Chapter 3
Description of Fluid Motion

3.1 Approaches and Basis Concepts

3.1.1 Microscopic View and Macroscopic View

The mean free path (λ): the mean distance that molecules travel between collisions with neighboring molecules is defined as the mean free path.

Continuous flow: a flow in which the mean free path (λ) of the molecules is small compared with the smallest physical length scale of the flow field. (Example: the diameter if a cylinder about which the fluid is flowing).

Free molecules flow: where the mean free path is the same order as the body scale.

Example: Vehicles such as the space shuttle encounter free molecule flow at the extreme outer edge of the atmosphere, where the air density is so low that the mean free path (λ) becomes order of the shuttle size.

Low density flows: exhibit characteristic of both continuous flow and free molecule flows.

We will be mainly concerned with continuous flows:

A flow can be considered a continuum only if there are a large number of molecules in the control volume we are considering (moving/fixed) such that the substance can be treated as being continuous.

A sufficient condition, though not a necessary condition, for the continuum assumption to be valid is:

$$\frac{1}{n} << \varepsilon << L^3$$

Where n is the number of molecules per unit volume. L is the smallest significant length scale in the flow field usually called the macroscopic length scale, and ε is a sufficient small volume.

Microscopic view (statistical mechanics): Approaches to solve the questions from the point of individual molecules motion and behaviors.

This method treats the fluid as consisting of molecules where motion is governed by the laws of dynamics.

The macroscopic phenomena are assumed to arise from the molecular motion of the molecules, and the theory attempts to predict the macroscopic behavior of the fluid from the viewpoints of mechanics and probability theory.

For a fluid which is in a state not too far from equilibrium. This approach yields the equations of mass, momentum and energy conservation.

The molecular approach also yields expression for the transport coefficient of viscosity and the thermal conductivity, in terms of molecular quantities such as the force acting between molecules.

The theory is well developed for light gases, but it is incomplete for polyatomic gas molecules and for liquids.

Example: Monotonic gas in 3 cm cube has 10^{20} atoms approximately under atmosphere pressure and temperature. To describe the velocity of each atom, the velocity components and to describe the position of each molecule, three coordinates must be specified. Thus, to completely describe the behavior of this system from a microscopic point of views, it would be necessary to deal with at least 6×10^{20} equations.

Macroscopic point of view (continuity concept):

In the continuity approach, individual molecules are ignored, and it is assumed that the fluid consists of continuous matter. In other words, we are concerned only with gross or average behavior of many molecules through measurable or observable properties knows as "fluid field variables" such as $P, T, \rho, \vec{V}$

Pressure (P): a gas exert on the walls of a container can be measured by a gauge. This pressure is the result of the change in momentum of the molecules as they collide with the wall. Here we are not assumed with the action of the individual molecules but with the time averaged force on given area.

If the matter can be considered as continuum, then, the continuous matters are required to obey the conservation laws of mass, momentum, energy, which give rise to a set of differential equations governing the field variables.

The solution to these differential equations then defines the variation of each field variable with space and time.

The continuous assumptions require that the mean free path (λ), of the molecules to be very small compared with the smallest physical length scale of the flow field (such as the diameter of cylinder or other body about which the fluid in flowing)

3.1.2 Basic Concepts

Ideal gas:

Ideal gas is composed of molecules which are small compared to the mean distance between them and so the potential energy arising from their mutual attraction may be neglected. Collisions between molecules or between molecules and the containing vessel are assumed to be perfectly elastic. The average distance a molecule travels before colliding with another is termed as the mean-free-path (λ). If the mean-free-path of the molecules approaches the order of magnitude of the dimensions of the vessel, then the concept of a continuum is not a valid assumption (ex. High vacuum technology, rarefied atmosphere). At temperature of 300K and above (room temperature and above) nitrogen and air behave as perfect or ideal gas up to pressure well above 1000 lb/in^2.

Pressure-*P*:

If a body is placed in a fluid, its surface is bombarded by a large number of molecules moving at random. When molecules bombard a surface they rebound, and by Newton's law the surface experiences a force equal and opposite to the time rate of change of momentum of the rebounding molecules. Thus, static pressure is the normal force per unit area exerted on a surface due to the time rate of change of momentum of the molecules impacting (or crossing) that surface

$$P = \lim_{dA \to dA'} \frac{dF_n}{dA}$$

Where *dA'* is the smallest area for which the system can be considered a continuum and dF_n is the force acting normal to that surface.
P is a point property and a scalar. *P* has units of (Force/Area), N/m^2, lb/ft^2, and lb/in^2. Most pressure and vacuum gages read the difference between the absolute pressure and the atmospheric pressure existing at the gage, and this is referred as gage pressure.

Density - ρ

$$\rho = \lim_{d\forall \to d\forall'} \frac{dm}{d\forall}$$

Where $d\forall'$ is the smallest volume for which the system can be considered a continuum and *dm* is the mass of that infinitesimal volume.

Specific volume

$$v = \lim_{d\forall \to d\forall'} \frac{d\forall}{dm}$$

Where $d\mathcal{V}'$ is the smallest volume for which the system can be considered a continuum and *dm* is the mass of that infinitesimal volume.

Temperature - *T*

The temperature of a gas (*T*) is directly proportional to the average kinetic energy of the molecules of the fluid.

Note: P, T, ρ are all static properties.
Example: A static temperature is that temperature measured by a common thermometer.

Viscosity - μ

Viscosity is that property of a fluid in ordered motion which causes their layer immediately adjacent to a surface to remain at rest.

Shear Stress $\tau \propto \dfrac{du}{dy}$

The constant of proportionality is μ.

If *T* is in Rankine:

$$\mu = 2.270\times10^{-8}\frac{T^{3/2}}{T+198.6} \qquad [slug\,/\,ft.\text{sec}]$$

If *T* is in Kelvin:

$$\mu = 1.456\times10^{-6}\frac{T^{3/2}}{T+110.3} \qquad [kg/m\cdot\text{sec}]$$

Steady Flow

If fluid properties at a point in a field do not change with time, then they are a function of space only. They are represented by: $\phi=\phi(q_1,q_2,q_3)$ Therefore for a steady flow $\dfrac{\partial\phi}{\partial t}=0$.

One-, Two-, and Three-Dimensional Flows

A flow is classified as one-, two-, or three-dimensional depending on the number of space coordinates required to specify all the fluid properties and the number of components of the velocity vector.

For example, a steady three-dimensional flow requires three space coordinates to specify the property and the velocity vector is given by: $\vec{V}=V_1\hat{e}_1+V_2\hat{e}_2+V_3\hat{e}_3$. Most real flows are three-dimensional in nature.

On the other hand, any property of a two-dimensional flow field requires only two space coordinates to describe it and its velocity has only two components along the two space coordinates that describe the field. The third component of velocity is identically zero everywhere. Steady channel flow between two parallel plates is a perfect example of two-dimensional flow if the viscous effects on the plates are neglected. The properties of the flow can be uniquely represented by $\phi = \phi(q_1, q_2)$ and the velocity vector can be written as $\vec{V} = V_1\hat{e}_1 + V_2\hat{e}_2$. The complexity of analysis increases considerably with the number of dimensions of the flow field.

In one-dimensional flow properties vary only as a function of one spatial coordinate and the velocity component in the other two directions are identically zero. In other words, $\phi = \phi(q_1)$ and $\vec{V} = V_1\hat{e}_1$.

Flux

Flux is defined as a rate of flow of any quantity per unit area across a control surface. Thus, mass flux is the rate of mass flow rate per unit area and heat flux is the heat flow rate per unit area across the control surface.

Incompressible

If density is constant, the flow is called incompressible. If the density is variable, it is called compressible flow. Flow of homogeneous liquid is treated as incompressible.

Boundary layer

The boundary layer is that region near the surface of a body where viscous effects are important.

Shear Stress $\tau_{wall} = \mu \frac{du}{dy}\Big|_{y=0}$ is large because $\frac{du}{dy}\Big|_{y=0}$ is large.

Effect of viscosity

The speed of flow which increases from zero at the surface of the body to the full streaming speed away from the body. (Velocity gradient inside the boundary layer).

Apparently steady force called the "skin friction drag" acting on the body in the direction of flow.

Newtonian versus Non-Newtonian fluid

Fluids for which the shear stress is directly proportional to the rate of strain are called Newtonian fluids. i.e., shear stress $\tau \propto \frac{du}{dy}$

For some fluids, however, the shear stress may not be directly proportional to the rate of strain.

Shear Stress (τ) not proportional to strain $\frac{du}{dy}$. These fluids are classified as non-Newtonian.

Examples: blood, certain plastics, clay-waste mixture.

Viscosity is important in the boundary layer, the separated flow region, and the wake region. Rest of the regions can be essentially treated as inviscid where $\mu = 0$.

Inviscid theory can adequately predict the pressure distribution and lift on a body. Inviscid theory also gives a valid representation of the streamlines and flow field away from the body. Inviscid theory can not predict any drag that depends on the friction.

Classification based on approximations of flow problems

Gas is a compressible, viscous, inhomogeneous substance, and the physical principles underlying its behavior are not completely enough understood to permit us to formulate exactly, any flow problem. Even if this were possible, the resulting equations would, in all probability be too difficult to solve. Hence all formulations are approximate at best.

Perfect fluid: homogeneous (not composed of discrete particles), incompressible (inelastic), inviscid fluid. The assumption of a perfect fluid gives good agreement with experiment for flows outside of boundary layer and wake of well-streamlined bodies moving with velocities of less than 200 mph at altitudes under about 100,000 ft.

Compressible, inviscid fluid: Fluid is considered compressible (elastic) and hence speed of sound characterizes the flow. It provides a good approximation for problems involving the flow outside of boundary layer and wake of bodies for all speeds at altitudes below about 100,000 ft.

Viscous, compressible fluid: Viscosity is included. Flow within the boundary layer and wake is amenable to accurate analysis, provided the flow is laminar (good for all speeds below altitudes of 100,000 ft).

3.2 Methods to Describe Fluid Motion

3.2.1 Lagrangian Method

Lagrangian method, a natural extension of particle mechanics, considers the individual molecules and obtains conversation equations (mass, momentum, and energy) based on individual molecular motion.

Attention is paid to what happens to the individual fluid particle (identified usually by its position at time *t=0*) in the course of time, what paths they described, what velocities or accelerations they possess, and so on.

The temperature in Lagrangian variables is given by $T = T(a,b,c,t)$, where (a,b,c) is the position of the particle at time *t=0*.

Also $\vec{R} = \vec{R}(a,b,c,t)$ is the position of the particle at time *t*, tagged by (a,b,c). t,a,b,c are the independent variables in Lagrangian frame. Since the fluid elements are continuously distributed, the values the parameters (a,b,c) will assume for the various elements that are continuous.

3.2.2 Eulerian Method

In Eulerian description, we describe the distribution of a macroscopic property as a function of space and time which we refer to as a Eulerian field of that property. Thus, we watch a fixed point (x,y,z) in space as time *t* proceeds.

The independent variables are the spatial coordinate (x,y,z) and time *t*.

For example:

The temperature of the fluid is given by $T = T(x,y,z,t)$

- At a given (x,y,z), $T = T(x,y,z,t)$ gives the time history of *T* at that point.
- At a given time *t*, $T = T(x,y,z,t)$ gives the spatial variation of *T*.
- In order words, for any fluid quantity *Q* can be expressed as $Q = Q(\vec{R},t)$ - a scalar or a vector field.

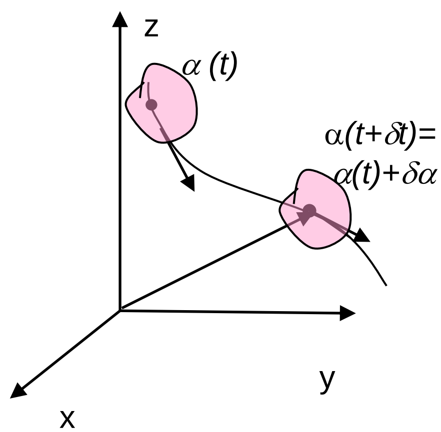
3.3 Substantial/Material Derivative

Substantial/material derivative gives the relation between the derivatives in Lagrangian and Eulerian derivatives.

In Eulerian viewpoint, since one attention is focused upon specific points in space at various times, the history of the individual fluid particles is not explicit. The substantial derivative allows us to express the time rate of change of a particle property in terms of the spatial (Eulerian) derivatives of that property at a given point.

Let α be any fluid variable such as density, velocity or energy. From Eulerian reference frame,

$$\alpha = \alpha(x, y, z, t) = \alpha(\vec{R}, t)$$



But if a specific fluid element is observed for a short period of time δt as it flows, its position will change by $\delta x, \delta y, \delta z$, and its value will change by an element amount $\delta \alpha$.

Thus, observed in Lagrangian reference frame, the independent variables are x_0, y_0, z_0 and time *t*, where x_0, y_0, z_0 are the initial are coordinates of the fluid particle.

Hence, x , y , z are no longer independent but function of time *t* as defined by the trajectory of the element.

Using differential calculus, the change in α can be calculated as

$\frac{\partial \alpha}{\partial t} \cdot \delta t + \frac{\partial \alpha}{\partial x} \cdot \delta x + \frac{\partial \alpha}{\partial y} \cdot \delta y + \frac{\partial \alpha}{\partial z} \cdot \delta z$, which equates the observed $\delta \alpha$ in Langrangian reference and divided by δt gives:

$$\frac{\delta \alpha}{\delta t} = \frac{\partial \alpha}{\partial t} + \frac{\partial \alpha}{\partial x} \cdot \frac{\delta x}{\delta t} + \frac{\partial \alpha}{\partial y} \cdot \frac{\delta y}{\delta t} + \frac{\partial \alpha}{\partial z} \cdot \frac{\delta z}{\delta t}$$

As $\delta t \rightarrow 0$

$$\lim_{\delta t \to 0} \frac{\delta \alpha}{\delta t} = \frac{D\alpha}{Dt}$$

Therefore:

$$\underbrace{\overbrace{\frac{D\alpha}{Dt}}^{\substack{Time\ derivatives\\ following\ a\ particle}}}_{Lagrangian\ frame} = \frac{\partial\alpha}{\partial t} + \frac{\partial\alpha}{\partial x}\cdot u + \frac{\partial\alpha}{\partial y}\cdot v + \frac{\partial\alpha}{\partial z}\cdot = \frac{\partial\alpha}{\partial t} + \vec{V}\bullet\nabla\alpha$$

$$= \underbrace{(\frac{\partial}{\partial t} + \vec{V}\bullet\nabla)\alpha}_{Expression\ in\ Eulerian\ domain}$$

$\frac{D}{Dt}$ is the change rate of a fluid property as the given fluid particle moves through space.

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{V}\bullet\nabla$$

$\frac{\partial}{\partial t}$ is the time change rate of fluid property of the given fixed point (i.e., local derivatives)

$\vec{V}\bullet\nabla$ is the time change rate of fluid property due to the movement of the fluid element from one location to another in the flow field where the flow property is spatially different (i.e., convective derivative).

2.4 Acceleration of a Fluid Particle

Acceleration of a fluid particle is by definition the change rate of its velocity at time *t*.

Lagrangian derivative of the velocity gives the acceleration of the fluid particle.

Since $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{V} \bullet \nabla$

$$\vec{V} = V_1\, \hat{e}_1 + V_2\, \hat{e}_2 + V_3\, \hat{e}_3$$

$$\nabla = \frac{\hat{e}_1}{h_1}\frac{\partial}{\partial q_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial q_2} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial q_3}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{V_1}{h_1}\frac{\partial}{\partial q_1} + \frac{V_2}{h_2}\frac{\partial}{\partial q_2} + \frac{V_3}{h_3}\frac{\partial}{\partial q_3}$$

$$\frac{D\vec{V}}{Dt} = (\frac{\partial}{\partial t} + \frac{V_1}{h_1}\frac{\partial}{\partial q_1} + \frac{V_2}{h_2}\frac{\partial}{\partial q_2} + \frac{V_3}{h_3}\frac{\partial}{\partial q_3})\vec{V}$$
$$= \frac{\partial}{\partial t}(V_1\, \hat{e}_1 + V_2\, \hat{e}_2 + V_3\, \hat{e}) + (\frac{V_1}{h_1}\frac{\partial}{\partial q_1} + \frac{V_2}{h_2}\frac{\partial}{\partial q_2} + \frac{V_3}{h_3}\frac{\partial}{\partial q_3})\vec{V}$$

Example:

In Cylindrical coordinate system
$\vec{V} = V_r\, \hat{e}_r + V_\theta\, \hat{e}_\theta + V_z\, \hat{e}_z$

Since unit vectors $\hat{e}_r, \hat{e}_\theta, \hat{e}_z$ do not change with time at a given point. Unit vectors change with respect to spatial coordinate in the θ - direction.

i.e., $\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta; \qquad + \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r$

Therefore,

$$\frac{\partial \vec{V}}{\partial t} = \frac{\partial V_r}{\partial t}\hat{e}_r + \frac{\partial V_\theta}{\partial t}\hat{e}_\theta + \frac{\partial V_Z}{\partial t}\hat{e}_z$$

$$(\vec{V}\bullet\nabla)\vec{V}=(V_r\frac{\partial}{\partial r}+\frac{V_\theta}{r}\frac{\partial}{\partial\theta}+V_z\frac{\partial}{\partial z})(V_r\,\hat{e}_r+V_\theta\,\hat{e}_\theta+V_z\,\hat{e})$$

$$=V_r\frac{\partial(V_r\,\hat{e}_r+V_\theta\,\hat{e}_\theta+V_z\,\hat{e})}{\partial r}+\frac{V_\theta}{r}\frac{\partial(V_r\,\hat{e}_r+V_\theta\,\hat{e}_\theta+V_z\,\hat{e})}{\partial\theta}+V_z\frac{\partial(V_r\,\hat{e}_r+V_\theta\,\hat{e}_\theta+V_z\,\hat{e})}{\partial z}$$

$$=V_r[\frac{\partial V_r}{\partial r}\hat{e}_r+V_r\frac{\partial\hat{e}_r}{\partial r}+\frac{\partial V_\theta}{\partial r}\hat{e}_\theta+V_\theta\frac{\partial\hat{e}_\theta}{\partial r}+\frac{\partial V_z}{\partial r}\hat{e}_z+V_z\frac{\partial\hat{e}_z}{\partial r}]$$

$$+\frac{V_\theta}{r}[\frac{\partial V_r}{\partial\theta}\hat{e}_r+V_r\frac{\partial\hat{e}_r}{\partial\theta}+\frac{\partial V_\theta}{\partial\theta}\hat{e}_\theta+V_\theta\frac{\partial\hat{e}_\theta}{\partial\theta}+\frac{\partial V_z}{\partial\theta}\hat{e}_z+V_z\frac{\partial\hat{e}_z}{\partial\theta}]$$

$$+V_z[\frac{\partial V_r}{\partial z}\hat{e}_r+V_r\frac{\partial\hat{e}_r}{\partial z}+\frac{\partial V_\theta}{\partial z}\hat{e}_\theta+V_\theta\frac{\partial\hat{e}_\theta}{\partial z}+\frac{\partial V_z}{\partial z}\hat{e}_z+V_z\frac{\partial\hat{e}_z}{\partial z}]$$

$$=V_r[\frac{\partial V_r}{\partial r}\hat{e}_r+\frac{\partial V_\theta}{\partial r}\hat{e}_\theta+\frac{\partial V_z}{\partial r}\hat{e}_z]+\frac{V_\theta}{r}[\frac{\partial V_r}{\partial\theta}\hat{e}_r+V_r\hat{e}_\theta+\frac{\partial V_\theta}{\partial\theta}\hat{e}_\theta-V_\theta\hat{e}_r+\frac{\partial V_z}{\partial\theta}\hat{e}_z]$$

$$+V_z[\frac{\partial V_r}{\partial z}\hat{e}_r+\frac{\partial V_\theta}{\partial z}\hat{e}_\theta+\frac{\partial V_z}{\partial z}\hat{e}_z]$$

$$=[V_r\frac{\partial V_r}{\partial r}+\frac{V_\theta}{r}\frac{\partial V_r}{\partial\theta}-\frac{V_\theta^{\,2}}{r}+V_z\frac{\partial V_r}{\partial z}]\hat{e}_r+[V\frac{\partial V_\theta}{\partial r}+\frac{V_\theta V_r}{r}+\frac{V_\theta}{r}\frac{\partial V_\theta}{\partial\theta}\hat{e}_\theta+V_z\frac{\partial V_\theta}{\partial z}]\hat{e}_\theta$$

$$+[V_r\frac{\partial V_z}{\partial r}+\frac{V_\theta}{r}\frac{\partial V_z}{\partial\theta}+V_z\frac{\partial V_z}{\partial z}]\hat{e}_z$$

Therefore:

$$\vec{a}=\frac{D\vec{V}}{Dt}=\left[\frac{\partial V_r}{\partial t}+V_r\frac{\partial V_r}{\partial r}+\frac{V_\theta}{r}\frac{\partial V_r}{\partial\theta}-\frac{V_\theta^{\,2}}{r}+V_z\frac{\partial V_r}{\partial z}\right]\hat{e}_r$$
$$+\left[\frac{\partial V_\theta}{\partial t}+V_r\frac{\partial V_\theta}{\partial r}+\frac{V_\theta V_r}{r}+\frac{V_\theta}{r}\frac{\partial V_\theta}{\partial\theta}+V_z\frac{\partial V_\theta}{\partial z}\right]\hat{e}_\theta$$
$$+\left[\frac{\partial V_z}{\partial t}+V_r\frac{\partial V_z}{\partial r}+\frac{V_\theta}{r}\frac{\partial V_z}{\partial\theta}+V_z\frac{\partial V_z}{\partial z}\right]\hat{e}_z\;=a_r\hat{e}_r+a_\theta\hat{e}_\theta+a_z\hat{e}_z$$

Another form of the Substantial derivative:

$$\frac{D\vec{V}}{Dt}=\frac{\partial\vec{V}}{\partial t}+\vec{V}\bullet\nabla\vec{V}=\frac{\partial\vec{V}}{\partial t}+\frac{1}{2}\nabla(\vec{V}\bullet\vec{V})-\vec{V}\times(\nabla\times\vec{V})$$

Proof:

Since $\nabla(\vec{A}\bullet\vec{B})=(\vec{A}\bullet\nabla)\vec{B}+(\vec{B}\bullet\nabla)\vec{A}+\vec{A}\times(\nabla\times\vec{B})+\vec{B}\times(\nabla\times\vec{A})$

$$\nabla\times(\vec{A}\times\vec{B})=\vec{A}(\nabla\bullet\vec{B})+(\vec{B}\bullet\nabla)\vec{A}-\vec{B}(\nabla\bullet\vec{A})-(\vec{A}\bullet\nabla)\vec{B}$$

Make $\vec{A}=\vec{A};$ $\qquad\vec{V}=\vec{B}$

Then

$$\nabla(\vec{A}\bullet\vec{V}) = (\vec{A}\bullet\nabla)\vec{V} + (\vec{V}\bullet\nabla)\vec{A} + \vec{A}\times(\nabla\times\vec{V}) + \vec{V}\times(\nabla\times\vec{A}) \qquad (1)$$

$$\nabla\times(\vec{A}\times\vec{V}) = \vec{A}(\nabla\bullet\vec{V}) + (\vec{V}\bullet\nabla)\vec{A} - \vec{V}(\nabla\bullet\vec{A}) - (\vec{A}\bullet\nabla)\vec{V} = -\nabla\times(\vec{V}\times\vec{A}) \qquad (2)$$

$(1)+(2)$

$$\nabla(\vec{V}\bullet\vec{A}) - \nabla\times(\vec{V}\times\vec{A}) = \vec{A}(\nabla\bullet\vec{V}) + 2(\vec{V}\bullet\nabla)\vec{A} - \vec{V}(\nabla\bullet\vec{A}) - (\vec{A}\bullet\nabla)\vec{V}$$
$$+ (\vec{A}\bullet\nabla)\vec{V} + \vec{A}\times(\nabla\times\vec{V}) + \vec{V}\times(\nabla\times\vec{A})$$

Thus,

$$(\vec{V}\bullet\nabla)\vec{A} = \frac{1}{2}[\nabla(\vec{V}\bullet\vec{A}) - \nabla\times(\vec{V}\times\vec{A}) - \vec{A}(\nabla\bullet\vec{V}) + \vec{V}(\nabla\bullet\vec{A}) - \vec{A}\times(\nabla\times\vec{V}) - \vec{V}\times(\nabla\times\vec{A})]$$

For $\vec{A}=\vec{V}$

Then

$$(\vec{V}\bullet\nabla)\vec{V} = \frac{1}{2}[\nabla(\vec{V}\bullet\vec{V}) - \nabla\times(\vec{V}\times\vec{V}) - \vec{V}(\nabla\bullet\vec{V}) + \vec{V}(\nabla\bullet\vec{V}) - \vec{V}\times(\nabla\times\vec{V}) - \vec{V}\times(\nabla\times\vec{V})]$$
$$= \frac{1}{2}\nabla(\vec{V}\bullet\vec{V}) - \vec{V}\times(\nabla\times\vec{V})$$

Therefore:

$$\frac{D\vec{V}}{Dt} = \frac{\partial\vec{V}}{\partial t} + \vec{V}\bullet\nabla\vec{V}$$
$$= \frac{\partial\vec{V}}{\partial t} + \frac{1}{2}\nabla(\vec{V}\bullet\vec{V}) - \vec{V}\times(\nabla\times\vec{V})$$

Example: acceleration of a fluid particle:

Lagrangian derivative of velocity gives the acceleration of a fluid particle.

Stagnation flow:

$$V_1 = CX_1$$
$$V_2 = -CX_2$$
$$V_3 = 0$$

$$\frac{DV_1}{Dt} = \frac{\partial V_1}{\partial t} + (V_1 \frac{\partial}{\partial X_1} + V_2 \frac{\partial}{\partial X_2} + V_3 \frac{\partial}{\partial X_3})V_1$$
$$= 0 + CX_1 \cdot C + (-CX_2 \cdot 0) + 0 \cdot 0$$
$$= C_1^{\ 2} X_1$$

$$\frac{DV_2}{Dt} = \frac{\partial V_2}{\partial t} + (V_1 \frac{\partial}{\partial X_1} + V_2 \frac{\partial}{\partial X_2} + V_3 \frac{\partial}{\partial X_3})V_2$$
$$= 0 + CX_1 \cdot 0 + (-CX_2) \cdot (-C) + 0 \cdot 0$$
$$= C_1^{\ 2} X_2$$

$$\frac{DV_3}{Dt} = \frac{\partial V_2}{\partial t} + (V_1 \frac{\partial}{\partial X_1} + V_2 \frac{\partial}{\partial X_2} + V_3 \frac{\partial}{\partial X_3})V_3$$
$$= 0 + CX_1 \cdot 0 + (-CX_2) \cdot 0 + 0 \cdot 0$$
$$= 0$$