$

Then note:

$$\frac{\partial}{\partial x^*} = \frac{\partial}{\partial (xL)} = \frac{1}{L}\frac{\partial}{\partial x} \Rightarrow \frac{\partial^2}{\partial x^{*2}} = \frac{1}{L^2}\frac{\partial^2}{\partial x^2}$$

$$\frac{D}{Dt^*} = \frac{D}{D(tL/V_\infty)} = \frac{V_\infty}{L}\frac{D}{Dt}$$

Boundary Layer Flows

Non-dimensional form of viscous flow equations

- And for pressure gradient

$$\frac{\partial p^*}{\partial x^*} = \frac{\partial}{\partial x^*}(p_\infty + \rho V_\infty^2 p) = \rho V_\infty^2 \frac{\partial p}{\partial x^*} = \frac{\rho V_\infty^2}{L}\frac{\partial p}{\partial x}$$

- Dimensional x-momentum

$$\rho \frac{Du^*}{Dt^*} = -\frac{\partial p^*}{\partial x^*} + \mu\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

$$\frac{\rho V_\infty}{L}\frac{D(uV_\infty)}{Dt} = -\frac{\rho V_\infty^2}{L}\frac{\partial p}{\partial x} + \mu\left(\frac{1}{L^2}\frac{\partial^2 (uV_\infty)}{\partial x^2} + \frac{1}{L^2}\frac{\partial^2 (uV_\infty)}{\partial y^2}\right)$$

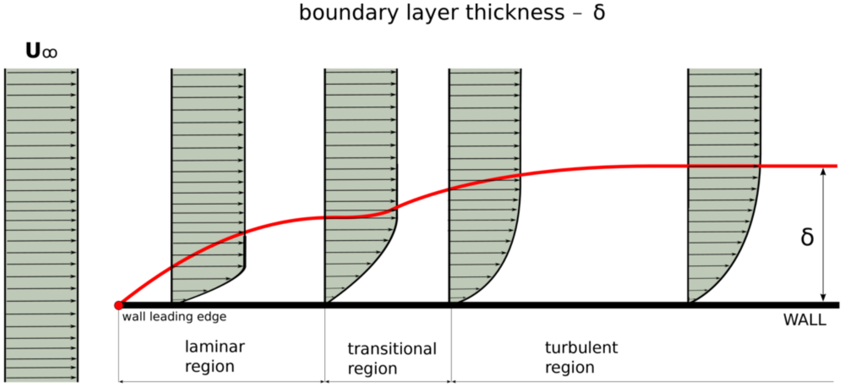
$$\frac{\rho V_\infty^2}{L}\frac{Du}{Dt} = -\frac{\rho V_\infty^2}{L}\frac{\partial p}{\partial x} + \frac{\mu V_\infty}{L^2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\boxed{\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\mu}{\rho V_\infty L}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}$$

Boundary Layer Flows

Non-dimensional form of viscous flow equations

$$\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

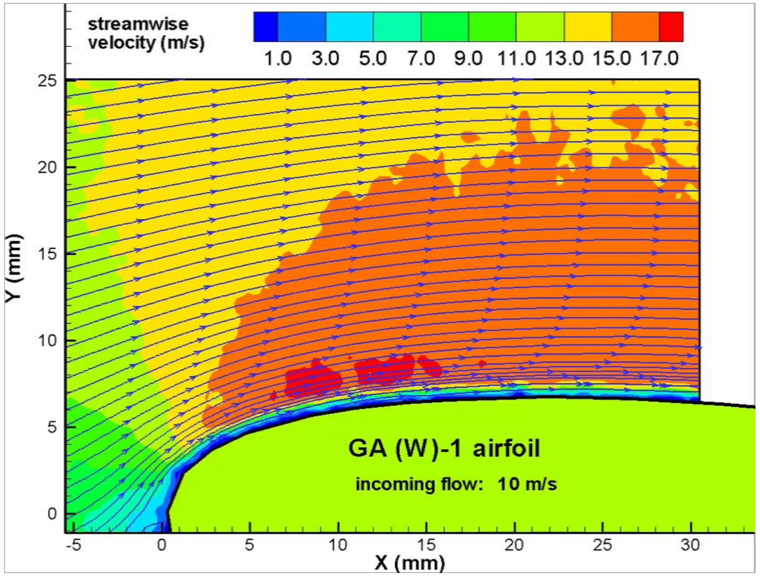


And similarly for y-momentum

$$\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

And continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$



Note that for freestream

$$u = 1, \qquad v = 0, \qquad p = 0$$

Boundary Layer Flows

$$\delta^* = \frac{\delta}{L} \ll 1$$

Term	Order
$\frac{\partial u^*}{\partial x^*}$	$\frac{(1)}{(1)} = 1$
$\frac{\partial v^*}{\partial y^*}$	$\frac{\delta^*}{\delta^*} = 1$
v^*	δ^*
$\frac{\partial v^*}{\partial x^*}$	$\frac{\delta^*}{1} = \delta^*$
$\frac{\partial^2 u^*}{\partial y^{*2}}$	$\frac{1}{\delta^{*2}}$
$\frac{du^*}{dt^*}$	$u^* \frac{\partial u^*}{\partial x^*} = 1$

$$\frac{\partial u^*}{\partial t} + u^* \frac{\partial u^*}{\partial x} + v^* \frac{\partial u^*}{\partial y} = -\frac{\partial P^*}{\partial x} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u^*}{\partial x^2} + \frac{\partial^2 u^*}{\partial y^2}\right)$$

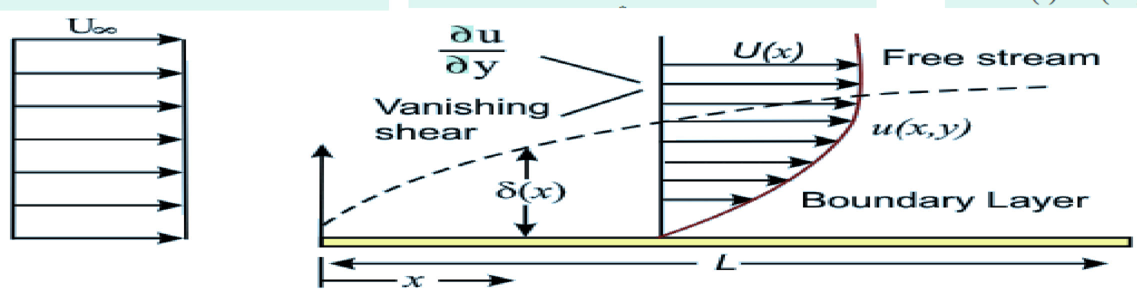
$$(1) \quad (1) \quad \delta^* \frac{1}{\delta^*} = 1 \quad \frac{(1)}{(1)} = 1 \quad \delta^{*2} \quad \frac{(1)}{(1)^2} \quad \frac{(1)}{(\delta^*)^2}$$

Neglect since of order $\frac{(1)}{(\delta^*)^2}$ >>>1

Also for y –direction

$$\frac{\partial v^*}{\partial t^*} + u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = -\frac{\partial P^*}{\partial y^*} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right)$$

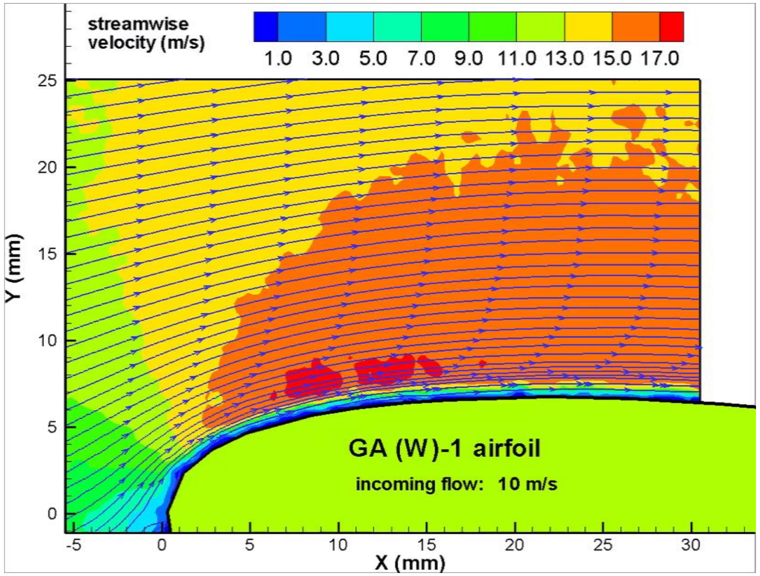
$$* \quad (1)\frac{(\delta^*)}{(1)} \quad (\delta^*)\frac{(\delta^*)}{(\delta^*)} \quad o\left(\frac{1}{\delta}\right) \quad (\delta^{*2})\left\{\frac{\delta^*}{(1)^2} + \frac{\delta^*}{(\delta^*)^2}\right\}$$



❑ Boundary Layer Flows

- ***Boundary layer flow over a flat plate***

- *Consider a steady uniform flow approaching the flat plate.*

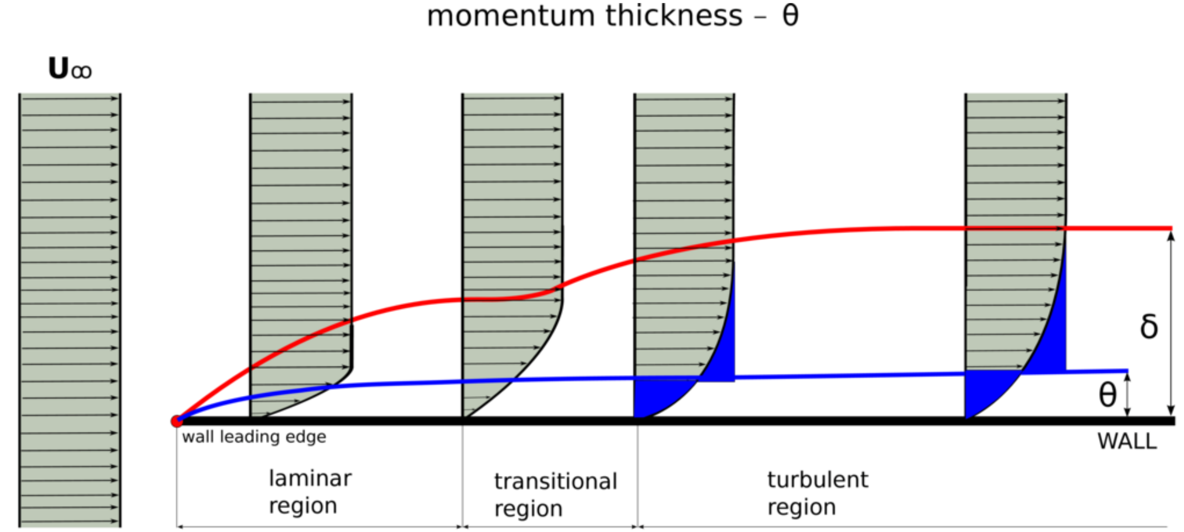


- *Continuity equation*

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u \sim O(1)$$
$$x \sim O(1)$$
$$y \sim O(\delta)$$

$$\Rightarrow v \sim O(\delta)$$

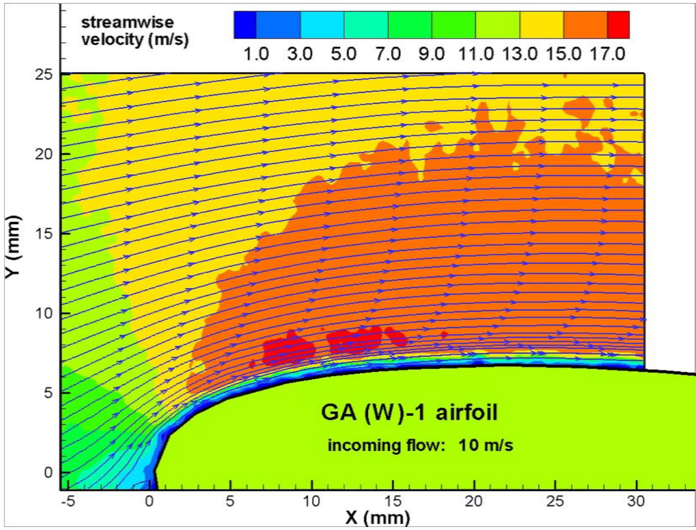


Boundary Layer Flows

Conservation of momentum

For x-momentum

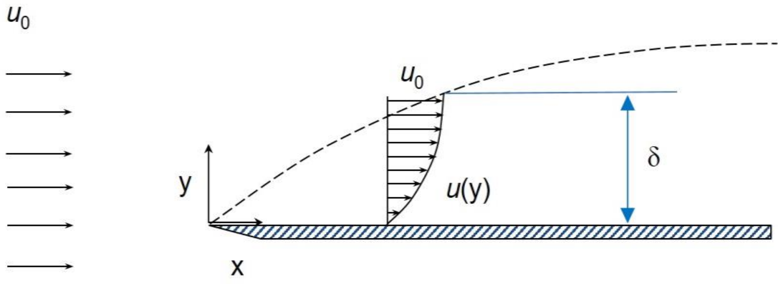
$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$



- On the left-hand side, the ratios are $O(1)$
- On the right-hand side

$$u\sim O(1), y\sim O(\delta)\Rightarrow\frac{\partial^2 u}{\partial y^2}\sim O\left(\frac{1}{\delta^2}\right)$$

$$u\sim O(1), x\sim O(\delta)\Rightarrow\frac{\partial^2 u}{\partial x^2}\sim O(1)$$



This means $\frac{\partial^2 u}{\partial x^2}\ll\frac{\partial^2 u}{\partial y^2}$

Therefore, unless $Re\ll 1$, we can ignore $\frac{\partial^2 u}{\partial x^2}$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{Re}\frac{\partial^2 u}{\partial y^2}$$

Boundary Layer Flows

- **Conservation of momentum**
- *For y-momentum*

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$u\frac{\partial v}{\partial x}\sim O(\delta), \qquad v\frac{\partial v}{\partial y}\sim O(\delta)$$

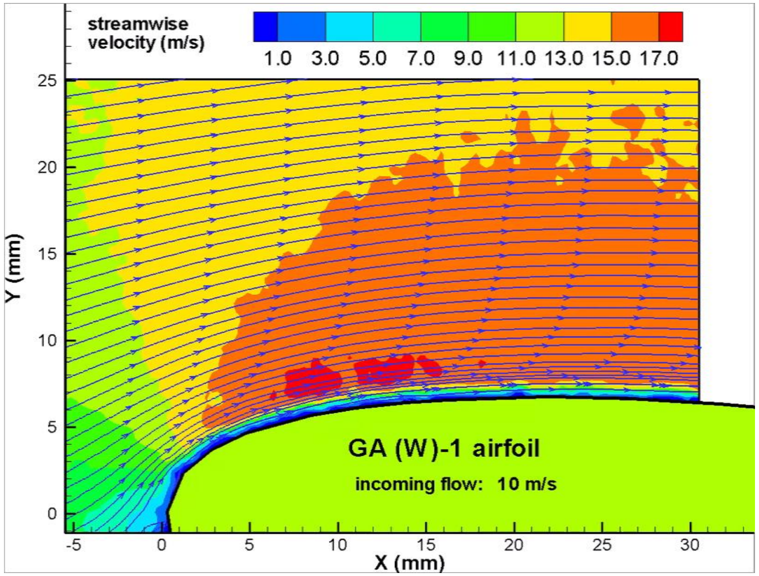
$$\frac{\partial p}{\partial y}\sim O\left(\frac{1}{\delta}\right)$$

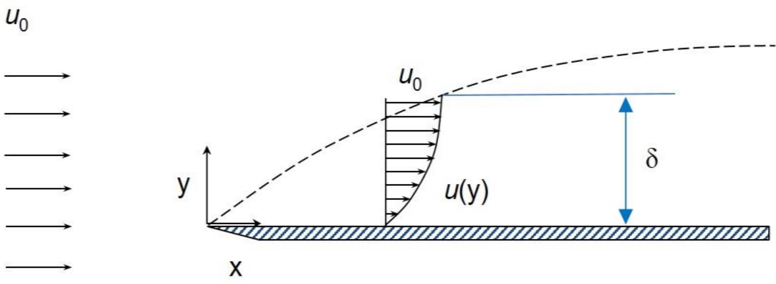
$$\frac{\partial^2 v}{\partial x^2}\sim O(\delta), \qquad \frac{\partial^2 v}{\partial y^2}\sim O\left(\frac{1}{\delta}\right)$$

If $Re \gg 1$, $\frac{\partial p}{\partial y} > \frac{1}{Re}\frac{\partial^2 v}{\partial y^2}$

So, y-momentum becomes

$$\frac{\partial p}{\partial y}=0$$





The Laminar Boundary Layer Equations — Derivation

- Rewriting the governing equations in terms of non-dimensional variables,

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \cancel{\frac{1}{Re_L} \frac{\partial^2 u^*}{\partial x^{*2}}} + \frac{\partial^2 u^*}{\partial y^{*2}}$$

$$\cancel{\frac{1}{Re_L} u^*} \frac{\partial v^*}{\partial x^*} + \cancel{\frac{1}{Re_L} v^*} \frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial y^*} + \cancel{\frac{1}{Re_L^2} \frac{\partial^2 v^*}{\partial x^{*2}}} + \cancel{\frac{1}{Re_L} \frac{\partial^2 v^*}{\partial y^{*2}}}$$

- As Re_L becomes large, terms with $1/Re_L$ can be neglected, and the system reduces to the classical Boundary Layer Equations:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{\partial^2 u^*}{\partial y^{*2}}$$

$$\frac{\partial p^*}{\partial y^*} = 0$$

The Laminar Boundary Layer Equations – Other Forms

- If we scale the variables by usual dimensions not involving the Reynolds number, i. e.

$$x^* = x/L,\ \ y^* = y/L,\ \ \rho^* = \rho/\rho_\infty,\ \ u^* = u/V_\infty,\ \ v^* = v/V_\infty,\ \ p^* = p/\rho_\infty V_\infty^2$$

- Then the equations become (note re-emergence of the Reynolds number in the momentum equation):

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{dp^*}{dx^*} + \frac{1}{Re_L}\frac{\partial^2 u^*}{\partial {y^*}^2}$$

- The dimensional form of the equations is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

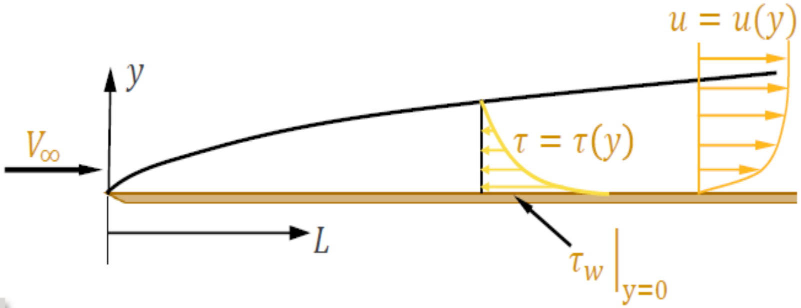
Boundary Layer Shear Stress

- The shear stress in 2D is:

$$\tau = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

- Applying the same dimensional considerations, we can show that in the boundary τ layer reduces to:

$$\tau = \mu\frac{\partial u}{\partial y}$$



- Along the wall, the shear stress is:

$$\tau_w\big|_{y=0} = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

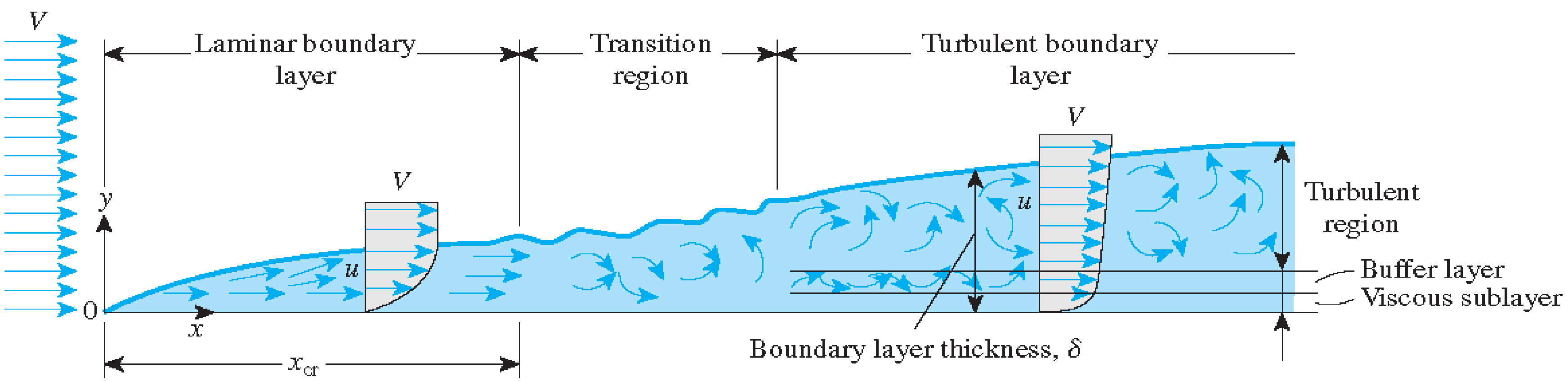
- Interestingly, this expression can give a hint as to where separation of the boundary layer may occur. As the shear stress changes sign past the separation point, a location where $\tau_w = 0$ can provide an indication of a potential point of separation.

Boundary Layer Theory

The Laminar Boundary Layer Equations – Summary

- Simplifications did not affect the continuity equation.
- Pressure gradient in the y-direction is zero, thus pressure is a function of x only, $p = p(x)$. As a result, pressure can be obtained from the Bernoulli's solution for the outer flow.
- With all second x derivatives vanished, the equations are now parabolic, which can simplify their numerical solution.

- Limitations of the boundary layer equations
  - Valid only for large Reynolds numbers $Re_x > 1000$.
  - Valid only for Reynolds numbers within the laminar regime, $Re_x < 10^6$. While a solution can be obtained for any Re, it will not be valid for flows in the turbulent regime.
  - The boundary layer theory is valid only for attached boundary layers, and it cannot describe separation of the boundary layer or the flow past the separation point.



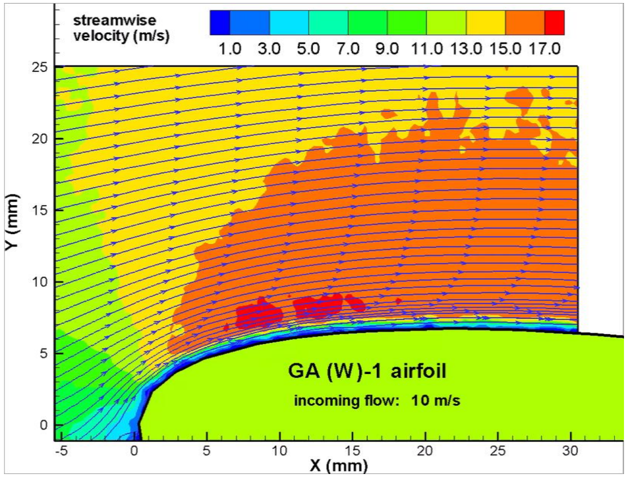
❑ Boundary Layer Flows

▪ *Summary of boundary layer equations*

- ***x-momentum:*** $$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\frac{\partial^2 u}{\partial y^2}$$

- ***y-momentum:*** $$\frac{\partial p}{\partial y} = 0$$

- ***Continuity:*** $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

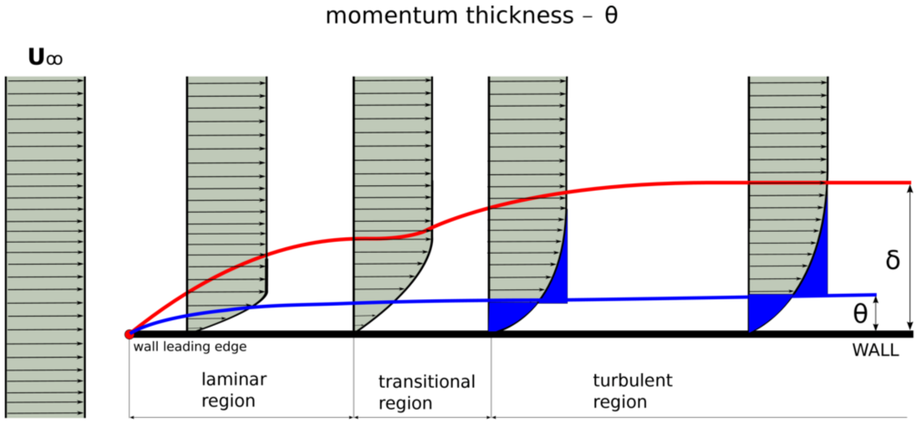


- ***Boundary conditions:***
  - ***No-slip at the wall***

    $$y = 0 \Rightarrow u = v = 0$$

  - ***Known far-field flow velocity***

    $$y \to \infty \Rightarrow u \to 1\,, p \to 0$$



Boundary Layer Theory

Solving the Boundary Layer Equations

- With the boundary layer equations, we can solve viscous flow problems by:
  1. Computing the pressure field dp/dx around the body using inviscid methods (e. g., potential flow).
  2. Computing the viscous flow field near the walls using the boundary layer equations.
- For the boundary layer equations, the following boundary conditions must be satisfied:

$$u(x,0) = 0$$
$$v(x,0) = 0$$

No-slip BC

$$u(x,y) \rightarrow V_\infty \;\; as\, y \rightarrow \infty$$

Freestream conditions

For a flat plate, the inviscid flow is uniform, and thus $\frac{\partial p^*}{\partial x^*} = 0$. This simplifies the equations, as will be highlighted in the Blasius Solution.