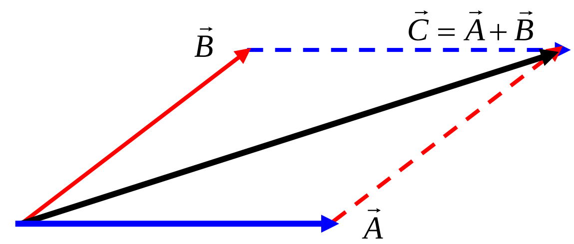
Chapter 2
Review of Vector Algebra

2.1 Definition of a Vector

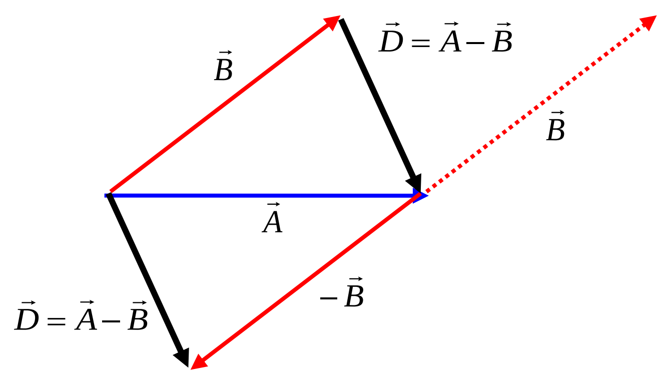
Definition: A vector is a quantity that posses both magnitude and direction, and obeys the parallelogram law of addition.

2.2 Vector Addition



$\vec{C} = \vec{A} + \vec{B}$

2.3 Vector Subtraction



$\vec{D} = \vec{A} - \vec{B}$

2.4 Properties of Vectors

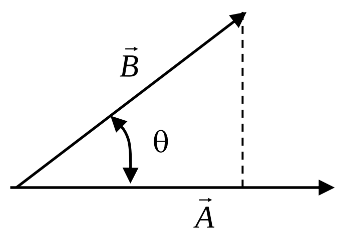
If s and t are two scalars and $\vec{A}$ and $\vec{B}$ are two vectors, then:

$$0\vec{A} = \vec{0}$$
$$+\vec{A} = \vec{A}$$
$$(-1)\vec{A} = -\vec{A}$$
$$(s+t)\vec{A} = s\vec{A} + t\vec{A}$$
$$s(\vec{A}+\vec{B}) = s\vec{A} + s\vec{B}$$
$$st(\vec{A}) = s(t\vec{A}) = t(s\vec{A})$$

Explanation: If *s* is scalar and $\vec{A}$ is vector, then $s\vec{A}$ is defined to be the vector having magnitude s times that of $\vec{A}$ and pointing in the same direction if $s>0$ and in the opposite direction if *s* is negative.

2.5 Scalar Product (Dot Product)

$\vec{A} \bullet \vec{B} = |\vec{A}||\vec{B}|\cos\theta$



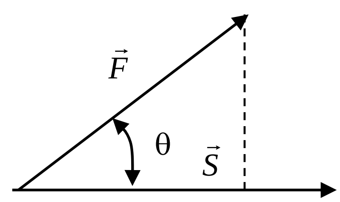
Where $|\vec{A}|, |\vec{B}|$ are the magnitude of the vectors $\vec{A}$ and $\vec{B}$.

θ ($0 \le \theta \le \pi$) is the angle between the vectors $\vec{A}$ and $\vec{B}$ when they are arranged "tail to tail".

- $|\vec{B}|\cos\theta$ is the projection of vector $\vec{B}$ to vector $\vec{A}$.
- If $\theta = \pi/2$, $\vec{A}$ and $\vec{B}$ are orthogonal to each other, and $\vec{A} \bullet \vec{B} = 0$
- Commutative: $\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}$

Example:

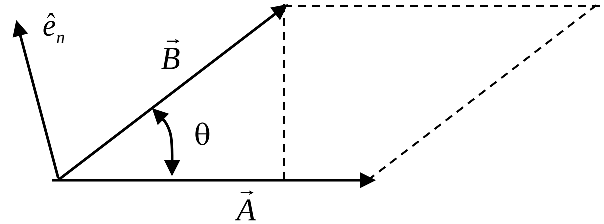
Work done by a force $\vec{F}$ during an infinitesimal displacement $\vec{S}$



2.6 Vector Product (Cross Product)

$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}|\sin\theta\, \hat{e}_n$

Where $\hat{e}_n$ is the unit vector normal to the plane containing $\vec{A}$ and $\vec{B}$. Direction is determined according to the "right-hand" rule. $0 \le \theta \le \pi$



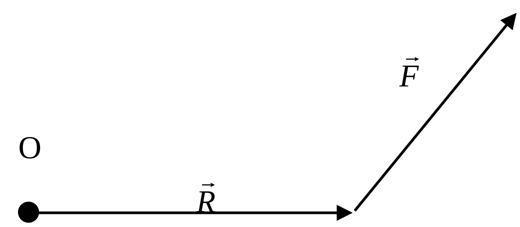
$|\vec{A} \times \vec{B}| =$ Area of the parallelogram

If the two vectors are parallel, that is if $\theta = 0$ or $\theta = \pi$, then $\vec{A} \times \vec{B} = \vec{0}$.

- Vector product is not commutative. i.e., $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$. However, $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

Application example:

Moment about *O*: $\vec{M}_O = \vec{R} \times \vec{F}$

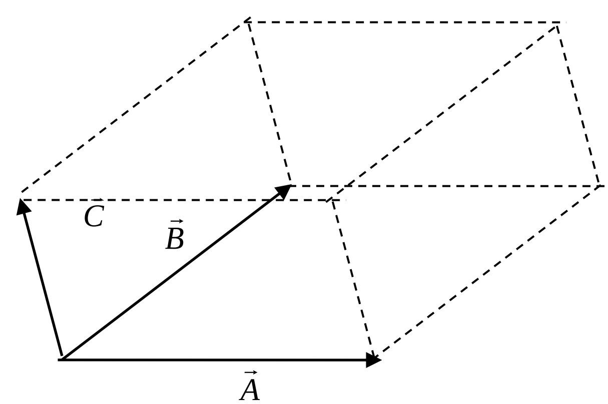


2.7 Triple Product:

2.7.1 Scalar Triple Product:

$\vec{A} \bullet (\vec{B} \times \vec{C}) = \vec{C} \bullet (\vec{A} \times \vec{B}) = \vec{B} \bullet (\vec{C} \times \vec{A})$

is the volume of the parallelepiped formed by the non-coplanar vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$.



2.7.2 Vector Triple Product:

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \bullet \vec{C})\vec{B} - (\vec{A} \bullet \vec{B})\vec{C}$$
$$= m\vec{B} - n\vec{C}$$

Where *m, n* are scalar parameters.

- $\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$

Proof:

$$(\vec{A}\times\vec{B})\times\vec{C} = -\vec{C}\times(\vec{A}\times\vec{B})$$
$$= -[(\vec{C}\bullet\vec{B})\vec{A} - (\vec{C}\bullet\vec{A})\vec{B}]$$
$$= (\vec{C}\bullet\vec{A})\vec{B} - (\vec{C}\bullet\vec{B})\vec{A}$$

Thus, vector $(\vec{A}\times\vec{B})\times\vec{C}$ is inside the plane of vectors $\vec{A}$ and $\vec{B}$, while the vector $\vec{A}\times(\vec{B}\times\vec{C})$ is inside the plane of vectors $\vec{B}$ and $\vec{C}$.

Therefore: $\vec{A}\times(\vec{B}\times\vec{C}) \neq (\vec{A}\times\vec{B})\times\vec{C}$

2.8 Unit Vector

A vector whose magnitude is 1 is called a unit vector:

$$\hat{e}_A = \frac{\vec{A}}{|\vec{A}|}$$

Where $|\vec{A}|$ is the magnitude of the vector $\vec{A}$, and $\hat{e}_A$ is a unit vector in the direction of $\vec{A}$.

2.9 Vector Differentiation

If $\vec{A}$ and $\vec{B}$ are differentiable vector , α, t are scalars, and $\vec{U} = \vec{A} + \vec{B}$, then,

$$\frac{d\vec{U}}{dt} = \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt}$$
$$\frac{d(\alpha\vec{U})}{dt} = \frac{d\alpha}{dt}\vec{U} + \alpha\frac{d\vec{U}}{dt}$$

2.10 Product Rules

$$\vec{A}\bullet\vec{A} = (|\vec{A}|)^2$$
$$\hat{e}_i \bullet \hat{e}_j = \begin{cases} 0 & if\ i \neq j \\ 1 & f\ i = j \end{cases}$$
$$|\hat{e}_i \times \hat{e}_j| = \begin{cases} 0 & if\ i = j \\ 1 & f\ i \neq j \end{cases}$$

2.11 Components of a Vector

In 3-D, a vector has 3 components. These 3 components are independent of each other. Consider three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$. In component form, these vectors in general can be written as:

$$\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$$
$$\vec{B} = B_1\hat{e}_1 + B_2\hat{e}_2 + B_3\hat{e}_3$$
$$\vec{C} = C_1\hat{e}_1 + C_2\hat{e}_2 + C_3\hat{e}_3$$

Based on the component form, the following relations can be established:

$$\vec{A} \bullet \vec{B} = A_1B_1 + A_2B_2 + A_3B_3$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$\vec{A} \bullet (\vec{B} \times \vec{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ A_1 & A_2 & A_3 \\ B_2C_3 - B_3C_2 & B_3C_1 - B_1C_3 & B_1C_2 - B_2C_1 \end{vmatrix}$$

In addition to Cartesian system, Cylindrical and Spherical coordinate systems are also the coordinate systems widely used. Their components forms are discussed next.

2.11.1 Cartesian Coordinate System

Rectangular coordinate system X, Y, Z c and the corresponding unit base vectors are $\hat{i},\ \hat{j},\ \hat{k}$

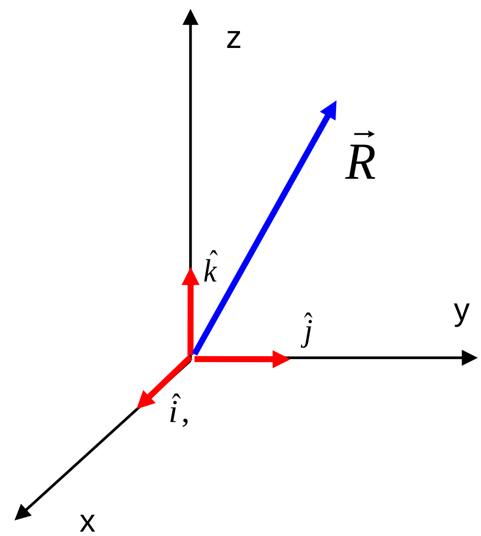
$\hat{i} \bullet \hat{j} = \hat{i} \bullet \hat{k} = \hat{j} \bullet \hat{k} = 0$

$\hat{i} \bullet \hat{i} = \hat{j} \bullet \hat{j} = \hat{k} \bullet \hat{k} = 1$

$\hat{i} \times \hat{j} = \hat{k}; \qquad \hat{j} \times \hat{k} = \hat{i}; \qquad \hat{k} \times \hat{i} = \hat{j}$

$\vec{A} = A_x\ \hat{i} + A_Y\ \hat{j} + A_z\ \hat{k}$

Where $A_x = \vec{A} \bullet \hat{i}; \qquad A_y = \vec{A} \bullet \hat{j}; \qquad A_z = \vec{A} \bullet \hat{k}$.



In other words, A_x, A_y, A_z are the components of vector $\vec{A}$, and there are the projections of $\vec{A}$ on X, Y, Z axes respectively.

The position vector in Cartesian system is given as:

$$\vec{R} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

2.11.2 Cylindrical Coordinate System

Variables in cylindrical coordinate system are (r,θ,z), and the corresponding unit base vectors are $\hat{e}_r, \hat{e}_\theta, \hat{e}_z$

$$\hat{e}_r \bullet \hat{e}_\theta = \hat{e}_r \bullet \hat{e}_z = \hat{e}_\theta \bullet \hat{e}_z = 0$$

$$\hat{e}_r \bullet \hat{e}_r = \hat{e}_\theta \bullet \hat{e}_\theta = \hat{e}_z \bullet \hat{e}_z = 1$$

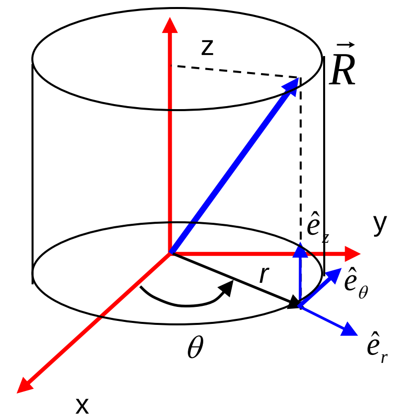
$$\hat{e}_r \times \hat{e}_\theta = \hat{e}_z; \qquad \hat{e}_\theta \times \hat{e}_z = \hat{e}_r; \qquad \hat{e}_z \times \hat{e}_r = \hat{e}_\theta$$

$$\vec{A} = A_r\ \hat{e}_r + A_\theta\ \hat{e}_\theta + A_z\ \hat{e}_z$$

Where:

$$A_r = \vec{A} \bullet \hat{e}_r; \qquad A_\theta = \vec{A} \bullet \hat{e}_\theta; \qquad A_z = \vec{A} \bullet\ \hat{e}_Z.$$

In other words, A_r, A_θ, A_z are the components of vector $\vec{A}$.



Cylindrical coordinate system *(r,θ, z)*

The position vector in Cartesian system is given as:

$$\vec{R} = r\hat{e}_r + z\ \hat{e}_z$$

2.11.3 Spherical Coordinate System

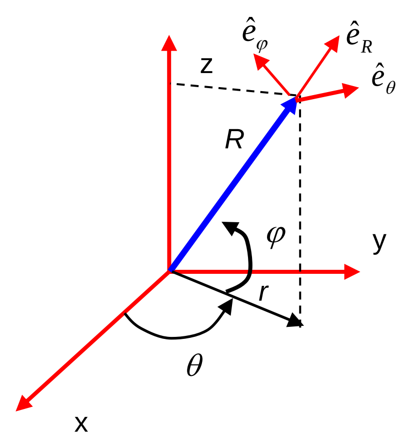
Variables in spherical coordinate system are (R,θ,φ), and the corresponding unit base vectors are $\hat{e}_R, \hat{e}_\theta, \hat{e}_\varphi$

$$\vec{A} = A_R\ \hat{e}_R + A_\theta\ \hat{e}_\theta + A_\varphi\ \hat{e}_\varphi$$

Where:

$$A_r = \vec{A} \bullet \hat{e}_r; \qquad A_\theta = \vec{A} \bullet \hat{e}_\theta; \qquad A_z = \vec{A} \bullet\ \hat{e}_Z.$$

In other words, A_r, A_θ, A_φ are the components of vector $\vec{A}$.



2.12 Relationship between Coordinate Systems

2.12.1 General Transformation

(q_1, q_2, q_3) are the general coordinates of a 3-D coordinate system.

$$q_1 = q_1(x, y, z)$$
$$q_2 = q_2(x, y, z)$$
$$q_3 = q_3(x, y, z)$$

2.12.2 General Inverse Transformation

$$x = x(q_1, q_2, q_3)$$
$$y = y(q_1, q_2, q_3)$$
$$z = z(q_1, q_2, q_3)$$

For example:
Transformation equations between the Cartesian coordinate and cylindrical coordinate system are:

$$r = x^2 + y^2 \qquad (0 \le r \le \infty)$$
$$\theta = \arctan(y/x) \quad (0 \le \theta \le 2\pi)$$
$$z = z \qquad (-\infty < z < \infty)$$

The inverse transformation equation will be:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

2.13 Scale factors, Unit Vectors and their Derivatives

2.13.1 Scale factors

Scale factor defines the relationship between coordinates and distance along coordinates.

2.13.2 General Coordinate System

A position vector $\vec{R}$ in Cartesian coordinate system is given by:

$$\vec{R} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

Using the inverse transformation, in a general coordinate system, the position vector can also be written as:

$$\vec{R} = x(q_1, q_2, q_3)\,\hat{i} + y(q_1, q_2, q_3)\,\hat{j} + z(q_1, q_2, q_3)\,\hat{k}$$

The variation of the position vector along the coordinate direction defines the following relations:

$$\frac{\partial \vec{R}}{\partial q_1} = h_1 \hat{e}_1$$
$$\frac{\partial \vec{R}}{\partial q_2} = h_2 \hat{e}_2$$
$$\frac{\partial \vec{R}}{\partial q_3} = h_3 \hat{e}_3$$

Where h_1, h_2, h_3 are the scale factors and $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are the unit vector in the q_1, q_2, q_3 direction, respectively.

2.14 Determination of Scale Factors and Derivatives of Unit Vectors

2.14.1 Cartesian Coordinate System

In Cartesian coordinate system, unit vectors are:

$$\hat{e}_1 = \hat{i}$$
$$\hat{e}_2 = \hat{j}$$
$$\hat{e}_3 = \hat{k}$$

A position vector $\vec{R}$ in Cartesian coordinate system is given by:

$$\vec{R} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

Therefore:

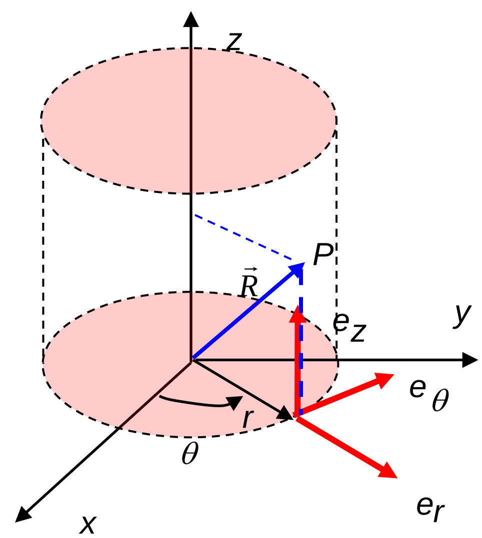
$$\frac{\partial \vec{R}}{\partial q_1} = h_1\,\hat{e}_1 = \hat{i} \quad \Rightarrow \quad h_1 = 1$$
$$\frac{\partial \vec{R}}{\partial q_2} = h_2\,\hat{e}_2 = \hat{j} \quad \Rightarrow \quad h_2 = 1\ ;$$
$$\frac{\partial \vec{R}}{\partial q_3} = h_3\,\hat{e}_3 = \hat{k} \quad \Rightarrow \quad h_3 = 1$$

Since the unit vectors are fixed in magnitude and direction in Cartesian coordinate system, therefore:

$$\frac{\partial \hat{i}}{\partial x} = \frac{\partial \hat{j}}{\partial x} = \frac{\partial \hat{k}}{\partial x} = 0$$
$$\frac{\partial \hat{i}}{\partial y} = \frac{\partial \hat{j}}{\partial y} = \frac{\partial \hat{k}}{\partial y} = 0$$
$$\frac{\partial \hat{i}}{\partial z} = \frac{\partial \hat{j}}{\partial z} = \frac{\partial \hat{k}}{\partial z} = 0$$

2.14.2 Cylindrical Coordinate System

In cylindrical coordinate system, a point P in space is given by a position vector $\vec{R}(r,\theta,z)$ with base $(\hat{e}_r, \hat{e}_\theta, \hat{e}_z)$.



Cylindrical system
(R,θ,z)

According to the definitions of the scale factors:

$$\frac{\partial \vec{R}}{\partial r} = h_r\ \hat{e}_r$$
$$\frac{\partial \vec{R}}{\partial \theta} = h_\theta\ \hat{e}_\theta$$
$$\frac{\partial \vec{R}}{\partial z} = h_z\ \hat{e}_z$$

The position vector can also be expressed as:

$$\vec{R} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

With the relationship between the Cartesian coordinate system and Cylindrical coordinate system as:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = Z \end{cases} \qquad r \ge 0, \quad 0 \le \theta \le 2\pi; \quad -\infty \le Z \le \infty .$$

Therefore:

$$\vec{R} = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} + z\,\hat{k}$$

r - Direction:

$$h_r\ \hat{e}_r = \frac{\partial \vec{R}}{\partial r} = \frac{\partial x}{\partial r}\hat{i} + \frac{\partial y}{\partial r}\hat{j} + \frac{\partial z}{\partial r}\hat{k} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} + 0\,\hat{k}$$

$$(h_r\ \hat{e}_r)\bullet(h_r\ \hat{e}_r) = (h_r)^2 = \cos^2\theta + \sin^2\theta = 1 \qquad \Rightarrow \qquad \begin{cases} h_r = 1 \\ \hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \end{cases}$$

θ - Direction:

$$h_\theta\ \hat{e}_\theta = \frac{\partial \vec{R}}{\partial \theta} = \frac{\partial x}{\partial \theta}\hat{i} + \frac{\partial y}{\partial \theta}\hat{j} + \frac{\partial z}{\partial \theta}\hat{k} = -r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j} + 0\,\hat{k}$$

$$(h_\theta\ \hat{e}_\theta)\bullet(h_\theta\ \hat{e}_\theta) = (h_\theta)^2 = r^2(\sin^2\theta + \cos^2\theta) = r^2 \qquad \Rightarrow \qquad \begin{cases} h_\theta = r \\ \hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} \end{cases}$$

Z - Direction:

$$h_Z \ \hat{e}_Z = \frac{\partial \vec{R}}{\partial Z} = \frac{\partial x}{\partial Z}\hat{i} + \frac{\partial y}{\partial Z}\hat{j} + \frac{\partial z}{\partial Z}\hat{k} = 0\,\hat{i} + 0\,\hat{j} + 1\,\hat{k}$$

$$(h_Z \ \hat{e}_Z) \bullet (h_Z \ \hat{e}_Z) = (h_Z)^2 = 1^2 \qquad \Rightarrow \qquad \begin{cases} h_Z = 1 \\ \hat{e}_Z = \hat{k} \end{cases}$$

Summarize:

$$\begin{aligned} \hat{e}_r &= \cos\theta\,\hat{i} + \sin\theta\,\hat{j} & & h_r = 1 \\ \hat{e}_\theta &= -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} & ; \quad & h_\theta = r \\ \hat{e}_Z &= \hat{k} & & h_z = 1 \end{aligned}$$

Transformation relationship

$$\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_Z \end{bmatrix} \quad ;or \quad \begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_Z \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

Derivatives of the unit vectors:

$$\begin{aligned} \frac{\partial \hat{e}_r}{\partial r} &= 0 & & \frac{\partial \hat{e}_\theta}{\partial r} = 0 & & \frac{\partial \hat{e}_Z}{\partial r} = 0 \\ \frac{\partial \hat{e}_r}{\partial \theta} &= \hat{e}_\theta & ; \quad & \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r & ; \quad & \frac{\partial \hat{e}_Z}{\partial \theta} = 0 \\ \frac{\partial \hat{e}_r}{\partial z} &= 0 & & \frac{\partial \hat{e}_\theta}{\partial z} = 0 & & \frac{\partial \hat{e}_Z}{\partial z} = 0 \end{aligned}$$

Example:

If $\vec{R} = \vec{R}(t) = r\hat{e}_r + z\hat{e}_z$ is the position vector of a particle in cylindrical coordinates, obtain expression for velocity vector, $\vec{V}$, and acceleration vector, $\vec{a}$, at that point.

Since $\hat{e}_r = \hat{e}_r(r,\theta,z)$, then, $d\hat{e}_r = \frac{\partial \hat{e}_r}{\partial \theta}d\theta + \frac{\partial \hat{e}_r}{\partial r}dr + \frac{\partial \hat{e}_r}{\partial z}dz$

Therefore $\frac{d\hat{e}_r}{dt} = \frac{\partial \hat{e}_r}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial \hat{e}_r}{\partial r}\frac{dr}{dt} + \frac{\partial \hat{e}_r}{\partial z}\frac{dz}{dt}$

Similarly, $\frac{d\hat{e}_\theta}{dt} = \frac{\partial \hat{e}_\theta}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial \hat{e}_\theta}{\partial r}\frac{dr}{dt} + \frac{\partial \hat{e}_\theta}{\partial z}\frac{dz}{dt}$

$$\frac{d\hat{e}_z}{dt} = \frac{\partial \hat{e}_z}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial \hat{e}_z}{\partial r}\frac{dr}{dt} + \frac{\partial \hat{e}_z}{\partial z}\frac{dz}{dt}$$

$$\vec{V} = \frac{d\vec{R}}{dt} = r\frac{d\hat{e}_r}{dt} + \frac{dr}{dt}\hat{e}_r + \frac{dz}{dt}\hat{e}_z + z\frac{d\hat{e}_z}{dt}$$
$$= r(\frac{\partial \hat{e}_r}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial \hat{e}_r}{\partial r}\frac{dr}{dt} + \frac{\partial \hat{e}_r}{\partial z}\frac{dz}{dt}) + \frac{dr}{dt}\hat{e}_r + \frac{dz}{dt}\hat{e}_z + z(\frac{\partial \hat{e}_z}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial \hat{e}_z}{\partial r}\frac{dr}{dt} + \frac{\partial \hat{e}_z}{\partial z}\frac{dz}{dt})$$
$$= r\hat{e}_\theta\frac{d\theta}{dt} + \frac{dr}{dt}\hat{e}_r + \frac{dz}{dt}\hat{e}_z$$

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d(r\hat{e}_\theta\frac{d\theta}{dt} + \frac{dr}{dt}\hat{e}_r + \frac{dz}{dt}\hat{e}_z)}{dt}$$
$$= \frac{dr}{dt}\hat{e}_\theta\frac{d\theta}{dt} + r\frac{d\hat{e}_\theta}{dt}\frac{d\theta}{dt} + r\hat{e}_\theta\frac{d^2\theta}{dt^2} + \frac{d^2r}{dt^2}\hat{e}_r + \frac{dr}{dt}\frac{d\hat{e}_r}{dt} + \frac{d^2z}{dt^2}\hat{e}_z + \frac{dz}{dt}\frac{d\hat{e}_z}{dt}$$
$$= \frac{dr}{dt}\frac{d\theta}{dt}\hat{e}_\theta + r\frac{d\theta}{dt}(\frac{\partial \hat{e}_\theta}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial \hat{e}_\theta}{\partial r}\frac{dr}{dt} + \frac{\partial \hat{e}_\theta}{\partial z}\frac{dz}{dt}) + r\hat{e}_\theta\frac{d^2\theta}{dt^2} + \frac{d^2r}{dt^2}\hat{e}_r +$$
$$\frac{dr}{dt}(\frac{\partial \hat{e}_r}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial \hat{e}_r}{\partial r}\frac{dr}{dt} + \frac{\partial \hat{e}_r}{\partial z}\frac{dz}{dt}) + \frac{d^2z}{dt^2}\hat{e}_z + \frac{dz}{dt}(\frac{\partial \hat{e}_z}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial \hat{e}_z}{\partial r}\frac{dr}{dt} + \frac{\partial \hat{e}_z}{\partial z}\frac{dz}{dt})$$
$$= \frac{dr}{dt}\frac{d\theta}{dt}\hat{e}_\theta - r\frac{d\theta}{dt}\frac{d\theta}{dt}\hat{e}_r + r\hat{e}_\theta\frac{d^2\theta}{dt^2} + \frac{d^2r}{dt^2}\hat{e}_r + \frac{dr}{dt}\frac{d\theta}{dt}\hat{e}_\theta + \frac{d^2z}{dt^2}\hat{e}_z$$
$$= [\frac{d^2r}{dt^2} - r(\frac{d\theta}{dt})^2]\hat{e}_r + [2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}]\hat{e}_\theta + \frac{d^2z}{dt^2}\hat{e}_z$$

Scale factors and unit vectors in Spherical coordinate system (R, φ, θ)

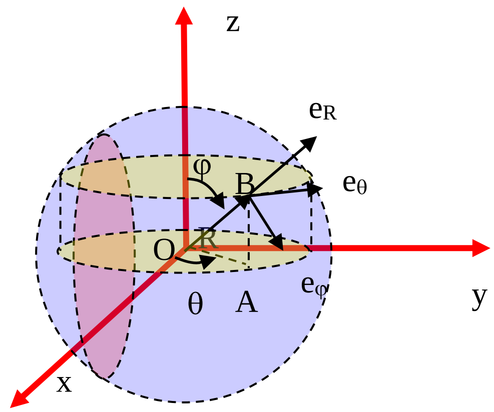
$\vec{OB} = R\,\hat{e}_R$

$OA = R\sin\varphi$

$x = R\sin\varphi\cos\theta$

$x = r\sin\varphi\sin\theta$

$z = r\cos\varphi$



$(R - \varphi - \theta)$

$h_R = 1$

$\hat{e}_R = \sin\varphi\cos\theta\,\hat{i} + \sin\varphi\sin\theta\,\hat{j} + \cos\varphi\,\hat{k}$

$h_\varphi = R$

$\hat{e}_\varphi = \cos\varphi\cos\theta\,\hat{i} + \cos\varphi\sin\theta\,\hat{j} - \sin\varphi\,\hat{k}$

$h_\theta = R\sin\varphi$

$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

2. 15 Vector Calculus

2.15.1 Del, the Vector Differential Operator:

$$\nabla = \hat{e}_1 \frac{\partial}{h_1 \partial q_1} + \hat{e}_2 \frac{\partial}{h_2 \partial q_2} + \hat{e}_3 \frac{\partial}{h_3 \partial q_3}$$

Where $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are three orthogonal unit vectors, h_1, h_2, h_3 are the scale factors along the coordinate axes q_1, q_2, q_3.

2.15.2. Cartesian Coordinate System

$$\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$

2.15.3. Cylindrical Coordinate System

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}$$

2.16 Scalar and Vector Field to Describe Physical Problems

Type of functions

- A scalar as a function of a scalar, for example: $\mu = \mu(T)$
- A vector as a function of a scalar, for example: $\vec{R} = \vec{R}(t)$
- A scalar as a function of a vector, for example: $T = T(\vec{R})$
- A vector as a function of a vector, for example: $\vec{V} = \vec{V}(\vec{R})$

General description: $\phi = \phi(\vec{R}, t)$ and $\vec{A} = \vec{A}(\vec{R}, t)$

Scalar field: A scalar quantity given as a function of coordinate space and time, *t,* is called scalar field.

For examples: $p = p(x, y, z, t) = p(\vec{R}, t)$ and $T = T(x, y, z, t) = T(\vec{R}, t)$

Vector field: A vector quantity given as a function of coordinate space and time, *t,* is called vector field.

For examples: $\vec{V} = \vec{V}(x, y, z, t) = \vec{V}(\vec{R}, t)$ and $\vec{M} = \vec{M}(x, y, z, t) = \vec{M}(\vec{R}, t)$

- In general, a field denotes a region throughout which a quantity is defined as a function of location within the region and time.

- If the quantity is independent of time, the field is steady or stationary.

2.17 Gradient

Gradient is a vector generated by the differentiation of a scalar function

Let $\phi = \phi(\vec{R}) = \phi(q_1, q_2, q_3)$

Since ϕ is a function of a vector $\vec{R}$, there are infinite number of directions in which to take the increment $\Delta\vec{R}$. The total change in ϕ, $d\phi$, would in general be different in different directions.

Spatial derivative of ϕ at a point is expressed as derivatives of ϕ in three independent directions. Gradient of a scalar is a vector.

2.17.1 Concept of Gradient

At any point, the gradient of a scalar function ϕ is equal in magnitude and direction to the greatest derivative of ϕ with respect to distance at the point.

Rate of change of scalar ϕ along two paths are of special importance:

1. Path along which the scalar is constant. (Isolines)
2. Path along which the rate of change of the scalar is the maximum (gradient line)

2.17.2 General Coordinate System:

$$\nabla\phi = \hat{e}_1 \frac{\partial\phi}{h_1\partial q_1} + \hat{e}_2 \frac{\partial\phi}{h_2\partial q_2} + \hat{e}_3 \frac{\partial\phi}{h_3\partial q_3}$$

2.17.2 Cartesian Coordinate System:

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

2.17. 3 Cylindrical Coordinate System

$$\nabla\phi = \hat{e}_r \frac{\partial\phi}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial\phi}{\partial\theta} + \hat{e}_z \frac{\partial\phi}{\partial z}$$

2.18 Concept of directional derivative

Consider the change of ϕ over the directed distance $d\vec{R}$ (i, e., $\vec{R} \to \vec{R} + \Delta\vec{R}$), find $d\phi = \lim_{\Delta\vec{R}\to 0} [\phi(\vec{R} + \Delta\vec{R}) - \phi(\vec{R})] = ?$

From the total differential formula of the calculus, the first order differential in ϕ will be

$$d\phi = \frac{\partial\phi}{\partial q_1} dq_1 + \frac{\partial\phi}{\partial q_2} dq_2 + \frac{\partial\phi}{\partial q_3} dq_3 + high \quad Orders \quad terms$$

$$\approx \frac{\partial\phi}{\partial q_1} dq_1 + \frac{\partial\phi}{\partial q_2} dq_2 + \frac{\partial\phi}{\partial q_3} dq_3 = \frac{1}{h_1}\frac{\partial\phi}{\partial q_1} h_1 dq_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial q_2} h_2 dq_2 + \frac{1}{h_3}\frac{\partial\phi}{\partial q_3} h_3 dq_3$$

$$= \frac{1}{h_1}\frac{\partial\phi}{\partial q_1} ds_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial q_2} ds_2 + \frac{1}{h_3}\frac{\partial\phi}{\partial q_3} ds_3$$

Since $d\vec{R} = d\vec{S} = ds_1\, \hat{e}_1 + ds_2\, \hat{e}_2 + ds_3\, \hat{e}_3$

Now introduce a vector $[\frac{1}{h_1}\frac{\partial\phi}{\partial q_1}, \frac{1}{h_2}\frac{\partial\phi}{\partial q_2}, \frac{1}{h_3}\frac{\partial\phi}{\partial q_3}]$ denoted by $\nabla\phi$ in the curvilinear orthogonal coordinate system with unit vector $(\hat{e}_1, \hat{e}_2 . \hat{e}_3)$, then,

$$d\phi = [\frac{1}{h_1}\frac{\partial\phi}{\partial q_1}, \frac{1}{h_2}\frac{\partial\phi}{\partial q_2}, \frac{1}{h_3}\frac{\partial\phi}{\partial q_3}] \bullet [ds_1, ds_2, ds_3]$$

$$= \nabla\phi \bullet d\vec{R} = \nabla\phi \bullet d\vec{S}$$

Since $d\vec{S} = dS \cdot \hat{e}_S$ therefore, $\frac{d\phi}{dS} = \nabla\phi \bullet \hat{e}_s$

- Directional derivative of $\phi(\vec{R})$ in any chosen direction is equal to the component of the gradient vector in that direction.
- $\frac{d\phi}{dS} = \nabla\phi \bullet \hat{e}_s$ is a maximum when $\nabla\phi \bullet \hat{e}_s$ is a maximum. i.e., when $\nabla\phi$ and $\hat{e}_s$ are in the same direction. In other words, $\nabla\phi$ is the direction of maximum changes of ϕ and $|\nabla\phi|$ is the magnitude of the change.
- The greatest rate of change of ϕ with respect to coordinate space at a point take place in the direction of $\nabla\phi$ and has the magnitude of the vector $\nabla\phi$.

2.19 Divergence of a Vector Field

Definition: The divergence of a vector ($\nabla \bullet \vec{B}$) at a point is the net outflow (efflux) of the vector field per unit volume enclosing the point.

$$\vec{V} = V_1\ \hat{e}_1 + V_2\ \hat{e}_2 + V_3\ \hat{e}_3$$

$$\nabla = \frac{\hat{e}_1}{h_1}\frac{\partial}{\partial q_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial q_2} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial q_3}$$

$$\nabla \bullet \vec{V} = \left(\frac{\hat{e}_1}{h_1}\frac{\partial}{\partial q_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial q_2} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial q_3}\right) \bullet (V_1\ \hat{e}_1 + V_2\ \hat{e}_2 + V_3\ \hat{e}_3)$$

Cartesian system:

$$\nabla \bullet \vec{V} = (\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}) \bullet (\hat{i}V_x + \hat{j}V_y + \hat{k}V_z)$$

$$= \hat{i} \bullet \frac{\partial(\hat{i}V_x + \hat{j}V_y + \hat{k}V_z)}{\partial x} + \hat{j} \bullet \frac{\partial(\hat{i}V_x + \hat{j}V_y + \hat{k}V_z)}{\partial y} + \hat{k} \bullet \frac{\partial(\hat{i}V_x + \hat{j}V_y + \hat{k}V_z)}{\partial z}$$

$$= \hat{i} \bullet [V_x \frac{\partial \hat{i}}{\partial x} + \hat{i}\frac{\partial V_x}{\partial x} + V_y \frac{\partial \hat{j}}{\partial x} + \hat{j}\frac{\partial V_y}{\partial x} + \frac{\partial \hat{k}}{\partial x}V_z + \hat{k}\frac{\partial V_z}{\partial x}] + \hat{j} \bullet \frac{\partial(\hat{i}V_x + \hat{j}V_y + \hat{k}V_z)}{\partial y} + \hat{k} \bullet \frac{\partial(\hat{i}V_x + \hat{j}V_y + \hat{k}V_z)}{\partial z}$$

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

Cylindrical system:

$$\nabla \bullet \vec{V} = (\hat{e}_r \frac{\partial}{\partial r} + \frac{\hat{e}_\theta}{r}\frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}) \bullet (V_r\ \hat{e}_r + V_\theta\ \hat{e}_\theta + V_z\ \hat{e}_z)$$

$$= \hat{e}_r \bullet \frac{\partial(V_r\ \hat{e}_r + V_\theta\ \hat{e}_\theta + V_z\ \hat{e}_z)}{\partial r} + \frac{\hat{e}_\theta}{r} \bullet \frac{\partial(V_r\ \hat{e}_r + V_\theta\ \hat{e}_\theta + V_z\ \hat{e}_z)}{\partial \theta} + \hat{e}_z \bullet \frac{\partial(V_r\ \hat{e}_r + V_\theta\ \hat{e}_\theta + V_z\ \hat{e}_z)}{\partial z}$$

$$Term\,1 = \hat{e}_r \bullet \frac{\partial(V_r\ \hat{e}_r + V_\theta\ \hat{e}_\theta + V_z\ \hat{e}_z)}{\partial r}$$

$$= \hat{e}_r \bullet (\frac{\partial V_r}{\partial r}\hat{e}_r + V_r \frac{\partial \hat{e}_r}{\partial r} + \frac{\partial V_r}{\partial r}\hat{e}_\theta + V_\theta \frac{\partial \hat{e}_\theta}{\partial r} + \frac{\partial V_z}{\partial r}\hat{e}_z + V_z \frac{\partial \hat{e}_z}{\partial r})$$

$$= \frac{\partial V_r}{\partial r}$$

$$Term\,2 = \frac{\hat{e}_\theta}{r} \bullet \frac{\partial(V_r\ \hat{e}_r + V_\theta\ \hat{e}_\theta + V_z\ \hat{e}_z)}{\partial \theta}$$

$$= \frac{\hat{e}_\theta}{r} \bullet (\frac{\partial V_r}{\partial \theta}\hat{e}_r + V_r \frac{\partial \hat{e}_r}{\partial \theta} + \frac{\partial V_r}{\partial \theta}\hat{e}_\theta + V_\theta \frac{\partial \hat{e}_\theta}{\partial \theta} + \frac{\partial V_z}{\partial \theta}\hat{e}_z + V_z \frac{\partial \hat{e}_z}{\partial \theta})$$

$$= \frac{\hat{e}_\theta}{r} \bullet (\frac{\partial V_r}{\partial \theta}\hat{e}_r + V_r\hat{e}_\theta + \frac{\partial V_r}{\partial \theta}\hat{e}_\theta - V_\theta\hat{e}_r + \frac{\partial V_z}{\partial \theta}\hat{e}_z)$$

$$= \frac{V_r}{r} + \frac{1}{r}\frac{\partial V_r}{\partial \theta}$$

$$Term\,3 = \hat{e}_z \bullet \frac{\partial(V_r\ \hat{e}_r + V_\theta\ \hat{e}_\theta + V_z\ \hat{e}_z)}{\partial z}$$

$$= \hat{e}_z \bullet (\frac{\partial V_r}{\partial z}\hat{e}_r + V_r \frac{\partial \hat{e}_r}{\partial z} + \frac{\partial V_r}{\partial z}\hat{e}_\theta + V_\theta \frac{\partial \hat{e}_\theta}{\partial z} + \frac{\partial V_z}{\partial z}\hat{e}_z + V_z \frac{\partial \hat{e}_z}{\partial z})$$

$$= \frac{\partial V_z}{\partial z}$$

Therefore,

$$\nabla \bullet \vec{V} = (\hat{e}_r \frac{\partial}{\partial r} + \frac{\hat{e}_\theta}{r}\frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}) \bullet (V_r\ \hat{e}_r + V_\theta\ \hat{e}_\theta + V_z\ \hat{e}_z)$$

$$= \frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z}$$

$$= \frac{1}{r}[\frac{\partial(rV_r)}{\partial r} + \frac{\partial V_\theta}{\partial \theta} + \frac{\partial(rV_z)}{\partial z}]$$

In general form:

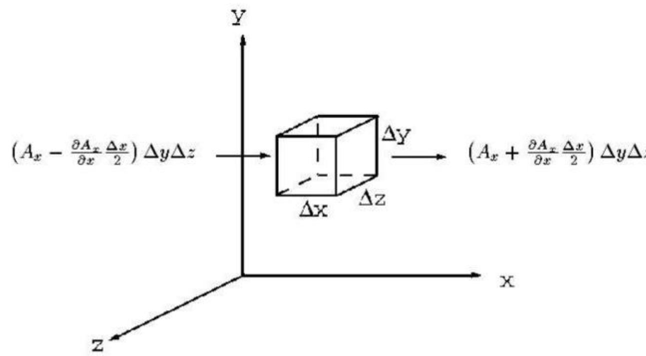
$$\nabla \bullet \vec{V} = \frac{1}{h_1 h_2 h_3}[\frac{\partial(h_2 h_3 V_1)}{\partial q_1} + \frac{\partial(h_1 h_3 V_2)}{\partial q_2} + \frac{\partial(h_1 h_2 V_3)}{\partial q_3}]$$

2. 20 Physical Meaning of Divergence of a Vector

The divergence of a vector at a point is the net outflow of the vector per unit volume enclosing the point.

Consider vector $\vec{A}$ with component A_x, A_y, A_z at a point in the vector field surround by an element control volume ∇V with an element surface ∇S.

For simplicity, the element control volume with its center having a vector and components A_x, A_y, A_z are oriented with edges parallel to x, y and z axes, respectively.



Outflow of $\vec{A}$ thorough any side = component of $\vec{A}$ in the direction normal to side × Area of the side.

Net outflow of $\vec{A}$ in X-direction (Net outflow of $\vec{A}$ from the X-direction)

$$=[(A_x + \frac{\partial A_x}{\partial x}\frac{\Delta x}{2}) - (A_x - \frac{\partial A_x}{\partial x}\frac{\Delta x}{2})]\Delta y \Delta z = \frac{\partial A_x}{\partial x}\Delta x \Delta y \Delta z = \frac{\partial A_x}{\partial x}\Delta V$$

Similarly, net outflow of $\vec{A}$ in Y-direction (Net outflow of $\vec{A}$ from the Y-direction)

$$=\frac{\partial A_y}{\partial y}\Delta x \Delta y \Delta z = \frac{\partial A_y}{\partial y}\Delta V$$

net outflow of $\vec{A}$ in Z-direction (Net outflow of $\vec{A}$ from the Z-direction)

$$=\frac{\partial A_z}{\partial z}\Delta x \Delta y \Delta z = \frac{\partial A_z}{\partial z}\Delta V$$

Therefore, the total net outflow of $\vec{A}$ at the point

$$=(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_Z}{\partial z})\Delta x \Delta y \Delta z = (\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_Z}{\partial z})\Delta V$$

$$\nabla \bullet \vec{A} = \lim_{\Delta V \to 0}[\frac{\text{total net outflow of } \vec{A} \text{ at the point in all direction}}{\Delta V}]$$
$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_Z}{\partial z}$$

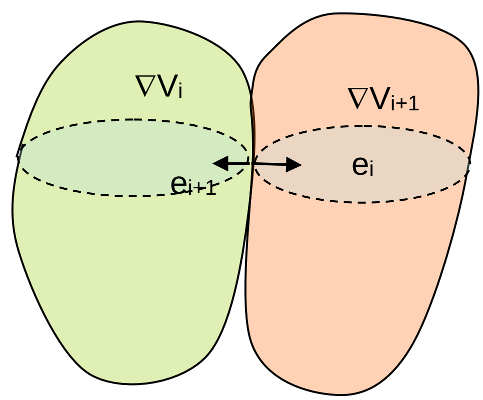
2.21 Gauss Divergence Theorem

Recall that: $\nabla \bullet \vec{B} = Div\vec{B} = \lim_{\Delta V \to 0} [\frac{\oint\!\!\oint_{\Delta S} \vec{B} \bullet d\vec{A}}{\Delta V}]$

can be approximated as: $\nabla \bullet \vec{B} = \frac{1}{\Delta V} \oint\!\!\oint_{\Delta S} \vec{B} \bullet d\vec{A}$ or $(\nabla \bullet \vec{B})\Delta V = \oint\!\!\oint_{\Delta S} \vec{B} \bullet d\vec{A}$ for an element control volume.

Now consider a finite control volume V in space subdivided into many smaller elemental sub-volumes.

Suppose $\nabla \bullet \vec{B}$ for all the sub volume are evaluated and summed:



$$\sum_{i=1}^{N} (\nabla \bullet \vec{B})_i \Delta V_i \approx \sum_{i=1}^{N} \oint\!\!\oint_{\Delta S} \vec{B} \bullet d\vec{A}$$

$$\underbrace{\lim_{\Delta V_i \to 0} \sum_{i=1}^{N} (\nabla \bullet \vec{B})_i \Delta V_i}_{volume\ integral\ by\ definition} \approx \lim_{\Delta V_i \to 0} \sum_{i=1}^{N} \oint\!\!\oint_{\Delta S} \vec{B} \bullet d\vec{A}$$

$$\oint\!\!\oint\!\!\oint_{\Delta V} (\nabla \bullet \vec{B})\, dV = \lim_{\Delta V_i \to 0} \sum_{i=1}^{N} \oint\!\!\oint_{\Delta S} \vec{B} \bullet d\vec{A}$$

The flow of $\vec{B}$ through the common faces of adjacent volumes canceled because the inflow through one face equals the outflow through the other.

Thus, if we now sum the net outflow of $\vec{B}$ of all the sub-volumes, only faces on the surface enclosing the region will contribute to the summation.

State in integral form the above statement becomes:

$$\lim_{\Delta V_i \to 0} \sum_{i=1}^{N} \oint\!\!\oint_{\Delta S} \vec{B} \bullet d\vec{A} = \oint\!\!\oint_{S} \vec{B} \bullet d\vec{A}$$

Thus, Gauss divergence theorem states:

$$\oint\!\!\oint\!\!\oint_{V} (\nabla \bullet \vec{B})\, dV = \oint\!\!\oint_{\Delta S} \vec{B} \bullet d\vec{A}$$

2.22 The Curl of a Vector Field

$\nabla \times \vec{B} = Curl\ \vec{B}$

$\nabla q_1 = \dfrac{\hat{e}_1}{h_1}; \qquad \nabla q_2 = \dfrac{\hat{e}_2}{h_2}; \qquad \nabla q_3 = \dfrac{\hat{e}_3}{h_3}$

$\vec{B} = B_1\ \hat{e}_1 + B_2\ \hat{e}_2 + B_3\ \hat{e}_3$

$\nabla \times \vec{B} = \nabla \times (B_1\ \hat{e}_1 + B_2\ \hat{e}_2 + B_3\ \hat{e}_3)$

Consider the first term

$\nabla \times (B_1\ \hat{e}_1) = \nabla \times (B_1\ h_1 \nabla q_1) = \nabla \times (B_1\ h_1 \nabla q_1)$

Since $\nabla \times (\phi\ \vec{A}) = \nabla \phi \times (\vec{A}) + \phi \nabla \times \vec{A}$

$$\begin{aligned}
\nabla \times (B_1\ \hat{e}_1) &= \nabla (B_1\ h_1) \times \nabla q_1 + (B_1\ h_1) \nabla \times \nabla q_1 \\
&= \nabla (B_1\ h_1) \times \nabla q_1 \\
&= [\frac{\hat{e}_1}{h_1} \frac{\partial (B_1\ h_1)}{\partial q_1} + \frac{\hat{e}_2}{h_2} \frac{\partial (B_1\ h_1)}{\partial q_2} + \frac{\hat{e}_3}{h_3} \frac{\partial (B_1\ h_1)}{\partial q_3}] \times (\frac{\hat{e}_1}{h_1}) \\
&= \frac{\hat{e}_2 \times \hat{e}_1}{h_2 h_1} \frac{\partial (B_1\ h_1)}{\partial q_2} + \frac{\hat{e}_3 \times \hat{e}_1}{h_3 h_1} \frac{\partial (B_1\ h_1)}{\partial q_3} \\
&= \frac{-\hat{e}_3}{h_2 h_1} \frac{\partial (B_1\ h_1)}{\partial q_2} + \frac{\hat{e}_2}{h_3 h_1} \frac{\partial (B_1\ h_1)}{\partial q_3} \\
&= \frac{1}{h_1} \{ \frac{\hat{e}_2}{h_3} \frac{\partial (B_1\ h_1)}{\partial q_3} - \frac{\hat{e}_3}{h_2} \frac{\partial (B_1\ h_1)}{\partial q_2} \}
\end{aligned}$$

Similarly,

$$\nabla \times (B_2\ \hat{e}_2) = \frac{1}{h_2} \{ \frac{\hat{e}_3}{h_1} \frac{\partial (B_2\ h_2)}{\partial q_1} - \frac{\hat{e}_1}{h_3} \frac{\partial (B_2\ h_2)}{\partial q_3} \}$$

$$\nabla \times (B_3\ \hat{e}_3) = \frac{1}{h_3} \{ \frac{\hat{e}_1}{h_2} \frac{\partial (B_2\ h_2)}{\partial q_2} - \frac{\hat{e}_2}{h_1} \frac{\partial (B_2\ h_2)}{\partial q_1} \}$$

Therefore,

$$\nabla\times\vec{B}=\nabla\times(B_1\ \hat{e}_1+B_2\ \hat{e}_2+B_3\ \hat{e}_3)$$

$$=\frac{1}{h_1}\{\frac{\hat{e}_2}{h_3}\frac{\partial(B_1\ h_1)}{\partial q_3}-\frac{\hat{e}_3}{h_2}\frac{\partial(B_1\ h_1)}{\partial q_2}\}$$

$$+\frac{1}{h_2}\{\frac{\hat{e}_3}{h_1}\frac{\partial(B_2\ h_2)}{\partial q_1}-\frac{\hat{e}_1}{h_3}\frac{\partial(B_2\ h_2)}{\partial q_3}\}$$

$$+\frac{1}{h_3}\{\frac{\hat{e}_1}{h_2}\frac{\partial(B_2\ h_2)}{\partial q_2}-\frac{\hat{e}_2}{h_1}\frac{\partial(B_2\ h_2)}{\partial q_1}\}$$

Or

$$\nabla\times\vec{B}=\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1\ B_1 & h_2\ B_2 & h_2\ B_2 \end{vmatrix}$$

2.23 Some Relations Involving the Vector Operator ∇

$\nabla\equiv\frac{\hat{e}_1}{h_1}\frac{\partial}{\partial q_1}+\frac{\hat{e}_2}{h_2}\frac{\partial}{\partial q_2}+\frac{\hat{e}_3}{h_3}\frac{\partial}{\partial q_3}$ is a vector operator and not a vector. Thus, it is necessary to present the orders in which ∇ appears with respect to the other terms.

For example: $\nabla\bullet\vec{A}\neq\vec{A}\bullet\nabla$

Some identities of interest: ϕ,ψ are scalar variables and $\vec{A},\vec{B}$ are vector variables:

- $\nabla(\phi\psi)=\phi\nabla\psi+\psi\nabla\phi$
- $\nabla\bullet(\phi\vec{A})=\nabla\phi\bullet\vec{A}+\phi\nabla\bullet\vec{A}$
- $\nabla\times(\phi\vec{A})=\nabla\phi\times\vec{A}+\phi\nabla\times\vec{A}$
- $\nabla(\vec{A}\bullet\vec{B})=(\vec{A}\bullet\nabla)\vec{B}+(\vec{B}\bullet\nabla)\vec{A}+\vec{A}\times(\nabla\times\vec{B})+\vec{B}\times(\nabla\times\vec{A})$
- $\nabla\bullet(\vec{A}\times\vec{B})=\vec{B}\bullet(\nabla\times\vec{A})-\vec{A}\bullet(\nabla\times\vec{B})$
- $\nabla\times(\vec{A}\times\vec{B})=\vec{A}(\nabla\bullet\vec{B})+(\vec{B}\bullet\nabla)\vec{A}-\vec{B}(\nabla\bullet\vec{A})-(\vec{A}\bullet\nabla)\vec{B}$
- $\nabla\bullet(\nabla\times\vec{A})=0$
- $\nabla\times(\nabla\phi)=0$
- $\nabla\times(\nabla\times\vec{A})=\nabla(\nabla\bullet\vec{A})-\nabla\bullet\nabla\vec{A}=\nabla(\nabla\bullet\vec{A})-\nabla^2\vec{A}$

Proof:

By simple expansion:

$$\nabla \bullet (\phi \vec{A}) = (\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}) \bullet (\hat{i}\phi A_x + \hat{j}\phi A_y + \hat{k}\phi A_z)$$

$$= \frac{\partial \phi A_x}{\partial x} + \frac{\partial \phi A_y}{\partial y} + \frac{\partial \phi A_z}{\partial z}$$

$$= A_x\frac{\partial \phi}{\partial x} + \phi\frac{\partial A_x}{\partial x} + \frac{\partial \phi}{\partial y}A_y + \phi\frac{\partial A_y}{\partial y} + \frac{\partial \phi}{\partial z}A_z + \phi\frac{\partial A_z}{\partial z}$$

$$= A_x\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}A_y + \frac{\partial \phi}{\partial z}A_z + \phi[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}]$$

$$= \nabla\phi \bullet \vec{A} + \phi\nabla \bullet \vec{A}$$

The vector and scalar in the identities are defined intrinsically - that is without reference to any special coordinate system. Verification of the above equations in any one coordinate system (e,g, Cartesian) is equivalent to verification of all coordinate system.

Determination of Laplacian equation $\nabla \bullet \nabla\psi = \nabla^2\psi$

Consider a scalar variable ψ

$$\nabla\psi = \frac{\hat{e}_1}{h_1}\frac{\partial\psi}{\partial q_1} + \frac{\hat{e}_2}{h_2}\frac{\partial\psi}{\partial q_2} + \frac{\hat{e}_3}{h_3}\frac{\partial\psi}{\partial q_3} = \vec{B}$$

i.e., $\vec{B} = [\frac{1}{h_1}\frac{\partial\psi}{\partial q_1};\ \ \frac{1}{h_2}\frac{\partial\psi}{\partial q_2};\ \ \frac{1}{h_3}\frac{\partial\psi}{\partial q_3}] = (B_1, B_2, B_3)$

$$\nabla \bullet \vec{B} = \frac{1}{h_1h_2h_3}\{\frac{\partial(B_1\ h_2h_3)}{\partial q_1} + \frac{\partial(h_1\ h_3B_2)}{\partial q_2} + \frac{\partial(h_1h_2B_3)}{\partial q_3}\}$$

$$= \frac{1}{h_1h_2h_3}\{\frac{\partial}{\partial q_1}(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial q_1}) + \frac{\partial}{\partial q_2}(\frac{h_1h_3}{h_2}\frac{\partial\psi}{\partial q_2}) + \frac{\partial}{\partial q_3}(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial q_3})\}$$

$$= \nabla^2\psi$$