Lecture # 39: Laminar Boundary Layer Flows

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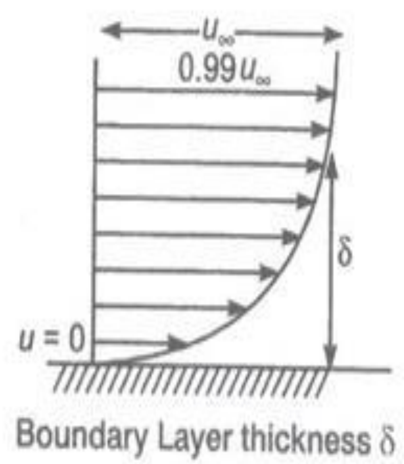
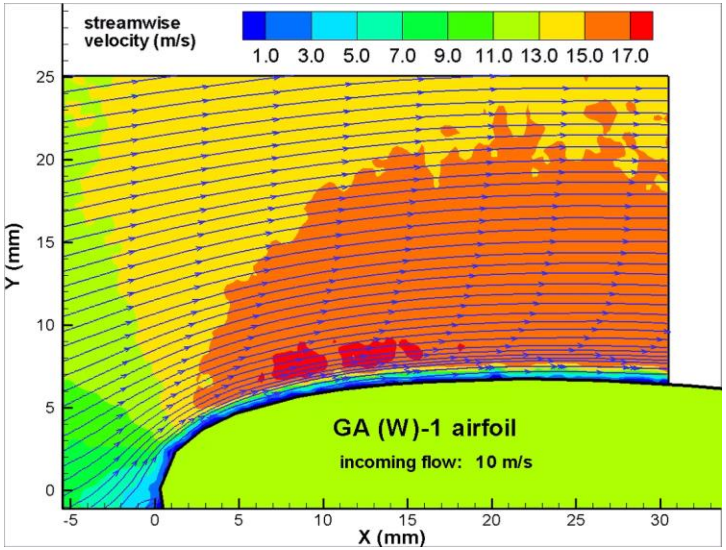
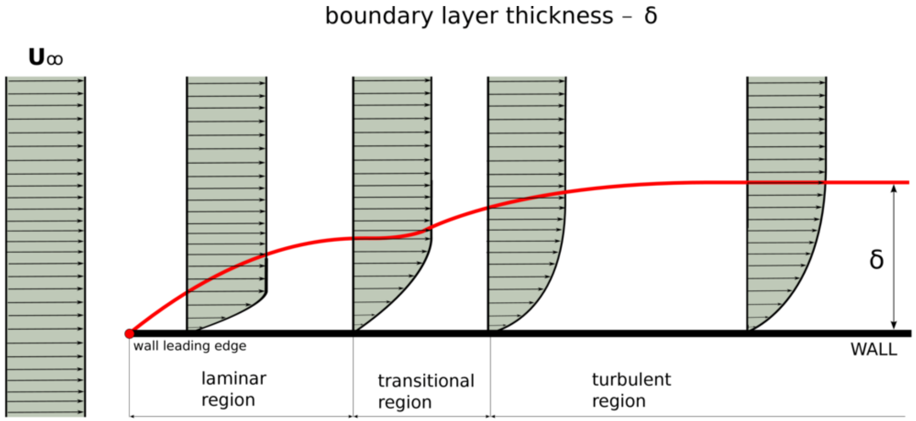
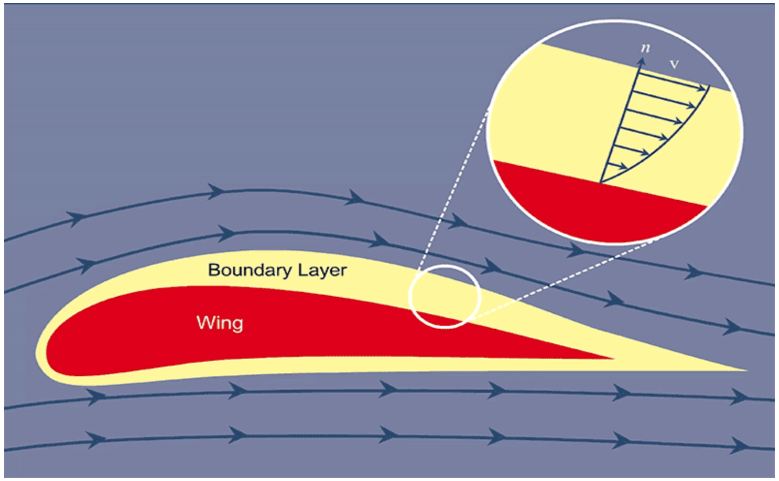
❑ Boundary Layer Flows

- *Viscous effects are limited to small region (thickness~δ) around the surface.*
- *Boundary layer thickness is defined at distance above the surface where velocity has reached 99% of the external flow.*









❑ Laminar Boundary Layer Flows

- ***Boundary layer equations - in non-dimensional form.***

- *x-momentum:* $$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\frac{\partial^2 u}{\partial y^2}$$

- *y-momentum:* $$\frac{\partial p}{\partial y} = 0$$

- *Continuity:* $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$Bernoulli's \quad equation: \frac{p}{\rho} + \frac{U^2}{2} = const.$$

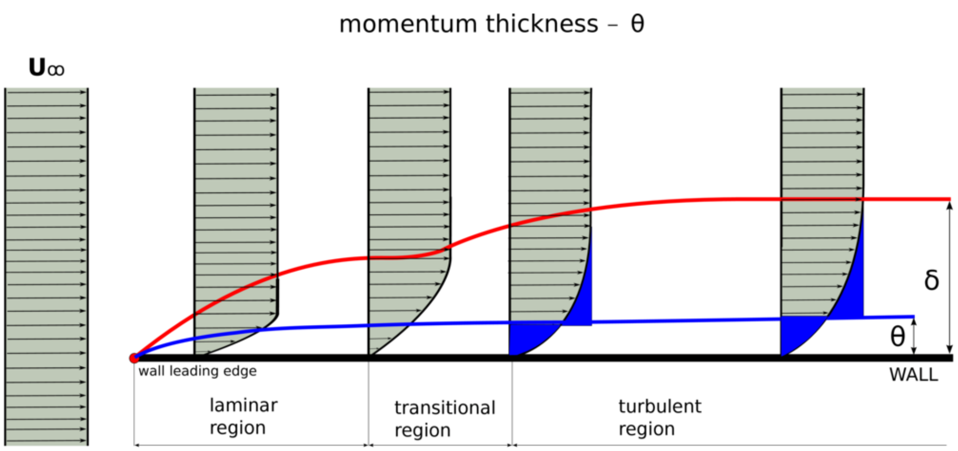
$$\Rightarrow \frac{1}{\rho}\frac{\partial p}{\partial x} = -U\frac{\partial U}{\partial x}$$

- *Boundary conditions:*
  - *No-slip at the wall*

    $$y = 0 \Rightarrow u = v = 0$$

  - *Known far-field flow velocity*

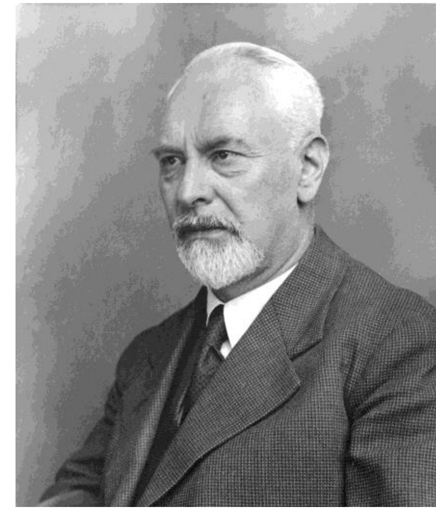
    $$y \rightarrow \infty \Rightarrow u \rightarrow 1\,, p \rightarrow 0$$



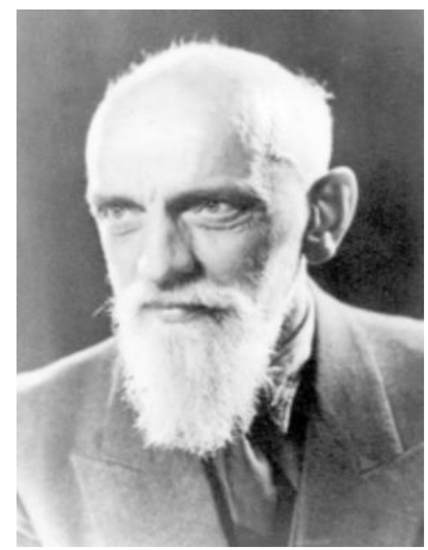
❑ Laminar Boundary Layer Flow – Blasius Solution

❑ The Blasius Solution

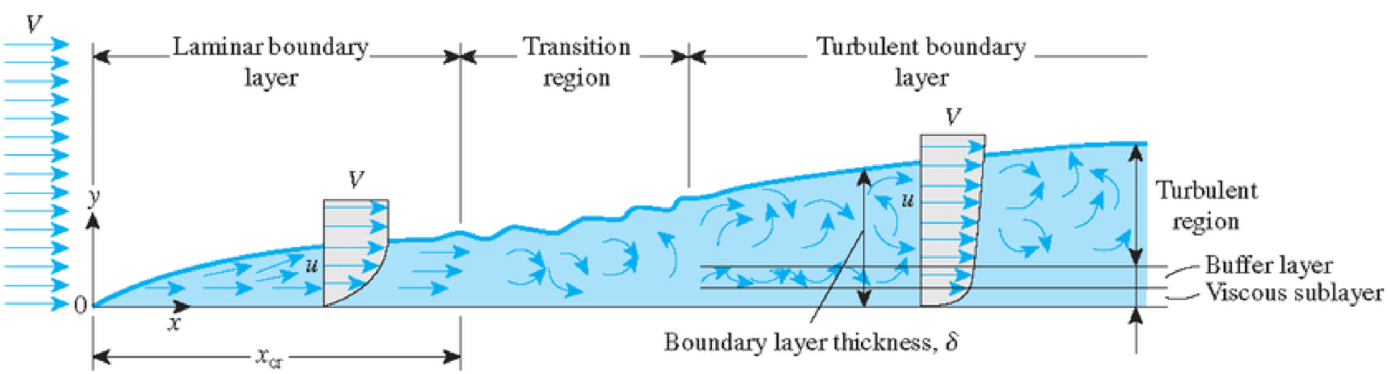
- ***The first person to obtain a solution to boundary layer equations was Paul RH Blasius (1883 - 1970), one of Prandtl's students at the University of Gottingen.***
- ***He developed an analytical technique for solving the equations in his Ph.D. thesis, published in 1909.***
- ***This work was groundbreaking given that general numerical solutions to the Navier-Stokes equations did not exist and most analytical solutions assumed ideal, inviscid flow.***
- ***We will present Blasius' basic analysis for a flat plate, and then provide the essential results, including correlations for boundary layer thickness, displacement thickness and skin friction.***
- ***His solution approach was later extended by Falkner and Skan to include additional effects and configurations.***



Ludwig Prandtl (1875-1953)



Paul Blasius (1883 - 1970)



❑ Laminar Boundary Layer Flow – Blasius Solution

Flat Plate Boundary Layer Governing Equations

- **Zero pressure gradient:** $\Rightarrow \dfrac{\partial p}{\partial x} = 0$

- The core of Blasius' analysis centers upon transforming the partial differential equations (PDEs) which comprise the flat plate boundary layer equations, with zero pressure gradient, into a single ordinary differential equation (ODE) by using a coordinate transformation approach.

- As noted previously, for a flat plate the boundary layer equation reduces to the pair of PDEs, which we will attempt to solve for the velocity components, u and v.

- For this analysis, it is assumed that the properties are known and hence we know the Reynolds Number, Re_x.

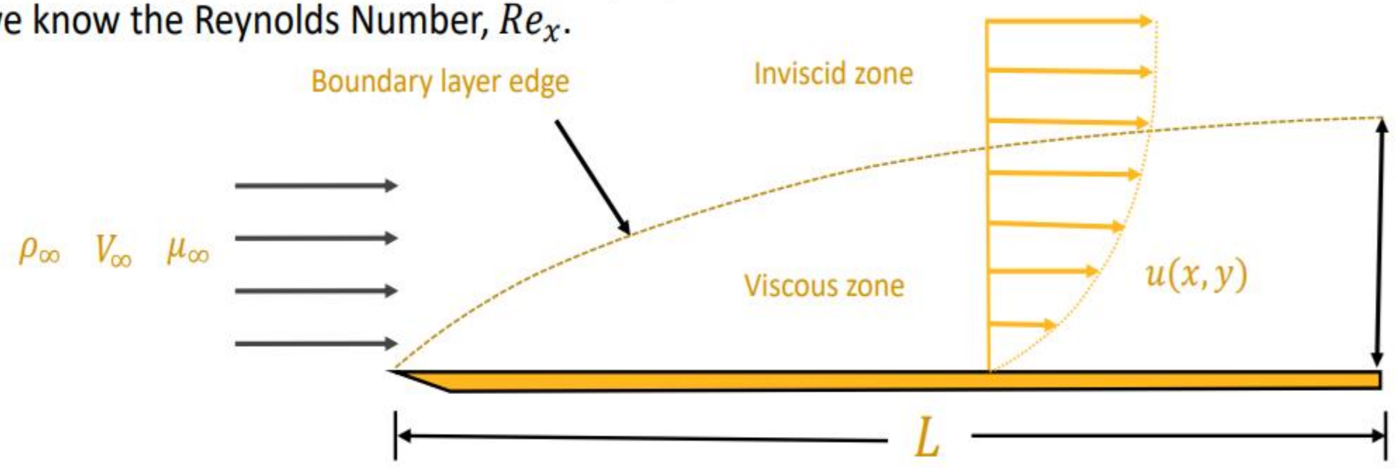
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

$$u(x,0) = 0$$

$$v(x,0) = 0$$

$$u(x,y) \rightarrow V_\infty \quad as\, y \rightarrow \infty$$



$$\delta(x) \sim \frac{x}{\sqrt{Re_x}}$$

$$Re_x = \frac{\rho_\infty V_\infty x}{\mu_\infty}$$

❑ Laminar Boundary Layer Flow – Blasius Solution

❑ *Laminar boundary layer-Blasius solution*

For a uniform flow over a flat plate, the streamlines are parallel to the plate and

$$\frac{\partial p}{\partial x} = 0$$

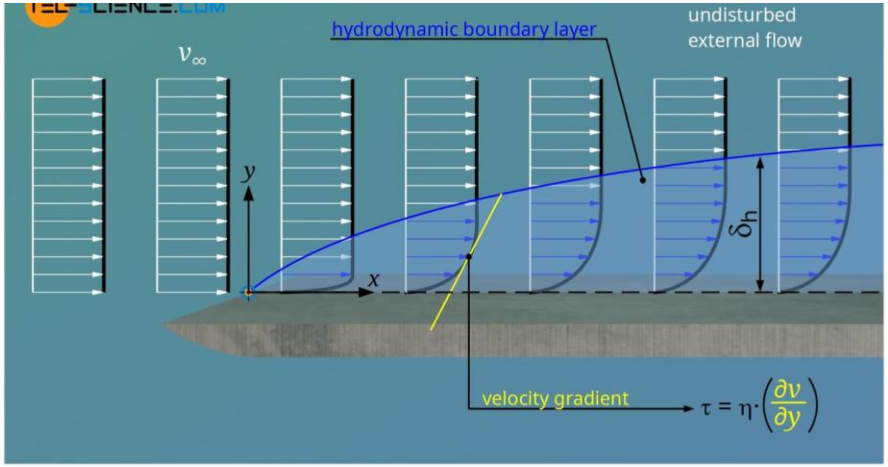
- x-momentum boundary layer equation (in the dimensional form) is then

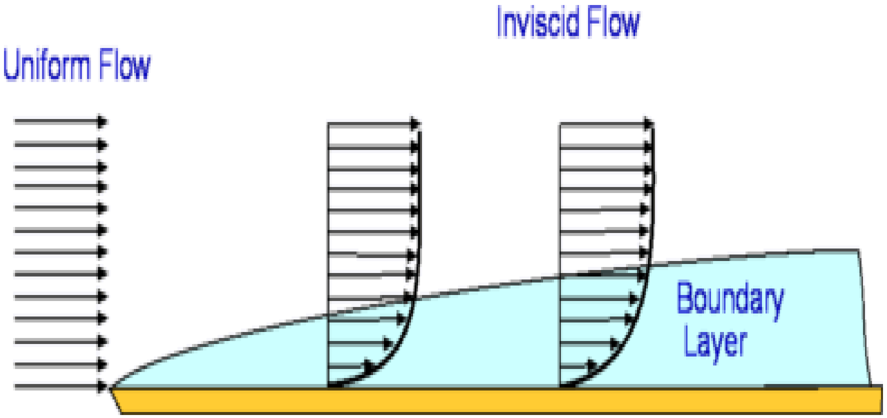
$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

Note $\nu = \mu/\rho$ is kinematic viscosity.

We are looking for a similarity solution in the form

$$\frac{u}{V_\infty} = F\left(\frac{y}{\delta}\right)$$

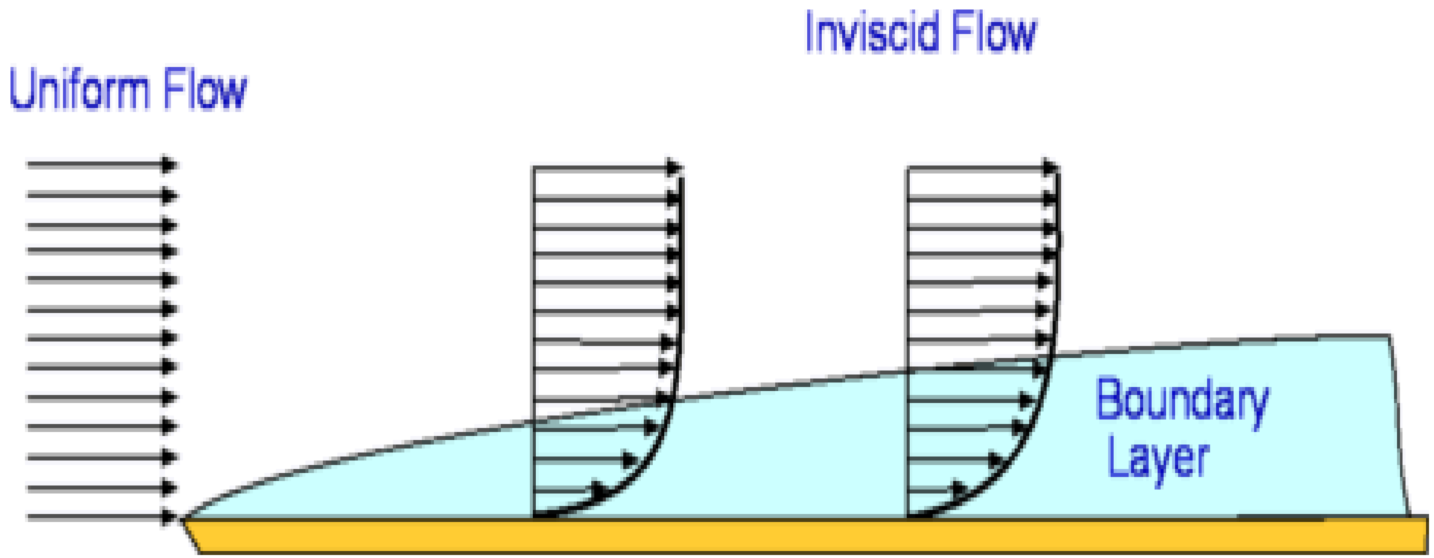




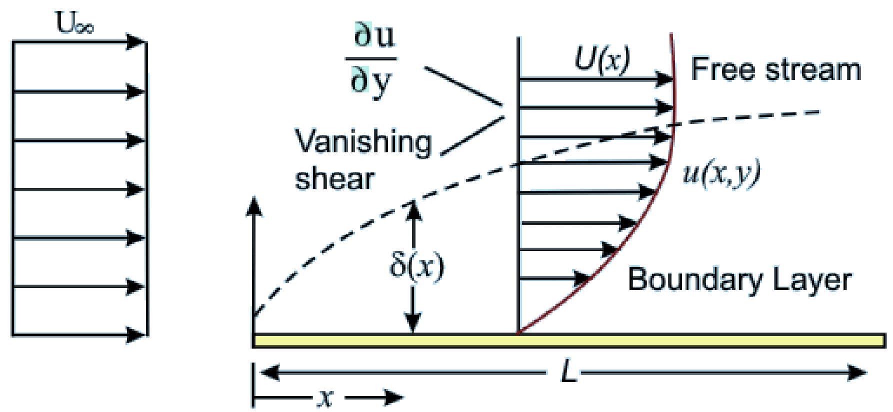
Laminar Boundary Layer Flow – Blasius Solution

- It is a solution where the form(shape) does not change in length or time scale.
- we expect that $\delta \sim \sqrt{\frac{2\nu x}{V_\infty}}$, therefore a transformation is defined as

$$\boxed{\delta \approx \sqrt{\frac{2\nu x}{V_\infty}} \quad \Rightarrow \quad \eta = \frac{y}{\delta} = y\sqrt{\frac{V_\infty}{2\nu x}}}$$



Laminar Boundary Layer Flow – Blasius Solution



Introducing the stream function

- Consider the stream function

$$u = \frac{\partial \psi}{\partial y}$$

We are looking for $\frac{u}{V_\infty} = F(\eta)$, therefore

$$\frac{\partial \psi}{\partial y} = V_\infty F(\eta) \Rightarrow \psi = \int V_\infty F(\eta) dy = \int V_\infty F(\eta) \delta d\eta$$

$$\Rightarrow \psi = V_\infty \delta \int F(\eta) d\eta = V_\infty \delta f(\eta)$$

$$\boxed{\delta \approx \sqrt{\frac{2\nu x}{V_\infty}} \quad \Rightarrow \quad \eta = \frac{y}{\delta} = y\sqrt{\frac{V_\infty}{2\nu x}}}$$

$$\boxed{f(\eta) = \int F(\eta) d\eta}$$

$$\psi = \sqrt{2\nu x V_\infty} f(\eta)$$

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left(\sqrt{2\nu x V_\infty} f(\eta)\right) = \sqrt{2\nu x V_\infty} \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} = V_\infty f'$$

$$f'(\eta) = \frac{u}{V_\infty}$$

$$\boxed{f'(\eta) = \frac{\partial f(\eta)}{\partial y} = \frac{u}{V_\infty}}$$

Laminar Boundary Layer Flow – Blasius Solution

- *x-momentum:* $$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{Re}\frac{\partial^2 u}{\partial y^2}$$

- *Zero-pressure gradient assumption:* $$\Rightarrow \frac{\partial p}{\partial x}=0$$

$$\eta=\frac{y}{\delta}=y\sqrt{\frac{V_\infty}{2\nu x}}$$

$$u(x)=V_\infty f'$$

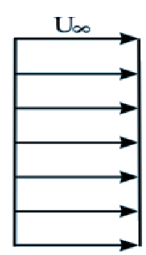
$$v(x)=\sqrt{\frac{\nu V_\infty}{2x}}(\eta f'-f)$$

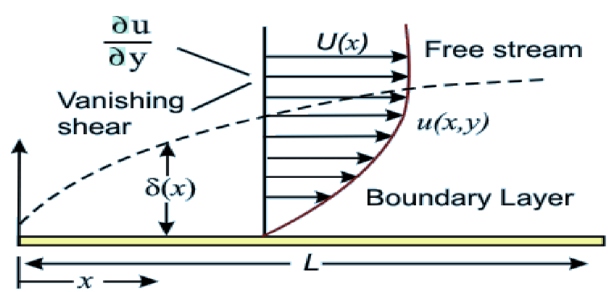
$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=+\nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\frac{\partial u}{\partial x}=-\frac{U}{2x}\eta\frac{d^2 f}{d\eta^2}\qquad \frac{\partial u}{\partial y}=U\sqrt{U/\nu x}\,\frac{d^2 f}{d\eta^2}\qquad \frac{\partial^2 u}{\partial y^2}=\frac{U^2}{\nu x}\frac{d^3 f}{d\eta^3}$$

$$2\frac{d^3 f}{d\eta^3}+f\frac{d^2 f}{d\eta^2}=2f'''+ff''=0$$





Laminar Boundary Layer Flow – Blasius Solution

- *ODE form of the BL equation :*

$$\boxed{2\frac{d^3 f}{d\eta^3} + f\,\frac{d^2 f}{d\eta^2} = 0 \quad or \quad 2f''' + f \cdot f'' = 0}$$

boundary conditions:

No slip ($u = 0$ @$y = 0$) $\Rightarrow f'(0) = 0$

No slip ($v = 0$ @$y = 0$) $\Rightarrow f(0) = 0$

Far field free stream ($u = V_\infty$ @$y \to \infty$) $\Rightarrow f'(\infty) = 1$

Solution Methods for the Blasius Equation

- Blasius used an analytical series solution technique to solve his equations in his original work. With the availability of computers, we can now develop a numerical solution and calculate it with a high degree of accuracy.
- The numerical method for calculating the solution involves casting the Blasius equation as a set of coupled first order ODEs.

$$f'(\eta) = g(\eta)$$
$$g'(\eta) = f''(\eta) = h(\eta)$$
$$h'(\eta) = f'''(\eta) = -\frac{1}{2}f(\eta)h(\eta)$$

The boundary conditions give us $f(0) = f'(0) = g(0) = 0$, but we don't have a value for $h(0)$. We can employ an iterative approach called the "shooting algorithm" to determine which initial condition for h yields the asymptotic condition $f'(\eta \to \infty) = 1$. The algorithm proceeds as follows:

1. Guess a value for $h(0)$.
2. Numerically solve the coupled ODEs by marching in η, and halt when $f'(\eta) = g(\eta)$ has stopped changing (within a small error tolerance). Note the value of $f'(\eta)$. If it is > 1.0 then decrease $h(0)$, otherwise increase $h(0)$.
3. With the updated $h(0)$ (increased or decreased) repeat 2. Continue until $f'(\eta)$ is sufficiently close to 1.0.

Laminar Boundary Layer Flow – Blasius Solution

- ***ODE form of the BL equation :***

$$2\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} = 0 \quad or \quad 2f''' + f \cdot f'' = 0$$

boundary conditions:

No slip $(u = 0 \; @y = 0) \Rightarrow f'(0) = 0$

No slip $(v = 0 \; @y = 0) \Rightarrow f(0) = 0$

Far field free stream $(u = V_\infty \; @y \to \infty) \Rightarrow f'(\infty) = 1$

η	f	f'	f''
0.00	0.00000	0.00000	0.33200
0.20	0.00664	0.06641	0.33193
0.40	0.02656	0.13277	0.33142
0.60	0.05974	0.19894	0.33003
0.80	0.10611	0.26472	0.32735
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3.60	1.93029	0.92370	0.09802
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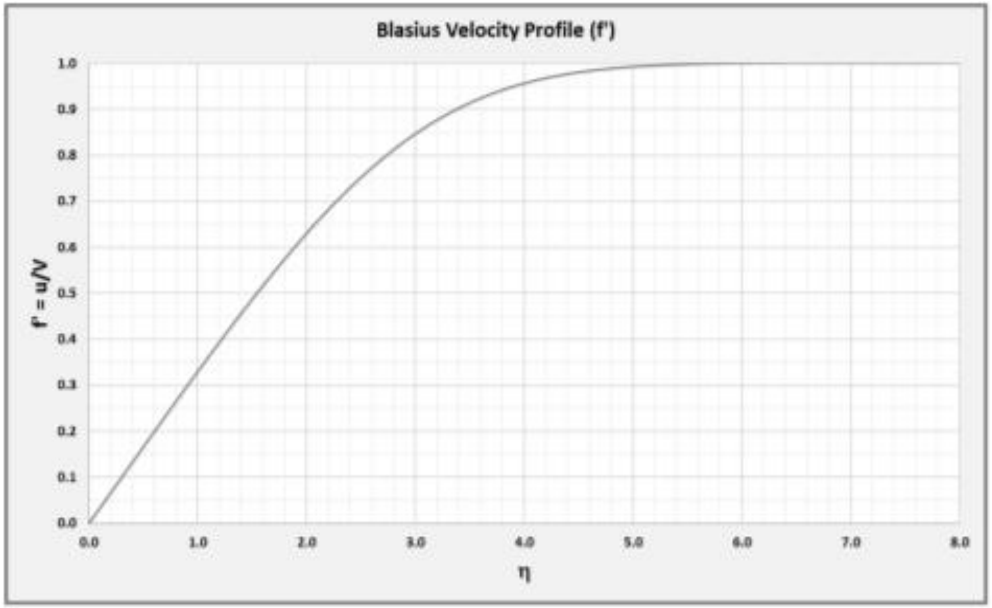
6.00	4.28206	0.99936	0.00233

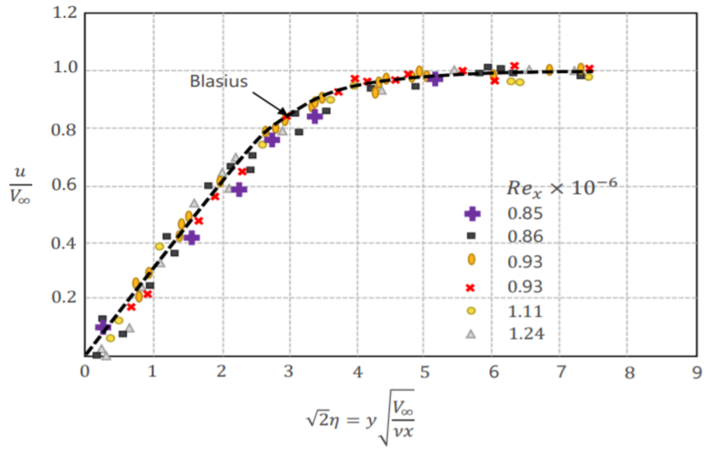


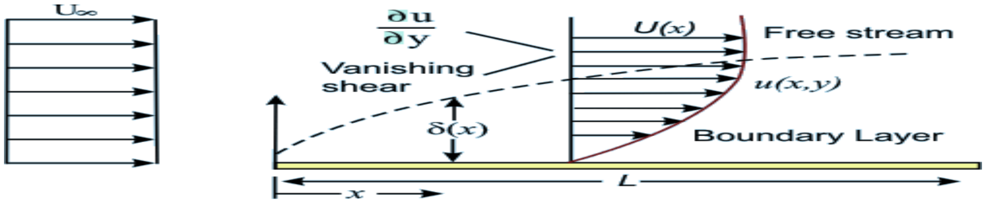
- Blasius solution calculated using a spreadsheet
- ODEs integrated numerically using Euler Predictor-Corrector method
- $f''(0) = \text{h}(0)$ determined by trial and error ("shooting method") as 0.332.

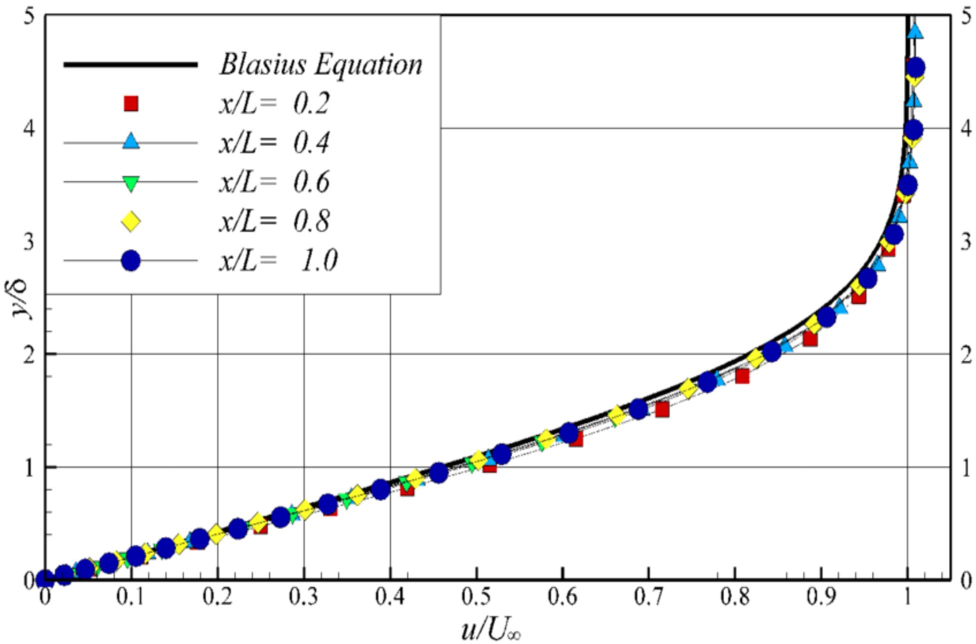
❑ Laminar Boundary Layer Flow – Blasius Solution

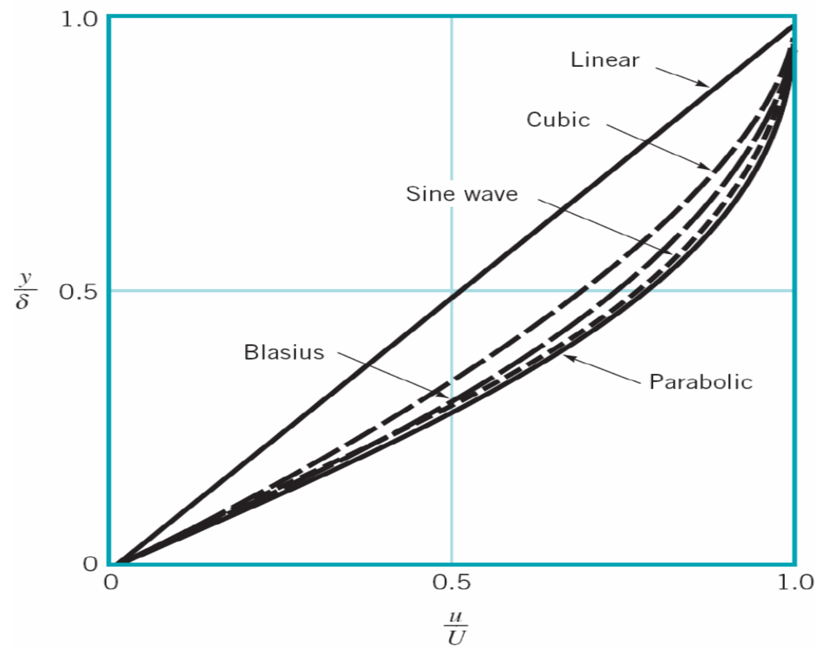
- BL approximation works very well, except very close to the leading edge where Re_x is small.

$$\boxed{u(x) = V_\infty f'}$$









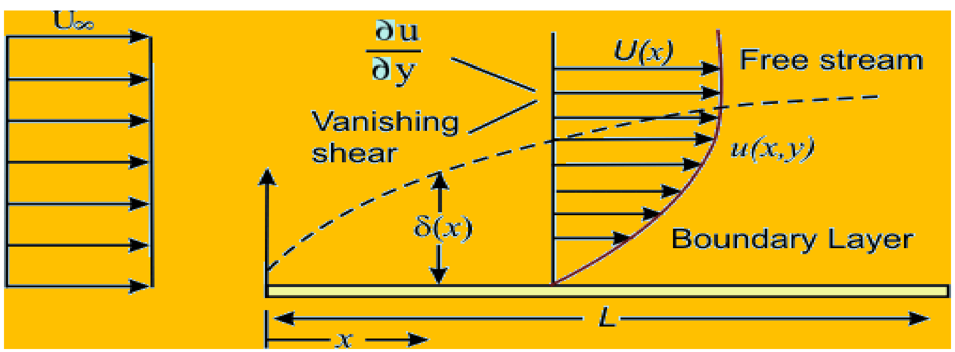
❑ Laminar Boundary Layer Flow – Blasius Solution

- ***Boundary layer thickness is obtained by noting from the tabulated results that $u(x) = f'V_\infty = 0.99\ V_\infty$ at $\eta \approx 5.0$.***

$$\boxed{\frac{u(x)}{V_\infty} = f' = 0.99 \Rightarrow \eta \approx 5.0}$$

$$\eta = \frac{y}{\delta} = y\sqrt{\frac{V_\infty}{2\nu x}} = 5.0$$

$$\Rightarrow \frac{\delta(x)}{x} = \frac{5.0}{\sqrt{\frac{V_\infty x}{\nu}}} = \frac{5.0}{\sqrt{\mathrm{Re}_x}}$$



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❑ Laminar Boundary Layer Flow – Blasius Solution

- The displacement and momentum thicknesses can be integrated (numerically) from the velocity profile:

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{V_\infty}\right) dy = \sqrt{\frac{\nu x}{V_\infty}} \int_0^\infty (1 - f')\, d\eta = 1.72 \sqrt{\frac{\nu x}{V_\infty}}$$

$$\theta = \int_0^\infty \frac{u}{V_\infty}\left(1 - \frac{u}{V_\infty}\right) dy = \sqrt{\frac{\nu x}{V_\infty}} \int_0^\infty f'(1 - f')\, d\eta = 0.664 \sqrt{\frac{\nu x}{V_\infty}}$$

or

$$\frac{\delta^*(x)}{x} = \frac{1.72}{\sqrt{Re_x}} \qquad \frac{\theta(x)}{x} = \frac{0.664}{\sqrt{Re_x}}$$

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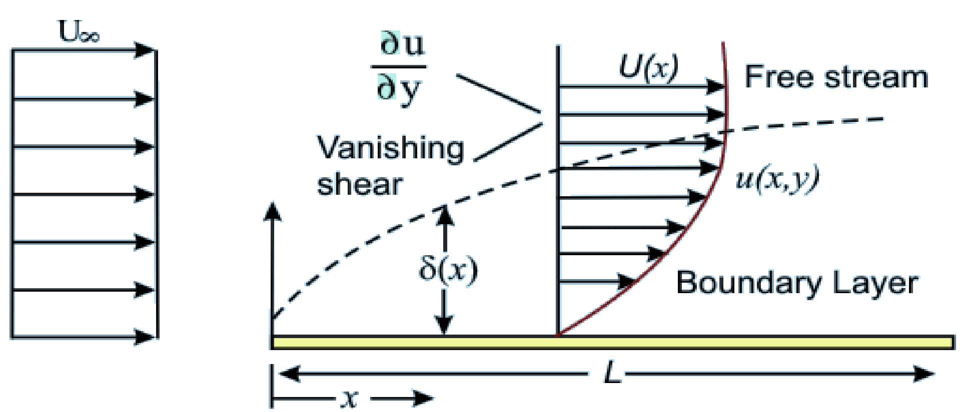
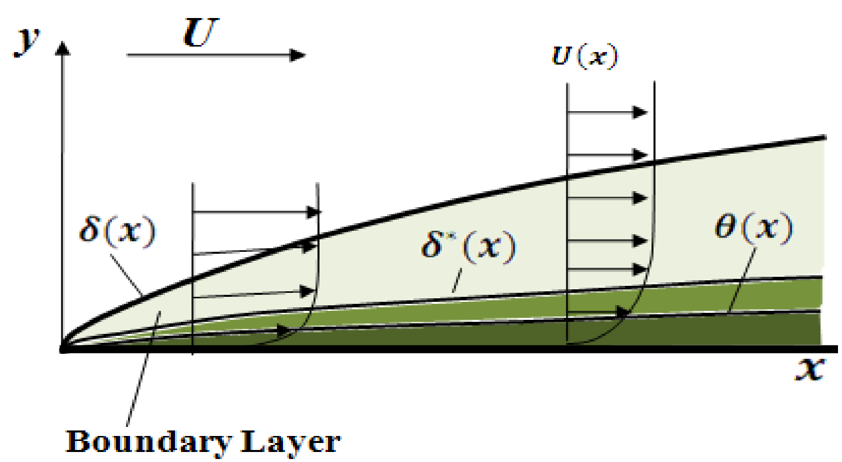




Laminar Boundary Layer Flow – Blasius Solution

Shear stress and skin friction

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \mu V_\infty \sqrt{\frac{V_\infty}{2\nu x}} f''(0)$$

- Wall shear stress is obtained from the velocity gradient at the wall $f''(0) = 0.332$:

$$\tau_W = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = 0.332 \frac{\mu V_\infty}{\sqrt{\nu x / V_\infty}}$$

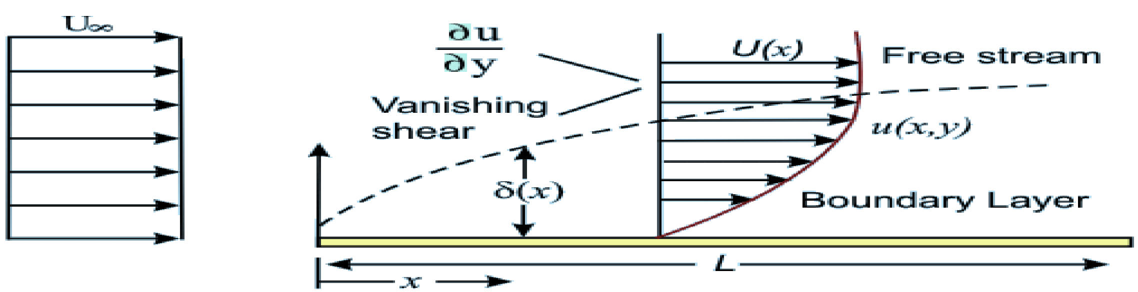
- The friction coefficient C_f can be computed as follows:

$$C_f = \frac{\tau_W}{\frac{1}{2}\rho_\infty V_\infty^2} = \frac{0.664}{\sqrt{Re_x}} = \frac{\theta}{x}$$

- The drag coefficient on a plate of length L is:

$$C_D(L) = \frac{1}{L}\int_0^L C_f \, dx = 2C_f(L) = \frac{1.328}{\sqrt{Re_L}}$$

drag on one side of the plate



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2.80	1.23132	0.81178	0.18400
3.00	1.39724	0.84634	0.16134
3.20	1.56963	0.87641	0.13909
3.40	1.74759	0.90211	0.11782
3.60	1.93029	0.92370	0.09802
3.80	2.11692	0.94151	0.08004
4.00	2.30676	0.95592	0.06414
4.20	2.49919	0.96736	0.05042
4.40	2.69365	0.97628	0.03887
4.60	2.88968	0.98309	0.02938
4.80	3.08689	0.98819	0.02177
5.00	3.28499	0.99194	0.01580
5.20	3.48373	0.99464	0.01124
5.40	3.68292	0.99655	0.00784
5.60	3.88244	0.99787	0.00535
5.80	4.08217	0.99876	0.00357
6.00	4.28206	0.99936	0.00233

Laminar Boundary Layer Flow – Blasius Solution

- Finally, let us compare results from the integral approach with those from the exact Blasius solution:

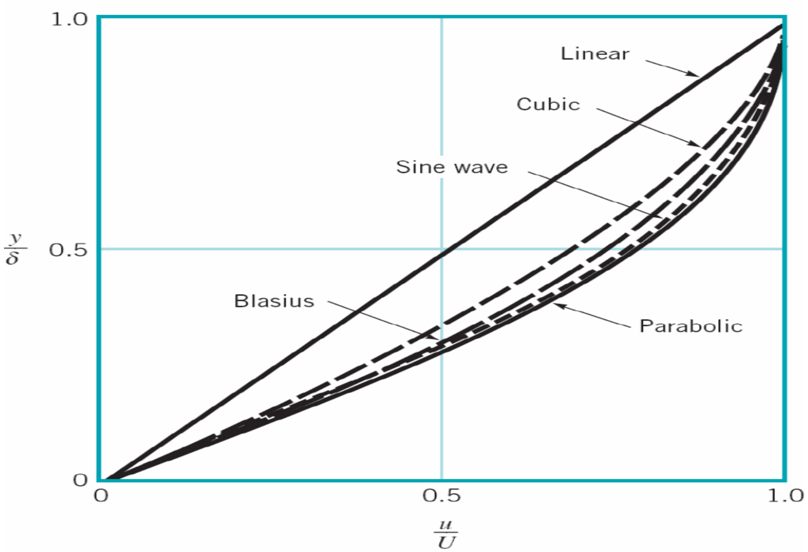
$$\frac{\delta}{x} = \frac{5.48}{\sqrt{Re_x}} \qquad \frac{\delta^*}{x} = \frac{1.83}{\sqrt{Re_x}} \qquad \frac{\theta}{x} = \frac{0.73}{\sqrt{Re_x}} \qquad C_D = \frac{1.46}{\sqrt{Re_L}}$$

Approximate solution from integral analysis

$$\frac{\delta}{x} = \frac{5.0}{\sqrt{Re_x}} \qquad \frac{\delta^*}{x} = \frac{1.72}{\sqrt{Re_x}} \qquad \frac{\theta}{x} = \frac{0.664}{\sqrt{Re_x}} \qquad C_D = \frac{1.328}{\sqrt{Re_L}}$$

Exact Blasius solution

- As was stated earlier, the integral analysis values are within 10% of the exact solution. Unlike the Blasius solution they were obtained without complicated math by examining the boundary layer flow from the first principles of conservation laws.



Flat Plate Momentum Integral Results for Various Assumed Laminar Flow Velocity Profiles

Profile Character	$\delta Re_x^{1/2}/x$	$c_f Re_x^{1/2}$	$C_{Df} Re_\ell^{1/2}$
a. Blasius solution	5.00	0.664	1.328
b. Linear $u/U = y/\delta$	3.46	0.578	1.156
c. Parabolic $u/U = 2y/\delta - (y/\delta)^2$	5.48	0.730	1.460
d. Cubic $u/U = 3(y/\delta)/2 - (y/\delta)^3/2$	4.64	0.646	1.292
e. Sine wave $u/U = \sin[\pi(y/\delta)/2]$	4.79	0.655	1.310

Laminar Boundary Layer Flow – Blasius Solution

❑ Laminar Boundary Layer Flows

❖ *Example problem:*

A small airplane flies at speed of 90 m/s at 1000 m altitude. The airplane wing has rectangular shape with chord length of 1m and span of 11m. Assume boundary layer over the wing surface is fully laminar and model the wing airfoil as a flat plate.

a) Estimate the boundary layer thickness at the trailing edge.

b) Estimate the displacement thickness at the trailing edge.

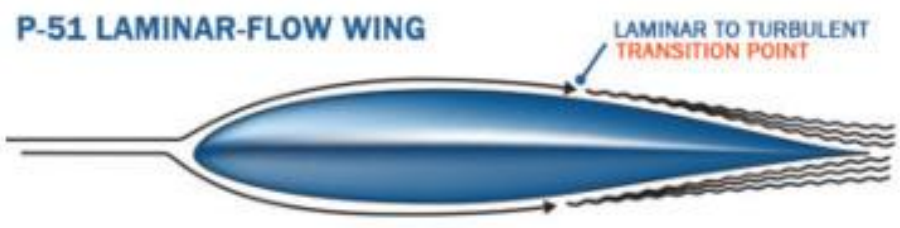
c) estimate the friction drag of the wing.

Laminar Boundary Layer Flows

a)

Reynolds number at trailing edge where $x = c = 1\,m$

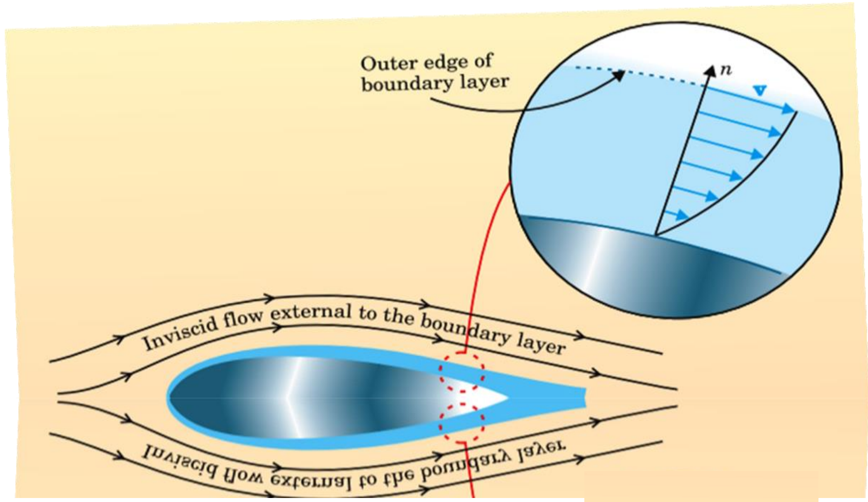
$$Re_c = \frac{\rho_\infty V_\infty c}{\mu_\infty} = \frac{1.112 \times 90 \times 1}{1.78 \times 10^{-5}} = 5.62 \times 10^6$$



Boundary layer thickness at chord length then

$$\frac{\delta_c}{c} = \frac{5}{\sqrt{Re_c}} = \frac{5}{\sqrt{5.62 \times 10^6}} = 0.0021$$

$$\delta_c = 2.1\,mm$$



b)

displacement thickness at chord length

$$\frac{\delta_c^*}{c} = \frac{1.72}{\sqrt{Re_c}} = \frac{1.72}{\sqrt{5.62 \times 10^6}} = 7.25 \times 10^{-4}$$

$$\delta_c^* = 0.0725\,mm$$

c)

drag coefficient assuming a fully laminar boundary layer over the wing:

$$C_D = \frac{1.328}{\sqrt{Re_c}} = \frac{1.328}{\sqrt{5.62 \times 10^6}} = 5.60 \times 10^{-4}$$

To calculate the friction drag force, note that there are boundary layers on both top and bottom surface of the wing, hence multiply the area by 2:

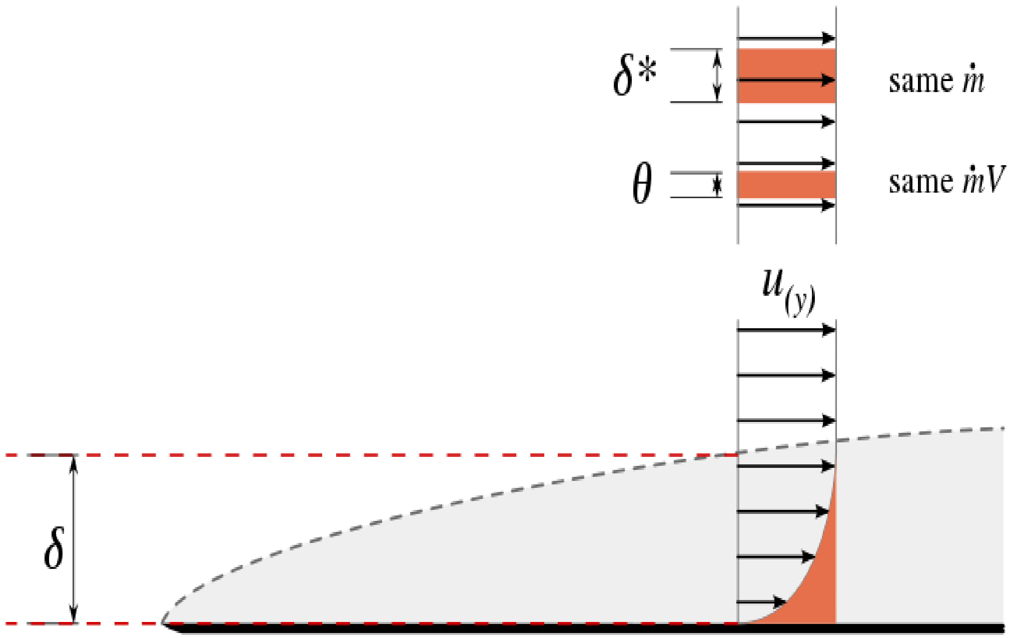
$$D_f = \frac{1}{2}\rho_\infty V_\infty^2 S C_D = \frac{1}{2} \times 1.112 \times 90^2 \times (11 \times 1 \times 2) \times 5.6 \times 10^{-4}$$

$$D_f = 55.5\,N$$

❑ Boundary Layer Flows

- ***Example #1***

- ***Problem 01 : Determine the displacement thickness, momentum thickness, shape factor and friction coefficient for the Blasius Boundary Layer.***



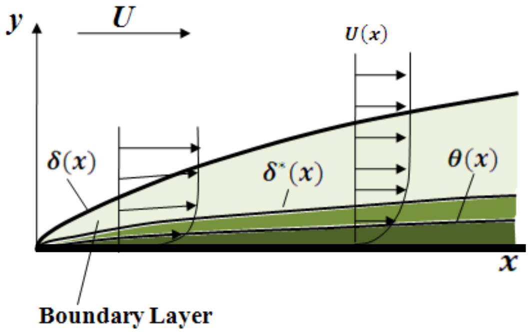
❑ Boundary Layer Flows

- ## *Example #1*

- ***Problem 01* : Determine the displacement thickness, momentum thickness, shape factor and friction coefficient for the Blasius Boundary Layer.**

From the Blasius solution we get

$$\eta(x,y) = \frac{y}{\sqrt{2\frac{\nu x}{U_\infty}}} \quad , \quad \frac{u}{U}(x,y) = f'[\eta(x,y)]$$



Hence

$$\delta_*(x) = \int_0^\infty \{1 - f'[\eta(x,y)]\}dy = \sqrt{\frac{2\nu x}{U}}\int_0^\infty [1 - f'(\eta)]d\eta \approx 1.721\sqrt{\frac{\nu x}{U}}$$

meaning that

$$\frac{\delta_*(x)}{x} \approx 1.721\sqrt{\frac{\nu}{Ux}} = \frac{1.721}{\sqrt{\mathrm{Re}_x}}$$

Next …

$$\theta(x) = \int_0^\infty f'[\eta(x,y)]\{1 - f'[\eta(x,y)]\}dy = \sqrt{\frac{2\nu x}{U}}\int_0^\infty f'(\eta)[1 - f'(\eta)]d\eta =$$

$$= \sqrt{\frac{2\nu x}{U}} f''(0) \approx 0.664\sqrt{\frac{\nu x}{U}}$$

so

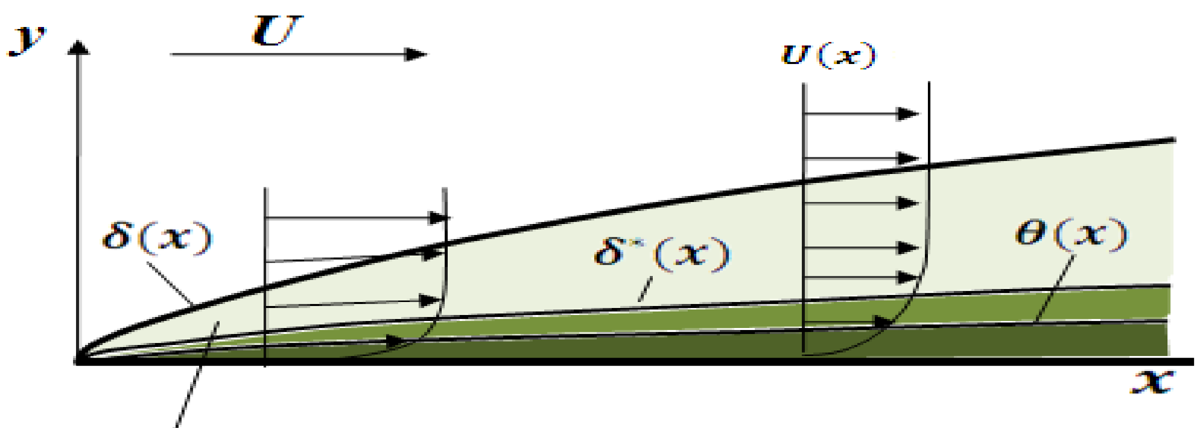
$$\frac{\theta(x)}{x} \approx 0.664\sqrt{\frac{\nu}{Ux}} = \frac{0.664}{\sqrt{\mathrm{Re}_x}} \quad , \quad H \approx 2.592 \text{ (constant – self-similarity!)}$$

❑ Boundary Layer Flows

Other parameters are

- wall shear stress $$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \frac{\mu U f''(0)}{\sqrt{2\nu x/U}}$$
- local friction coefficient $$C_f = \frac{2\tau_w}{\rho U^2} \approx \frac{\sqrt{2}f''(0)}{\sqrt{\mathrm{Re}_x}} = \frac{0,664}{\sqrt{\mathrm{Re}_x}} = \frac{\theta}{x}$$
- (global) friction coefficient for the flat plate having the length L

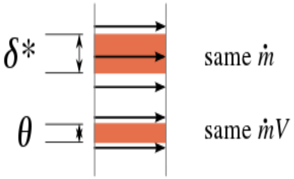
$$C_D(L) = \frac{1}{L}\int_0^L C_f(x)dx = \frac{0.664}{L}\sqrt{\frac{\nu}{U}}\int_0^L x^{-1/2}dx = \frac{0.664}{L}\sqrt{\frac{\nu}{U}}2\sqrt{L} = 2C_f(L) = \frac{1.328}{\sqrt{\mathrm{Re}_x}}$$



❑ Laminar Boundary Layer Flows

- ***Boundary layer thickness ($0.99V_\infty$)***

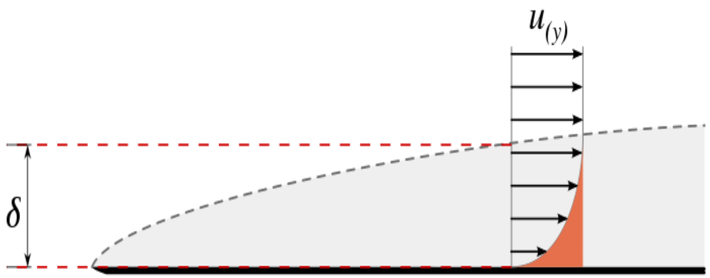
$$\frac{\delta}{x} = \frac{4.95}{\sqrt{\frac{V_\infty x}{\nu}}} \approx \frac{5}{\sqrt{\mathrm{Re}_x}}$$



Displacement and momentum thickness

- Displacement thickness

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{V_\infty}\right) dy = \sqrt{\frac{2\nu x}{V_\infty}} \int_0^\infty (1 - f') d\eta$$

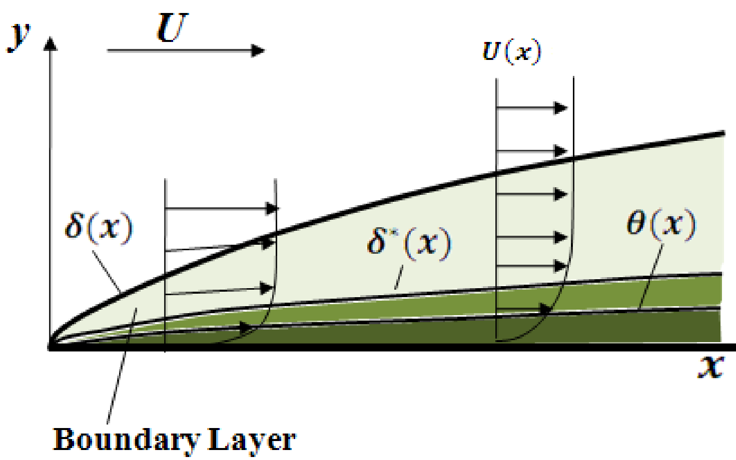


$$\frac{\delta^*}{x} = \frac{1.72}{\sqrt{Re_x}}$$

- Momentum thickness

$$\theta = \int_0^\infty \frac{u}{V_\infty}\left(1 - \frac{u}{V_\infty}\right) dy = \sqrt{\frac{2\nu x}{V_\infty}} \int_0^\infty f'(1 - f') d\eta$$

$$\boxed{\frac{\theta}{x} = \frac{0.664}{\sqrt{Re_x}}}$$



Laminar Boundary Layer Flows

Wall shear stress-Flat plate drag

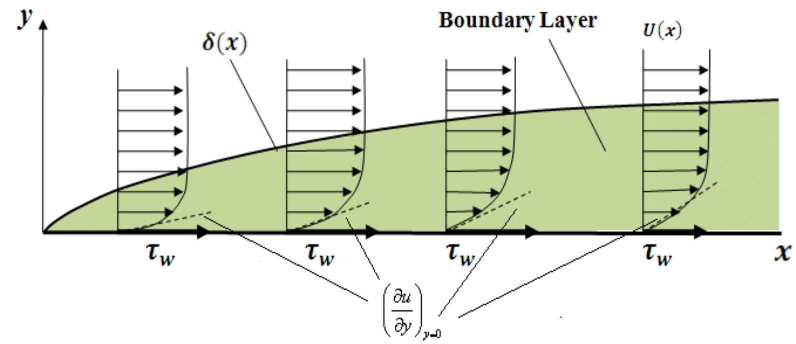
$$\eta = \frac{y}{\delta} = y\sqrt{\frac{V_\infty}{2\nu x}}$$

$$f' = \frac{df(\eta)}{dx} = \frac{u}{V_\infty}$$

Shear stress and skin friction

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \mu V_\infty \sqrt{\frac{V_\infty}{2\nu x}} f''(0)$$

$$C_f = \frac{\tau_w}{1/2\rho V_\infty^2} = \frac{0.664}{\sqrt{Re_x}}$$



Drag coefficient (for a plate with length L)

$$D = \int_0^L \tau_w dx \Rightarrow C_D = \frac{1}{1/2\rho V_\infty^2 L}\int_0^L \tau_w dx = \frac{1}{L}\int_0^L C_f dx$$

$$C_D = \frac{1.328}{\sqrt{Re}}$$

Note $D \sim V_\infty^{3/2} \times L^{1/2}$

Laminar Boundary Layer Flow – Blasius Solution

- ***ODE form of the BL equation :***

$$\boxed{2\frac{d^3 f}{d\eta^3} + f\,\frac{d^2 f}{d\eta^2} = 0}$$

$$\boxed{\eta = \frac{y}{\delta} = y\sqrt{\frac{V_\infty}{2\nu x}}}$$

$$\boxed{u(x) = V_\infty f'}$$

$$\boxed{v(x) = \sqrt{\frac{\nu V_\infty}{2x}}(\eta f' - f)}$$

boundary conditions:

No slip ($u = 0$ @$y = 0$) $\Rightarrow f'(0) = 0$

No slip ($v = 0$ @$y = 0$) $\Rightarrow f(0) = 0$

Far field free stream ($u = V_\infty$ @$y \to \infty$) $\Rightarrow f'(\infty) = 1$

- ***This is a third order non-linear <u>ordinary differential equation</u>. It can be solved numerically, e.g., with the <u>shooting method</u>.***

  - ***The limiting form for small η <<1 is:***

$$\boxed{f(\eta) = \frac{1}{2}\alpha\eta^2 + O(\eta^5),}$$

$$\alpha = 0.332057336215196$$

  - ***The limiting form for large η <<1 is:***

$$\boxed{f(\eta) = \eta - \beta + O\left((\eta-\beta)^{-2} e^{-\frac{1}{2}(\eta-\beta)^2}\right)}$$

$$\beta = 1.7207876575205$$