Chapter 4
Conservation Equations

4.1 Three Basic Laws

System: A system is a region enclosed by a rigid or flexible boundary with a quantity of matter of fixed mass and identity. Heat and work can cross the boundary of a system.

Control volume: A control volume is a finite region in space that may be fixed or moving in space. Mass, momentum, heat and work can cross the boundary of the region called the control surface.

If the laws of physics are written for a fixed region of space, i.e., for different fluid particles occupy this region at different times, then, the frame of reference is said to be **Eulerian**. However, if the laws govern the same fluid particles in a particular region that moves with the fluid, the laws are written in **Lagrangian** reference frame.

Basic conservation laws can be applied more easily to an arbitrary collection of matter of fixed identity (a system composed of the same quantity of matter at all times) than to a volume fixed in space. Applying conservation laws to matter of fixed identity gives rise to Lagrangian reference frame and the associated substantial derivatives of volume integrals. However, control volumes of fixed shape is preferred (Eulerian reference frame) for the following reasons of difficulty with the Lagrangian reference frame.

- The fluid media is capable of continuous distortion and deformation, since it is often extremely difficult to identify and follow the same mass of fluid at all times as must be done in Lagrangian reference frame.

- Also, our primary interest often is not in the motion of a given mass of fluid, but rather in the effect of the overall fluid motion on some device or structure.

Thus the basic laws applied to a fixed mass in Lagrangian reference frame must be transformed to equivalent expressions in Eulerian reference frame. The theorem which permits this transformation is called Reynolds' transport theorem.

Conservation of mass:
The conservation of mass simply states that the mass, *M,* of the system is constant.
Writing as an equation, one obtained:

$$\frac{dM_S}{dt} = 0 \qquad or \qquad \frac{d}{dt}(\int_S dm) = \frac{d}{dt}(\int_S \rho \, d\forall) = 0$$

Conservation of Linear Momentum (Newton's Second Law):

Sum of all external forces acting on the system is equal to the time change rate of the linear momentum of the system.

$$\vec{F}\ \Big|_{acting\,on\,a\,system} = \frac{d}{dt}\left(\int_S \vec{V}\, dm\right) = \frac{d}{dt}\left(\int_S \vec{V}\,\rho\, d\forall\right)$$

Conservation of Energy (First Law of Thermodynamics):

$$\delta W\ \Big|_{system} + \delta Q\ \Big|_{system} = \delta E\ \Big|_{system} \qquad or \qquad \dot{W}\ \Big|_{system} + \dot{Q}\ \Big|_{system} = \frac{dE}{dt}\ \Big|_{system}$$

$$E\ \Big|_{system} = \int_{system} e\, dm = \int_{\forall} e\,\rho\, d\forall$$

Where e is the specific inner energy.

4.2 Reynolds Transport Theorem

Let N be any extensive property of the identifiable fixed mass (system) such as total mass, momentum, or energy. The corresponding intensive property (extensive property per unit mass) will be designated as α .

Then if :

$$N \Big|_{system} = \int_{system} \alpha \, dm = \int_{\forall} \alpha \, \rho \, d\forall$$

Let us follow this identifiable specific mass of fluid for a short period of time δt as it flows.

Since a specific mass of fluid is being considered and since X_0, Y_0, Z_0 and t are the independent variables in the Lagrangian framework, the quantity N will not be a function of t only as the specific mass moves:

i.e., $N \Big|_{system} = N(X_0, Y_0, Z_0, t)$ in general

For a specific mass, it becomes

$N \Big|_{system} = N(t)$ only as X_0, Y_0, Z_0 are fixed.

Note: N is not a function of x, y, z - the coordinates in the Eulerian space.

The rate of change of N can be written:

$$\frac{DN}{Dt} \Big|_{system} = \lim_{\delta t \to 0} \{ \frac{1}{\delta t} [\int_{\forall(t+\delta t)} \alpha(t+\delta t) \, \rho(t+\delta t) \, d\forall - \int_{\forall(t)} \alpha(t) \, \rho(t) \, d\forall] \}$$

$\alpha(t)\rho(t)$ can be thought of as an extensive property per unit volume, and denoted by β . Therefore, the above integral becomes:

$$\frac{DN}{Dt} \Big|_{system} = \lim_{\delta t \to 0} \{ \frac{1}{\delta t} [\int_{\forall(t+\delta t)} \beta(t+\delta t) \, d\forall - \int_{\forall(t)} \beta(t) \, d\forall] \}$$

Where $\forall$ is the volume of the specified mass and it may change the size and shape as it moves.

The quantity $\beta(t+\delta t)$ integrated over volume $\forall(t)$ will be subtracted and then added again inside the above limit to yield:

$$\frac{DN}{Dt}\Big|_{system} = \frac{D\int_{\mathcal{V}} \beta \, d\mathcal{V}}{Dt} = \lim_{\delta t \to 0}\{\frac{1}{\delta t}[\int_{\mathcal{V}(t+\delta t)} \beta(t+\delta t)\, d\mathcal{V} - \int_{\mathcal{V}(t)} \beta(t+\delta t)\, d\mathcal{V} + \int_{\mathcal{V}(t)} \beta(t+\delta t)\, d\mathcal{V} - \int_{\mathcal{V}(t)} \beta(t)\, d\mathcal{V}]\}$$

$$= \lim_{\delta t \to 0}\{\frac{1}{\delta t}[\overbrace{\int_{\mathcal{V}(t+\delta t)} \beta(t+\delta t)\, d\mathcal{V} - \int_{\mathcal{V}(t)} \beta(t+\delta t)\, d\mathcal{V}}^{firsttwo} + \overbrace{\int_{\mathcal{V}(t)} \beta(t+\delta t)\, d\mathcal{V} - \int_{\mathcal{V}(t)} \beta(t)\, d\mathcal{V}]\}}^{sencondtwo}$$

The first two integrals inside the limit correspond to holdup the integrand fixed and permitting the control volume $\mathcal{V}$ to vary, while the second two integrals correspond to holding $\mathcal{V}$ fixed and permitting the integrand β to vary.

The second integrals is by definition the local derivative in Eulerian reference frame.

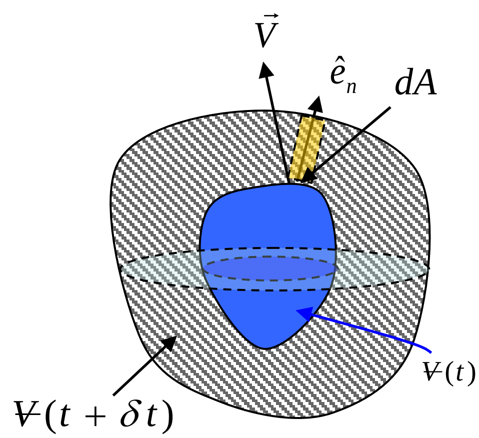
Therefore,

$$\frac{D\int_{\mathcal{V}} \beta \, d\mathcal{V}}{Dt} = \lim_{\delta t \to 0}\{\frac{1}{\delta t}(\overbrace{\int_{\mathcal{V}(t+\delta t)} \beta(t+\delta t)\, d\mathcal{V} - \int_{\mathcal{V}(t)} \beta(t+\delta t)\, d\mathcal{V}}^{firsttwo})\} + \lim_{\delta t \to 0}\{\frac{\overbrace{\int_{\mathcal{V}(t)} \beta(t+\delta t)\, d\mathcal{V} - \int_{\mathcal{V}(t)} \beta(t)\, d\mathcal{V}]\}}^{sencondtwo}}{\delta t}\}$$

$$= \lim_{\delta t \to 0}\{\frac{1}{\delta t}(\overbrace{\int_{\mathcal{V}(t+\delta t)} \beta(t+\delta t)\, d\mathcal{V} - \int_{\mathcal{V}(t)} \beta(t+\delta t)\, d\mathcal{V}}^{firsttwo})\} + \frac{\partial}{\partial t}\int_{\mathcal{V}} \beta \, d\mathcal{V}$$

The remaining limit corresponds to holdup the integrand β fixed, and permitting the control volume $\mathcal{V}$ to vary.

$$\frac{D\int_{\mathcal{V}} \beta \, d\mathcal{V}}{Dt} = \lim_{\delta t \to 0}\{\frac{1}{\delta t}(\int_{\mathcal{V}(t+\delta t)-\mathcal{V}(t)} \beta(t+\delta t)\, d\mathcal{V})\} + \frac{\partial}{\partial t}\int_{\mathcal{V}} \beta \, d\mathcal{V}$$



Geometrically, this can be visualized as the adjacent figure, and thus the limit is to be carried in the shadowed region.

In the limiting sense, the perpendicular distance from any point on the inner surface to the outer surface is $\vec{V} \bullet \hat{e}_n \delta t$ and therefore, $d\mathcal{V}$ can be equate to $\vec{V} \bullet \hat{e}_n \delta t \, dS$.

Thus, the volume integral may be transformed to surface integral.

$$\frac{D\int_{\mathcal{V}} \alpha\,\rho\, d\mathcal{V}}{Dt} = \frac{D\int_{\mathcal{V}} \beta\, d\mathcal{V}}{Dt} = \lim_{\delta t\to 0}\{\frac{1}{\delta t}(\int_{\mathcal{V}(t+\delta t)-\mathcal{V}(t)} \beta(t+\delta t)\, d\mathcal{V})\} + \frac{\partial}{\partial t}\int_{\mathcal{V}} \beta\, d\mathcal{V}$$

$$= \lim_{\delta t\to 0}\{\frac{1}{\delta t}(\int_{S(t)} \beta(t+\delta t)(\vec{V}\bullet\hat{e}_n)\delta t\, dA)\} + \frac{\partial}{\partial t}\int_{\mathcal{V}} \beta\, d\mathcal{V}$$

$$= \lim_{\delta t\to 0}(\int_{S(t)} \beta(t+\delta t)(\vec{V}\bullet\hat{e}_n)\, dA) + \frac{\partial}{\partial t}\int_{\mathcal{V}} \beta\, d\mathcal{V}$$

$$= \int_{C.S(t)} \beta(t)(\vec{V}\bullet\hat{e}_n)\, dA) + \frac{\partial}{\partial t}\int_{\mathcal{V}} \beta\, d\mathcal{V} = \int_{S} \beta(\vec{V}\bullet\hat{e}_n)\, dA + \frac{\partial}{\partial t}\int_{\mathcal{V}} \beta\, d\mathcal{V}$$

$$= \int_{C.S} \beta\vec{V}\bullet d\vec{A} + \frac{\partial}{\partial t}\int_{\mathcal{V}} \beta\, d\mathcal{V} = \int_{C.S} \alpha\,\rho\vec{V}\bullet d\vec{A} + \frac{\partial}{\partial t}\int_{\mathcal{V}} \alpha\,\rho\, d\mathcal{V}$$

The Lagrangian derivative of a volume integral has been converted into a surface integral in which the integrands contain only Eularian derivatives.

Physical meaning:

$\frac{DN_s}{Dt} = \frac{D\int_{\mathcal{V}} \alpha\,\rho\, d\mathcal{V}}{Dt}$ is the total rate of change of any arbitrary extensive property of the system.

$\frac{\partial}{\partial t}\int_{\mathcal{V}} \alpha\,\rho\, d\mathcal{V}$ is the time rate of change of the arbitrary extensive property *N* with the control volume evaluated by an observer fixed in the moving control volume.

$\int_{C.S.} \alpha\,\rho\vec{V}\bullet d\vec{A}$ is the net rate of efflux of the extensive property N through the control surface to the control volume.

The equation:

$$\frac{DN_s}{Dt} = \frac{D\int_{\mathcal{V}} \alpha\,\rho\, d\mathcal{V}}{Dt} = \frac{\partial}{\partial t}\int_{C.V.} \alpha\,\rho\, d\mathcal{V} + \int_{C.S.} (\alpha\,\rho\vec{V})\bullet d\vec{A}$$

Where α is any intensive property corresponding to *N*. (i.e., α = *N* per unit mass), and it can be used for different quantities as follows.

N_S	α
Mass	1
Linear momentum	$\vec{V}$
Angular momentum	$\vec{R}\times\vec{V}$
Energy	e
Entropy	s

4.3 Conservation of Mass

The conservation of mass simply states that the mass, *M,* of the system is constant.

$$\frac{DM_s}{Dt} = 0$$

Using Reynolds' transport theory, this can be converted to Euler formulation as:

$$\frac{DN_s}{Dt} = \frac{D\int_{C.V.} \alpha \rho \, d\forall}{Dt} = \frac{\partial}{\partial t}\int_{C.V.} \alpha \rho \, d\forall + \int_{C.S.} (\alpha \rho \vec{V}) \bullet d\vec{A} \qquad \text{when} \quad \alpha = 1, \text{ then}$$

$$\frac{DM_s}{Dt} = \frac{\partial}{\partial t}\int_{C.V.} \rho \, d\forall + \int_{C.S.} (\rho \vec{V}) \bullet d\vec{A} = 0$$

Physically,

$$\begin{bmatrix} \text{Rate of change of mass} \\ \text{inside a control volume} \end{bmatrix} + \begin{bmatrix} \text{Net rate of mass efflux (outflow)} \\ \text{through th e control surface} \end{bmatrix} = 0$$

Since the control volume is fixed with respect to a coordinate system attached to it, the limits of integration are also fixed. Hence, the time derivative can be placed inside the volume integral, and the equation can be re-written as:

$$\int_{C.\forall.} \frac{\partial \rho}{\partial t} \, d\forall + \int_{C.S.} (\rho \vec{V}) \bullet d\vec{A} = 0$$

States the conservation of mass law in a finite space.

Applying Guess divergence theorem, we convert the surface integral to volume integral to obtain:

$$\int_{C.\forall.} \frac{\partial \rho}{\partial t} \, d\forall + \int_{C.S.} (\rho \vec{V}) \bullet d\vec{A} = \int_{C.V.} \frac{\partial \rho}{\partial t} \, d\forall + \int_{C.V.} \nabla \bullet (\rho \vec{V}) \, d\forall = \int_{C.V.} [\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \vec{V})] \, d\forall = 0$$

Therefore:

$$\int_{C.V.} [\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \vec{V})] \, d\forall = 0$$

Since the control volume $\forall$ was arbitrarily chosen, the only way this equation can be satisfied is for the integrand to be zero at all points within the control volume. Thus, by setting the integrand to zero, we have the partial equation of conservation law:

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \vec{V}) = 0$$

Mathematically, the variation of the above equation can be:

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \vec{V}) = \frac{\partial \rho}{\partial t} + \vec{V} \bullet \nabla \rho + \rho \nabla \bullet \vec{V}$$

$$= \frac{D\rho}{Dt} + \rho \nabla \bullet \vec{V} = 0$$

Simplifications:
Form incompressible flows:

ρ is constant, then $\frac{\partial \rho}{\partial t} = 0;$ $\nabla \rho = 0$

Therefore, $\nabla \bullet \vec{V} = 0$

4.4 Conservation of Momentum

Newton's second law:

[Time change rate of momentum of a system] = [Resultant external force acting on the system]

i.e., $$\frac{d\vec{M}_S}{dt} = \sum \vec{F}_S = \sum \vec{F}_{Surface} + \sum \vec{F}_{body}$$

Use $\alpha = \vec{V}$ in Reynolds' transport theorem apply to $\vec{M}_S$ where $\vec{M}_S = \int_{C.V.} \vec{V} \rho \, d\forall$ = total momentum

$$\frac{d\vec{M}_s}{dt} = \frac{d}{dt}\int_{C.V.} \vec{V} \rho \, d\forall = \frac{\partial}{\partial t}\int_{C.V.} \vec{V} \rho \, d\forall + \int_{C.S.} (\vec{V} \rho \vec{V}) \bullet d\vec{A}$$

Therefore;

$$\frac{\partial}{\partial t}\int_{C.V.} \vec{V} \rho \, d\forall + \int_{C.S.} (\vec{V} \rho \vec{V}) \bullet d\vec{A} = \vec{F}_{surface} + \vec{F}_{body}$$

$\vec{F}_{surface}$ Surface forces such as pressure and shear stress. The surface forces usually can be expressed as $\vec{F}_{surface} = \int_{C.S.} \tilde{P} \bullet d\vec{A}$, where $\tilde{P}$ is the stress tensor exerted by the surroundings on the particle surface. $\tilde{P} = -P\tilde{I} + \tilde{\tau}$

$\vec{F}_{body}$ Body forces such as electromagnetic, gravitational forces. Usually the body force can be expressed as $\vec{F}_{body} = \int_{C.V.} \rho \vec{f} \, d\forall$, where $\vec{f}$ is a vector which references the resultant force per unit mass.

Then, the momentum equation reduces to:

$$\frac{\partial}{\partial t}\int_{C.V.} \rho \vec{V} \, d\forall + \int_{C.S.} (\rho \vec{V} \vec{V}) \bullet d\vec{A} = \int_{C.S.} \tilde{P} \bullet d\vec{A} + \int_{C.V.} \rho \vec{f} \, d\forall$$

Using divergence theorem for the control surface integrals, we obtained following equation after noting that the limits do not change.

$$\frac{\partial}{\partial t}\int_{C.V.}\rho\vec{V}\,d\forall + \int_{C.S.}\nabla\bullet(\rho\vec{V}\vec{V})\,d\forall = \int_{C.S.}\nabla\bullet\tilde{P}\,d\forall + \int_{C.V.}\rho\,\vec{f}\,d\forall$$

$$\Rightarrow \int_{C.V.}[\frac{\partial(\rho\vec{V})}{\partial t} + \nabla\bullet(\rho\vec{V}\vec{V}) - \nabla\bullet\tilde{P} - \rho\,\vec{f}\,]d\forall = 0$$

$$\Rightarrow \frac{\partial(\rho\vec{V})}{\partial t} + \nabla\bullet(\rho\vec{V}\vec{V}) - \nabla\bullet\tilde{P} - \rho\,\vec{f} = 0$$

Expand the above equation using $\nabla\bullet(\phi\vec{A}) = (\vec{A}\bullet\nabla)\phi + \phi\nabla\bullet\vec{A}$

$$\frac{\partial(\rho\vec{V})}{\partial t} + \nabla\bullet(\rho\vec{V}\vec{V}) - \nabla\bullet\tilde{P} - \rho\,\vec{f} = 0$$

$$\Rightarrow \vec{V}\frac{\partial\rho}{\partial t} + \rho\frac{\partial\vec{V}}{\partial t} + \vec{V}\nabla\bullet(\rho\vec{V}) + (\rho\vec{V}\bullet\nabla)\vec{V} - \nabla\bullet\tilde{P} - \rho\,\vec{f} = 0$$

$$\Rightarrow \vec{V}[\frac{\partial\rho}{\partial t} + \nabla\bullet(\rho\vec{V})] + \rho\frac{\partial\vec{V}}{\partial t} + (\rho\vec{V}\bullet\nabla)\vec{V} - \nabla\bullet\tilde{P} - \rho\,\vec{f} = 0$$

$$\Rightarrow \rho\frac{\partial\vec{V}}{\partial t} + (\rho\vec{V}\bullet\nabla)\vec{V} - \nabla\bullet\tilde{P} - \rho\,\vec{f} = 0$$

$$\Rightarrow \rho\frac{D\vec{V}}{Dt} - \nabla\bullet\tilde{P} - \rho\,\vec{f} = 0$$

Therefore, the differential form of the momentum equation is:

$$\rho\frac{D\vec{V}}{Dt} - \nabla\bullet\tilde{P} - \rho\,\vec{f} = 0$$

4.5 The Navier-Stokes Equations

$$\frac{\partial(\rho\vec{V})}{\partial t} + \nabla \bullet (\rho\vec{V}\vec{V}) - \nabla \bullet \tilde{P} - \rho\vec{f} = 0$$

Where $\tilde{P} = -P\tilde{I} + \tilde{\tau}$, and tensor $\tilde{I} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$; and $\tilde{\tau} = \begin{vmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{vmatrix}$

Re-writing the equation after substitution leads to:

$$\frac{\partial(\rho\vec{V})}{\partial t} + \nabla \bullet (\rho\vec{V}\vec{V}) - \nabla \bullet (-P\tilde{I} + \tilde{\tau}) - \rho\vec{f} = 0$$

$$\frac{\partial(\rho\vec{V})}{\partial t} + \nabla \bullet (\rho\vec{V}\vec{V} + P\tilde{I} - \tilde{\tau}) - \rho\vec{f} = 0$$

Since $\nabla \bullet (P\tilde{I}) = P\nabla \bullet \tilde{I} + (\tilde{I} \bullet \nabla)P = \nabla P$

Therefore:

$$\frac{\partial(\rho\vec{V})}{\partial t} + \nabla \bullet (\rho\vec{V}\vec{V}) + \nabla P - \nabla \bullet \tilde{\tau} - \rho\vec{f} = 0$$

Another form of Navier-Stokes equation:

$$\frac{\partial(\rho\vec{V})}{\partial t} + \nabla \bullet (\rho\vec{V}\vec{V}) + \nabla P - \nabla \bullet \tilde{\tau} - \rho\vec{f} = 0$$

$$\Rightarrow \rho\frac{\partial\vec{V}}{\partial t} + \vec{V}\frac{\partial\rho}{\partial t} + \vec{V}\nabla \bullet (\rho\vec{V}) + (\rho\vec{V} \bullet \nabla)\vec{V} + \nabla P - \nabla \bullet \tilde{\tau} - \rho\vec{f} = 0$$

$$\Rightarrow \vec{V}[\underbrace{\frac{\partial\rho}{\partial t} + \nabla \bullet (\rho\vec{V})]}_{continuity\quad equation} + \rho\frac{\partial\vec{V}}{\partial t} + (\rho\vec{V} \bullet \nabla)\vec{V} + \nabla P - \nabla \bullet \tilde{\tau} - \rho\vec{f} = 0$$

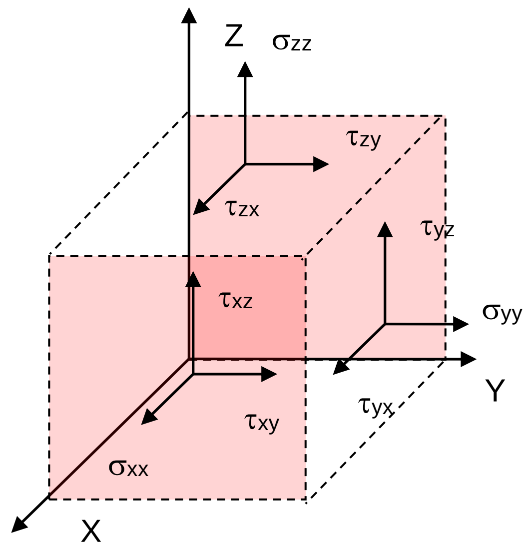
$$\Rightarrow \rho(\underbrace{\frac{\partial\vec{V}}{\partial t} + \vec{V} \bullet \nabla\vec{V}}_{substtantial\quad derivative}) + \nabla P - \nabla \bullet \tilde{\tau} - \rho\vec{f} = 0 \Rightarrow \rho\frac{D\vec{V}}{Dt} + \nabla P - \nabla \bullet \tilde{\tau} - \rho\vec{f} = 0$$

i.e., $\rho\frac{D\vec{V}}{Dt} + \nabla P - \nabla \bullet \tilde{\tau} - \rho\vec{f} = 0$

Stress Tensor

The stress tensor has nine components:

$$\tilde{\tau} = \begin{vmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{vmatrix}$$



Newtonian fluid,

$$\tilde{\tau} = \mu[\nabla\vec{V} + (\nabla\vec{V})^T - \frac{2}{3}(\nabla \bullet \vec{V})\tilde{I}]$$

For incompressible flow, in Cartesian coordinate system

$$\tau_{ij} = \mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$$

$$\tau_{xy} = \tau_{yx}; \qquad \tau_{xz} = \tau_{zx} \qquad \tau_{zy} = \tau_{yz}$$

4.6 Expansion of the Navier-Stokes Equation in Cartesian Coordinate

The vector form of the Navier-Stokes equation is :

$$\rho \frac{D\vec{V}}{Dt} + \nabla P - \nabla \bullet \tilde{\tau} - \rho \vec{f} = 0$$

For incompressible flow, Navier-Stokes equation in Cartesian coordinate system will be:

x-direction :

$$\begin{cases} \rho \dfrac{Du}{Dt} = -\dfrac{\partial p}{\partial x} + \rho f_x + \mu[\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}] \\ \qquad or \\ \rho[\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z}] = -\dfrac{\partial p}{\partial x} + \rho f_x + \mu[\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}] \end{cases}$$

y-direction:

$$\begin{cases} \rho \dfrac{Dv}{Dt} = -\dfrac{\partial p}{\partial y} + \rho f_y + \mu[\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} + \dfrac{\partial^2 v}{\partial z^2}] \\ \qquad or \\ \rho[\dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z}] = -\dfrac{\partial p}{\partial x} + \rho f_y + \mu[\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} + \dfrac{\partial^2 v}{\partial z^2}] \end{cases}$$

z-direction:

$$\begin{cases} \rho \dfrac{Dw}{Dt} = -\dfrac{\partial p}{\partial z} + \rho f_z + \mu[\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2}] \\ \qquad or \\ \rho[\dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + v\dfrac{\partial w}{\partial y} + w\dfrac{\partial w}{\partial z}] = -\dfrac{\partial p}{\partial z} + \rho f_z + \mu[\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2}] \end{cases}$$

4.7 Expansion of the Navier-Stokes Equation in Cylindrical Coordinate

$$\frac{\partial(\rho\vec{V})}{\partial t} + \nabla \bullet (\rho \vec{V}\vec{V}) + \nabla P - \nabla \bullet \tilde{\tau} - \rho \vec{f} = 0$$

$$\vec{V} = V_r \hat{e}_r + V_\theta \hat{e}_\theta + V_Z \hat{e}_Z$$

$$\frac{\partial(\rho\vec{V})}{\partial t} = \frac{\partial[\rho(V_r \hat{e}_r + V_\theta \hat{e}_\theta + V_Z \hat{e}_Z]}{\partial t} = \frac{\partial(\rho V)_r}{\partial t}\hat{e}_r + \frac{\partial(\rho V)_\theta}{\partial t}\hat{e}_\theta + \frac{\partial(\rho V)_Z}{\partial t}\hat{e}_Z$$

$$\nabla \bullet (\rho \vec{V}\vec{V}) = \nabla \bullet [\rho(V_r \hat{e}_r + V_\theta \hat{e}_\theta + V_Z \hat{e}_Z)(V_r \hat{e}_r + V_\theta \hat{e}_\theta + V_Z \hat{e}_Z)]$$

$$= \nabla \bullet \begin{bmatrix} \overbrace{\rho V_r V_r \hat{e}_r \hat{e}_r}^{B_1} + \overbrace{\rho V_\theta V_r \hat{e}_\theta \hat{e}_r}^{B_2} + \overbrace{\rho V_Z V_r \hat{e}_Z \hat{e}_r}^{B_3} \\ + \overbrace{\rho V_r V_\theta \hat{e}_r \hat{e}_\theta}^{B_4} + \overbrace{\rho V_\theta V_\theta \hat{e}_\theta \hat{e}_\theta}^{B_5} + \overbrace{\rho V_Z V_\theta \hat{e}_Z \hat{e}_\theta}^{B_6} \\ + \underbrace{\rho V_r V_Z \hat{e}_r \hat{e}_Z}_{B7} + \underbrace{\rho V_\theta V_Z \hat{e}_\theta \hat{e}_Z}_{B_8} + \underbrace{\rho V_Z V_Z \hat{e}_Z \hat{e}_Z}_{B_9} \end{bmatrix}$$

$$\nabla = \overbrace{\hat{e}_r \frac{\partial}{\partial r}}^{A_1} + \overbrace{\frac{\hat{e}_\theta}{r}\frac{\partial}{\partial \theta}}^{A_2} + \overbrace{\hat{e}_Z \frac{\partial}{\partial Z}}^{A_3}$$

Then:

$$\nabla \bullet (\rho \vec{V}\vec{V}) = \nabla \bullet \begin{bmatrix} \overbrace{\rho V_r V_r \hat{e}_r \hat{e}_r}^{B_1} + \overbrace{\rho V_\theta V_r \hat{e}_\theta \hat{e}_r}^{B_2} + \overbrace{\rho V_Z V_r \hat{e}_Z \hat{e}_r}^{B_3} \\ + \overbrace{\rho V_r V_\theta \hat{e}_r \hat{e}_\theta}^{B_4} + \overbrace{\rho V_\theta V_\theta \hat{e}_\theta \hat{e}_\theta}^{B_5} + \overbrace{\rho V_Z V_\theta \hat{e}_Z \hat{e}_\theta}^{B_6} \\ + \underbrace{\rho V_r V_Z \hat{e}_r \hat{e}_Z}_{B7} + \underbrace{\rho V_\theta V_Z \hat{e}_\theta \hat{e}_Z}_{B_8} + \underbrace{\rho V_Z V_Z \hat{e}_Z \hat{e}_Z}_{B_9} \end{bmatrix}$$

$$= (A_1 + A_2 + A_3) \bullet (B_1 + B_2 + B_3 + \ldots + B_9)$$

$$\nabla \bullet (\rho \vec{V}\vec{V}) = (A_1 + A_2 + A_3) \bullet (B_1 + B_2 + B_3 + \ldots + B_9)$$

$$= \hat{e}_r \left[\overbrace{\frac{\partial(\rho V_r V_r)}{\partial r}}^{A_1B_1} + \overbrace{\frac{\partial(\rho V_\theta V_r)}{r\partial \theta}}^{A_2B_4} + \overbrace{\frac{\partial(\rho V_Z V_r)}{\partial r}}^{A_3B_7} + \rho \overbrace{\frac{V_r V_r}{r}}^{A_2B_1} - \rho \overbrace{\frac{V_\theta V_\theta}{r}}^{A_2B_5} \right]$$

$$+ \hat{e}_\theta \left[\overbrace{\frac{\partial(\rho V_r V_\theta)}{\partial r}}^{A_1B_2} + \overbrace{\frac{\partial(\rho V_\theta V_\theta)}{r\partial \theta}}^{A_2B_5} + \overbrace{\frac{\partial(\rho V_Z V_\theta)}{\partial r}}^{A_3B_8} + \rho \overbrace{\frac{V_r V_\theta}{r}}^{A_2B_2} + \rho \overbrace{\frac{V_\theta V_r}{r}}^{A_2B_4} \right]$$

$$+ \hat{e}_z \left[\overbrace{\frac{\partial(\rho V_r V_z)}{\partial r}}^{A_1B_3} + \overbrace{\frac{\partial(\rho V_\theta V_z)}{r\partial \theta}}^{A_2B_6} + \overbrace{\frac{\partial(\rho V_Z V_z)}{\partial r}}^{A_3B_9} + \rho \overbrace{\frac{V_r V_z}{r}}^{A_2B_1} \right]$$

For example

$$A_1 \bullet B_1 = \hat{e}_r \bullet \frac{\partial(\rho V_r V_r \hat{e}_r \hat{e}_r)}{\partial r} = \hat{e}_r \bullet [\frac{\partial(\rho V_r V_r)}{\partial r} \hat{e}_r \hat{e}_r + \rho V_r V_r \frac{\partial(\hat{e}_r \hat{e}_r)}{\partial r}]$$

$$= \hat{e}_r \bullet \{\frac{\partial(\rho V_r V_r)}{\partial r} \hat{e}_r \hat{e}_r + \rho V_r V_r [\frac{\partial \hat{e}_r}{\partial r} \hat{e}_r + \hat{e}_r \frac{\partial \hat{e}_r}{\partial r}]\}$$

$$= \hat{e}_r \bullet [\frac{\partial(\rho V_r V_r)}{\partial r} \hat{e}_r \hat{e}_r + \rho V_r V_r (0+0)]$$

$$= \frac{\partial(\rho V_r V_r)}{\partial r} \hat{e}_r \bullet \hat{e}_r \hat{e}_r$$

$$= \frac{\partial(\rho V_r V_r)}{\partial r} \hat{e}_r$$

$$A_2 \bullet B_1 = \frac{\hat{e}_\theta}{r} \bullet \frac{\partial(\rho V_r V_r \hat{e}_r \hat{e}_r)}{\partial \theta} = \frac{\hat{e}_\theta}{r} \bullet [\frac{\partial(\rho V_r V_r)}{\partial \theta} \hat{e}_r \hat{e}_r + \rho V_r V_r \frac{\partial(\hat{e}_r \hat{e}_r)}{\partial \theta}]$$

$$= \frac{\hat{e}_\theta}{r} \bullet \{\frac{\partial(\rho V_r V_r)}{\partial \theta} \hat{e}_r \hat{e}_r + \rho V_r V_r [\frac{\partial \hat{e}_r}{\partial \theta} \hat{e}_r + \hat{e}_r \frac{\partial \hat{e}_r}{\partial \theta}]\}$$

$$= \frac{\hat{e}_\theta}{r} \bullet [\frac{\partial(\rho V_r V_r)}{\partial \theta} \hat{e}_r \hat{e}_r + \rho V_r V_r (\hat{e}_\theta \hat{e}_r + \hat{e}_r \hat{e}_\theta)]$$

$$= \frac{\partial(\rho V_r V_r)}{\partial \theta} \frac{\hat{e}_\theta}{r} \bullet \hat{e}_r \hat{e}_r + \rho V_r V_r (\frac{\hat{e}_\theta}{r} \bullet \hat{e}_\theta \hat{e}_r + \frac{\hat{e}_\theta}{r} \bullet \hat{e}_r \hat{e}_\theta)]$$

$$= 0 + \frac{\rho V_r V_r}{r} (\hat{e}_r + 0)$$

$$= \frac{\rho V_r V_r}{r} \hat{e}_r$$

$$A_2 \bullet B_2 = \frac{\hat{e}_\theta}{r} \bullet \frac{\partial(\rho V_\theta \hat{e}_\theta V_r \hat{e}_r)}{\partial\theta} = \frac{\hat{e}_\theta}{r} \bullet [\frac{\partial(\rho V_r V_\theta)}{\partial\theta} \hat{e}_r \hat{e}_\theta + \rho V_r V_\theta \frac{\partial(\hat{e}_r \hat{e}_\theta)}{\partial\theta}]$$

$$= \frac{\hat{e}_\theta}{r} \bullet \{\frac{\partial(\rho V_r V\theta)}{\partial\theta} \hat{e}_r \hat{e}_\theta + \rho V_r V_\theta [\frac{\partial\hat{e}_r}{\partial\theta} \hat{e}_\theta + \hat{e}_r \frac{\partial\hat{e}_\theta}{\partial\theta}]\}$$

$$= \frac{\hat{e}_\theta}{r} \bullet [\frac{\partial(\rho V_r V_\theta)}{\partial\theta} \hat{e}_r \hat{e}_\theta + \rho V_r V_\theta (\hat{e}_\theta \hat{e}_\theta - \hat{e}_r \hat{e}_r)]$$

$$= \frac{\partial(\rho V_r V_\theta)}{\partial\theta} \frac{\hat{e}_\theta}{r} \bullet \hat{e}_r \hat{e}_\theta + \rho V_r V_\theta (\frac{\hat{e}_\theta}{r} \bullet \hat{e}_\theta \hat{e}_\theta - \frac{\hat{e}_\theta}{r} \bullet \hat{e}_r \hat{e}_r)]$$

$$= 0 + \frac{\rho V_r V_\theta}{r} (\hat{e}_\theta + 0)$$

$$= \frac{\rho V_r V_\theta}{r} \hat{e}_\theta$$

$$\nabla P = \hat{e}_r \frac{\partial P}{\partial r} + \frac{\hat{e}_\theta}{r} \frac{\partial P}{\partial\theta} + \hat{e}_Z \frac{\partial P}{\partial Z}$$

$$\rho \vec{f} = \hat{e}_r \vec{f}_r + \hat{e}_\theta f_\theta + \hat{e}_Z f_Z$$

$$\tilde{\tau} = \mu[\nabla\vec{V} + (\nabla\vec{V})^T - \frac{2}{3}(\nabla \bullet \vec{V}) \tilde{I}]$$

$$\nabla\vec{V} = (\hat{e}_r \frac{\partial}{\partial r} + \frac{\hat{e}_\theta}{r} \frac{\partial}{\partial\theta} + \hat{e}_Z \frac{\partial}{\partial Z})(\hat{e}_r V_r + \hat{e}_\theta V_\theta + \hat{e}_Z V_Z)$$

$$= (A_1 + A_2 + A_3)(B_1 + B_2 + B_3)$$

$$= \hat{e}_r \frac{\partial(\hat{e}_r V_r + \hat{e}_\theta V_\theta + \hat{e}_Z V_Z)}{\partial r} + \frac{\hat{e}_\theta}{r} \frac{\partial(\hat{e}_r V_r + \hat{e}_\theta V_\theta + \hat{e}_Z V_Z)}{\partial\theta} + \hat{e}_Z \frac{\partial(\hat{e}_r V_r + \hat{e}_\theta V_\theta + \hat{e}_Z V_Z)}{\partial Z}$$

$$= \hat{e}_r \{\frac{\partial\hat{e}_r}{\partial r} V_r + \hat{e}_r \frac{\partial V_r}{\partial r} + \frac{\partial(\hat{e}_\theta)}{\partial r} V_\theta + \frac{\partial V_\theta}{\partial r} \hat{e}_\theta + \hat{e}_Z \frac{\partial V_Z}{\partial r} + \frac{\partial\hat{e}_Z}{\partial r} V_Z]$$

$$+ \frac{\hat{e}_\theta}{r} \{\frac{\partial\hat{e}_r}{\partial\theta} V_r + \hat{e}_r \frac{\partial V_r}{\partial\theta} + \frac{\partial(\hat{e}_\theta)}{\partial\theta} V_\theta + \frac{\partial V_\theta}{\partial\theta} \hat{e}_\theta + \hat{e}_Z \frac{\partial V_Z}{\partial\theta} + \frac{\partial\hat{e}_Z}{\partial\theta} V_Z]$$

$$+ \hat{e}_Z \{\frac{\partial\hat{e}_r}{\partial Z} V_r + \hat{e}_r \frac{\partial V_r}{\partial Z} + \frac{\partial(\hat{e}_\theta)}{\partial Z} V_\theta + \frac{\partial V_\theta}{\partial Z} \hat{e}_\theta + \hat{e}_Z \frac{\partial V_Z}{\partial Z} + \frac{\partial\hat{e}_Z}{\partial Z} V_Z]$$

$$= \hat{e}_r [\frac{\partial\hat{e}_r}{\partial r} V_r + \hat{e}_r \frac{\partial V_r}{\partial r} + \frac{\partial(\hat{e}_\theta)}{\partial r} V_\theta + \frac{\partial V_\theta}{\partial r} \hat{e}_\theta + \hat{e}_Z \frac{\partial V_Z}{\partial r} + \frac{\partial\hat{e}_Z}{\partial r} V_Z]$$

$$+ \frac{\hat{e}_\theta}{r} [\frac{\partial\hat{e}_r}{\partial\theta} V_r + \hat{e}_r \frac{\partial V_r}{\partial\theta} + \frac{\partial(\hat{e}_\theta)}{\partial\theta} V_\theta + \frac{\partial V_\theta}{\partial\theta} \hat{e}_\theta + \hat{e}_Z \frac{\partial V_Z}{\partial\theta} + \frac{\partial\hat{e}_Z}{\partial\theta} V_Z]$$

$$+ \hat{e}_Z \{\frac{\partial\hat{e}_r}{\partial Z} V_r + \hat{e}_r \frac{\partial V_r}{\partial Z} + \frac{\partial(\hat{e}_\theta)}{\partial Z} V_\theta + \frac{\partial V_\theta}{\partial Z} \hat{e}_\theta + \hat{e}_Z \frac{\partial V_Z}{\partial Z} + \frac{\partial\hat{e}_Z}{\partial Z} V_Z]$$

$$\nabla\vec{V} = \hat{e}_r\hat{e}_r\frac{\partial V_r}{\partial r} + \frac{\partial V_\theta}{\partial r}\hat{e}_r\hat{e}_\theta + \hat{e}_r\hat{e}_Z\frac{\partial V_Z}{\partial r} + \frac{\hat{e}_\theta}{r}\hat{e}_\theta V_r + \frac{\hat{e}_\theta}{r}\hat{e}_r\frac{\partial V_r}{\partial\theta} - \frac{\hat{e}_\theta}{r}\hat{e}_r V_\theta + \frac{\partial V_\theta}{\partial\theta}\frac{\hat{e}_\theta}{r}\hat{e}_\theta + \frac{\hat{e}_\theta}{r}\hat{e}_Z\frac{\partial V_Z}{\partial\theta}$$
$$+\hat{e}_Z\hat{e}_r\frac{\partial V_r}{\partial Z} + \frac{\partial V_\theta}{\partial Z}\hat{e}_Z\hat{e}_\theta + \hat{e}_Z\hat{e}_Z\frac{\partial V_Z}{\partial Z}$$
$$= \begin{pmatrix} \frac{\partial V_r}{\partial r} & \frac{\partial V_\theta}{\partial r} & \frac{\partial V_Z}{\partial r} \\ \frac{1}{r}\frac{\partial V_r}{\partial\theta} - \frac{V_\theta}{r} & \frac{V_r}{r} + \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} & \frac{1}{r}\frac{\partial V_Z}{\partial\theta} \\ \frac{\partial V_r}{\partial Z} & \frac{\partial V_\theta}{\partial Z} & \frac{\partial V_Z}{\partial Z} \end{pmatrix}$$

$$(\nabla\vec{V})^T = \begin{pmatrix} \frac{\partial V_r}{\partial r} & \frac{1}{r}\frac{\partial V_r}{\partial\theta} - \frac{V_\theta}{r} & \frac{\partial V_r}{\partial Z} \\ \frac{\partial V_\theta}{\partial r} & \frac{V_r}{r} + \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} & \frac{\partial V_\theta}{\partial Z} \\ \frac{\partial V_Z}{\partial r} & \frac{1}{r}\frac{\partial V_Z}{\partial\theta} & \frac{\partial V_Z}{\partial Z} \end{pmatrix}$$

$$-\frac{2}{3}(\nabla\bullet\vec{V})\tilde{I} = -\frac{2}{3}(\nabla\bullet\vec{V})\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus,

$$\tilde{\tau} = \mu[\nabla\vec{V} + (\nabla\vec{V})^T - \frac{2}{3}\mu(\nabla\bullet\vec{V})\tilde{I}]$$
$$= \mu\begin{pmatrix} 2\frac{\partial V_r}{\partial r} - \frac{2}{3}(\nabla\bullet\vec{V}) & \frac{1}{r}\frac{\partial V_r}{\partial\theta} - \frac{V_\theta}{r} + \frac{\partial V_\theta}{\partial r} & \frac{\partial V_r}{\partial Z} + \frac{\partial V_Z}{\partial r} \\ \frac{1}{r}\frac{\partial V_r}{\partial\theta} - \frac{V_\theta}{r} + \frac{\partial V_\theta}{\partial r} & 2(\frac{V_r}{r} + \frac{1}{r}\frac{\partial V_\theta}{\partial\theta}) - \frac{2}{3}(\nabla\bullet\vec{V}) & \frac{\partial V_\theta}{\partial Z} + \frac{1}{r}\frac{\partial V_Z}{\partial\theta} \\ \frac{\partial V_Z}{\partial r} + \frac{\partial V_r}{\partial Z} & \frac{1}{r}\frac{\partial V_Z}{\partial\theta} + \frac{\partial V_\theta}{\partial Z} & \frac{\partial V_Z}{\partial Z} - \frac{2}{3}(\nabla\bullet\vec{V}) \end{pmatrix}$$
$$= \begin{pmatrix} \sigma_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \sigma_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{Z\theta} & \tau_{ZZ} \end{pmatrix}$$

$$\nabla \bullet \tilde{\tau} = (\hat{e}_r \frac{\partial}{\partial r} + \frac{\hat{e}_\theta}{r}\frac{\partial}{\partial \theta} + \hat{e}_Z \frac{\partial}{\partial Z}) \bullet \begin{pmatrix} \sigma_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \sigma_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{Z\theta} & \tau_{ZZ} \end{pmatrix}$$

$$= (\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{\partial}{r\partial \theta} + \hat{e}_Z \frac{\partial}{\partial Z}) \bullet (\hat{e}_r\hat{e}_r\sigma_{rr} + \hat{e}_r\hat{e}_\theta\tau_{r\theta} + \hat{e}_r\hat{e}_Z\tau_{rz}$$
$$+ \sigma_{\theta\theta}\hat{e}_\theta\hat{e}_\theta + \hat{e}_\theta\hat{e}_r\tau_{\theta z} + \hat{e}_Z\hat{e}_r\tau_{zr} + \tau_{z\theta}\hat{e}_Z\hat{e}_\theta + \sigma_{zz}\hat{e}_Z\hat{e}_Z\,]$$
$$= \hat{e}_r[\frac{\partial\sigma_{rr}}{\partial r} + \frac{\partial\tau_{r\theta}}{r\partial\theta} + \frac{\partial\tau_{zr}}{\partial Z} + \frac{\sigma_{rr}}{r} - \frac{\sigma_{\theta\theta}}{r}]$$
$$+ \hat{e}_\theta[\frac{\partial\tau_{r\theta}}{\partial r} + \frac{\partial\sigma_{\theta\theta}}{r\partial\theta} + \frac{\partial\tau_{z\theta}}{\partial Z} + \frac{\tau_{r\theta}}{r} + \frac{\tau_{\theta r}}{r}]$$
$$+ \hat{e}_Z[\frac{\partial\tau_{rZ}}{\partial r} + \frac{\partial\tau_{\theta Z}}{r\partial\theta} + \frac{\partial\sigma_{zz}}{\partial Z} + \frac{\tau_{rz}}{r}]$$
$$= \hat{e}_r[\frac{\partial(r\sigma_{rr})}{r\partial r} + \frac{\partial\tau_{r\theta}}{r\partial\theta} + \frac{\partial\tau_{zr}}{\partial Z} - \frac{\sigma_{\theta\theta}}{r}]$$
$$+ \hat{e}_\theta[\frac{\partial(r\tau_{r\theta})}{r\partial r} + \frac{\partial\sigma_{\theta\theta}}{r\partial\theta} + \frac{\partial\tau_{z\theta}}{\partial Z} + \frac{\tau_{\theta r}}{r}]$$
$$+ \hat{e}_Z[\frac{\partial(r\tau_{rZ})}{r\partial r} + \frac{\partial\tau_{\theta Z}}{r\partial\theta} + \frac{\partial\sigma_{zz}}{\partial Z}]$$

Therefore, the Navier-Stokes equation in cylindrical coordinate system will be:

$$\frac{\partial(\rho V_r)}{\partial t} + \frac{\partial(\rho V_r V_r)}{\partial r} + \frac{\partial(\rho V_\theta V_r)}{r\partial\theta} + \frac{\partial(\rho V_Z V_r)}{\partial Z} + \rho\frac{V_r V_r}{r} - \rho\frac{V_\theta V_\theta}{r}$$
$$= -\frac{\partial p}{\partial r} + [\frac{\partial(r\sigma_{rr})}{r\partial r} + \frac{\partial\tau_{r\theta}}{r\partial\theta} + \frac{\partial\tau_{zr}}{\partial Z} - \frac{\sigma_{\theta\theta}}{r}] + \rho f_r$$

r-direction: *or*

$$\frac{\partial(\rho V_r)}{\partial t} + \frac{\partial(\rho r V_r V_r)}{r\,\partial r} + \frac{\partial(\rho V_\theta V_r)}{r\partial\theta} + \frac{\partial(\rho V_Z V_r)}{\partial Z} - \rho\frac{V_\theta V_\theta}{r}$$
$$= -\frac{\partial p}{\partial r} + [\frac{\partial(r\sigma_{rr})}{r\partial r} + \frac{\partial\tau_{r\theta}}{r\partial\theta} + \frac{\partial\tau_{zr}}{\partial Z} - \frac{\sigma_{\theta\theta}}{r}] + \rho f_r$$

$$\frac{\partial(\rho V_\theta)}{\partial t}+\frac{\partial(\rho V_r V_\theta)}{\partial r}+\frac{\partial(\rho V_\theta V_\theta)}{r\partial\theta}+\frac{\partial(\rho V_Z V_\theta)}{\partial Z}+\rho\frac{V_r V_\theta}{r}$$
$$=-\frac{1}{r}\frac{\partial p}{\partial\theta}+[\frac{\partial(r\tau_{r\theta})}{r\partial r}+\frac{\partial\sigma_{\theta\theta}}{r\partial\theta}+\frac{\partial\tau_{z\theta}}{\partial Z}+\frac{\tau_{\theta r}}{r}]+\rho f_\theta$$

θ-direction: *or*

$$\frac{\partial(\rho V_\theta)}{\partial t}+\frac{\partial(\rho r V_r V_\theta)}{r\partial r}+\frac{\partial(\rho V_\theta V_\theta)}{r\partial\theta}+\frac{\partial(\rho V_Z V_\theta)}{\partial Z}+\rho\frac{V_r V_\theta}{r}$$
$$=-\frac{1}{r}\frac{\partial p}{\partial\theta}+[\frac{\partial(r\tau_{r\theta})}{r\partial r}+\frac{\partial\sigma_{\theta\theta}}{r\partial\theta}+\frac{\partial\tau_{z\theta}}{\partial Z}+\frac{\tau_{\theta r}}{r}]+\rho f_\theta$$

z-direction:

$$\frac{\partial(\rho V_Z)}{\partial t}+\frac{\partial(\rho V_r V_Z)}{\partial r}+\frac{\partial(\rho V_\theta V_Z)}{r\partial\theta}+\frac{\partial(\rho V_Z V_Z)}{\partial Z}=-\frac{\partial p}{\partial Z}+[\frac{\partial(r\tau_{rZ})}{r\partial r}+\frac{\partial\tau_{\theta Z}}{r\partial\theta}+\frac{\partial\sigma_{zz}}{\partial Z}]+\rho f_Z$$

4.7 Discussion of Flow Governing Equations

The properties and the flow pattern of a moving fluid are governed by the fundamental laws of physics expression:

- Conservation of mass
- Conservation of momentum
- Conservation of energy
- Equation of state

When the mathematical equations expression these laws are solved satisfying the approximate initial and boundary conditions, the fluid properties and the flow pattern results.

These conservation equations involved three scalar fields (i.e., ρ, P, T) and one vector field (i.e., $\vec{V}$) as the unknown functions.

Conservation laws for	Equations	Number of Eqns.	Order of Eqns.	Total order
Mass	$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\vec{V} = 0$	1	1	1
Momentum	$\frac{\partial}{\partial t}\left(\rho\vec{V}\right) + \nabla\cdot(\rho\vec{V}\vec{V}) = -\nabla p + \nabla\cdot\tilde{\tau} + \rho\vec{f}$	3	2	6
Energy	–	1	2	2
Equation of state	$p = f(\rho, T)$	1	0	0
	Total	6	9	9

- Independent variables: q_1, q_2, q_3, t
- Dependent variables: $\rho, P, T, V_1, V_2, V_3$
- Prescribed quantities: $\vec{f}, \mu(T), c_p(T), R$, etc.
- There are six equations and six dependent variables $\Rightarrow$ Equations can be solved.
- The sum of the order of the differential equations is equal to nine and we need nine boundary conditions.
- The conservation equations are nonlinear, that is coefficients of some of the derivatives are dependent variables. Need an interactive solution.
- All equations are coupled and hence must be solved for simultaneously.

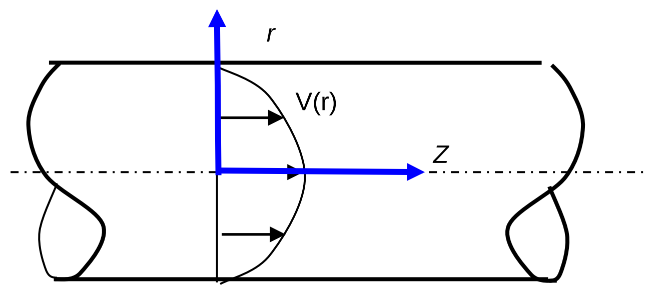
In general, exact solutions to the conservation equations are unknown because they are nonlinear and no general method is presently known for solving nonlinear differential equations. However when restrictions are placed on the flow geometry and the fluid properties, several exact solutions to the conservation equations are possible.

4.8 Hagan-Poiseuille Flow

Hagan-Poisuille flow results when the flow through a circular pipe has attained what is called a fully developed profile.

Assumptions:

1. Steady flow: $\Rightarrow$ $\frac{\partial}{\partial t} = 0$
2. Incompressible flows $\Rightarrow$ ρ, μ are constant.
3. Axial symmetry $\Rightarrow$ $\frac{\partial}{\partial \theta} = 0$
4. In addition, we impose on velocity $V_r = 0$
5. No body forces.



Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \vec{V}) = 0 \qquad \Rightarrow \qquad \nabla \bullet \vec{V} = 0$$

$$\frac{1}{r}[\frac{\partial (rV_r)}{\partial r} + \frac{\partial V_\theta}{\partial \theta} + \frac{\partial (rV_z)}{\partial z}] = 0$$

$$\Rightarrow \frac{\partial (rV_r)}{\partial r} + \frac{\partial V_\theta}{\partial \theta} + \frac{\partial (rV_z)}{\partial z} = 0$$

$$\Rightarrow \frac{\partial (rV_z)}{\partial z} = 0 \Rightarrow \frac{\partial V_z}{\partial z} = 0$$

$$\Rightarrow V_z = V_z(r)$$

V_z is only the function of *r*, not functions of *Z* and θ [due to the axial symmetry of $\frac{\partial}{\partial \theta} = 0$].

Momentum equation:

r-direction:

$$\frac{\partial (\rho V_r)}{\partial t} + \frac{\partial (\rho r V_r V_r)}{r \partial r} + \frac{\partial (\rho V_\theta V_r)}{r \partial \theta} + \frac{\partial (\rho V_Z V_r)}{\partial Z} - \rho \frac{V_\theta V_\theta}{r}$$
$$= -\frac{\partial p}{\partial r} + [\frac{\partial (r \sigma_{rr})}{r \partial r} + \frac{\partial \tau_{r\theta}}{r \partial \theta} + \frac{\partial \tau_{zr}}{\partial Z} - \frac{\sigma_{\theta\theta}}{r}] + \rho f_r$$

$$\begin{pmatrix} \sigma_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \sigma_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{Z\theta} & \tau_{ZZ} \end{pmatrix} = \mu \begin{pmatrix} 2\frac{\partial V_r}{\partial r} - \frac{2}{3}(\nabla \bullet \vec{V}) & \frac{1}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta}{r} + \frac{\partial V_\theta}{\partial r} & \frac{\partial V_r}{\partial Z} + \frac{\partial V_Z}{\partial r} \\ \frac{1}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta}{r} + \frac{\partial V_\theta}{\partial r} & 2(\frac{V_r}{r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta}) - \frac{2}{3}(\nabla \bullet \vec{V}) & \frac{\partial V_\theta}{\partial Z} + \frac{1}{r}\frac{\partial V_Z}{\partial \theta} \\ \frac{\partial V_Z}{\partial r} + \frac{\partial V_r}{\partial Z} & \frac{1}{r}\frac{\partial V_Z}{\partial \theta} + \frac{\partial V_\theta}{\partial Z} & \frac{\partial V_Z}{\partial Z} - \frac{2}{3}(\nabla \bullet \vec{V}) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & \mu\frac{\partial V_Z}{\partial r} \\ 0 & 0 & 0 \\ \mu\frac{\partial V_Z}{\partial r} & 0 & 0 \end{pmatrix}$$

therefore

$$0+0+0+0-0 = -\frac{\partial p}{\partial r} + [0+0+\frac{\partial(\mu\frac{\partial V_Z}{\partial r})}{\partial Z} - 0] + 0$$

$$\Rightarrow \frac{\partial p}{\partial r} = \frac{\partial(\mu\frac{\partial V_Z}{\partial r})}{\partial Z} = \mu\frac{\partial^2 V_Z}{\partial Z \partial r} = \mu\frac{\partial}{\partial r}(\frac{\partial V_Z}{\partial Z}) = 0$$

$$\Rightarrow \frac{\partial p}{\partial r} = 0$$

Leads to the conclusion of $P = P(Z)$

θ-direction:

$$\frac{\partial(\rho V_\theta)}{\partial t} + \frac{\partial(\rho r V_r V_\theta)}{r\partial r} + \frac{\partial(\rho V_\theta V_\theta)}{r\partial\theta} + \frac{\partial(\rho V_Z V_\theta)}{\partial Z} + \rho\frac{V_r V_\theta}{r}$$

$$= -\frac{1}{r}\frac{\partial p}{\partial \theta} + [\frac{\partial(r\tau_{r\theta})}{r\partial r} + \frac{\partial \sigma_{\theta\theta}}{r\partial\theta} + \frac{\partial \tau_{z\theta}}{\partial Z} + \frac{\tau_{\theta r}}{r}] + \rho f_\theta$$

$$\Rightarrow 0+0+0+0+0 = -0+[0+0+0+0]+0$$

$$\Rightarrow 0 = 0$$

Z-direction:

$$\frac{\partial(\rho V_Z)}{\partial t}+\frac{\partial(\rho V_r V_Z)}{\partial r}+\frac{\partial(\rho V_\theta V_Z)}{r\partial\theta}+\frac{\partial(\rho V_Z V_Z)}{\partial Z}=-\frac{\partial P}{\partial Z}+[\frac{\partial(r\tau_{rZ})}{r\partial r}+\frac{\partial\tau_{rZ}}{r\partial\theta}+\frac{\partial\sigma_{zz}}{\partial Z}]+\rho f_Z$$

$$\Rightarrow 0+0+0+0=-\frac{\partial P}{\partial Z}+[\frac{\partial(r\mu\frac{\partial V_Z}{\partial r})}{r\partial r}+0+0]+0$$

$$\Rightarrow \underbrace{\frac{\partial P}{\partial Z}}_{\text{Only a function of Z}} = \underbrace{\mu\frac{\partial(r\frac{\partial V_Z}{\partial r})}{r\partial r}}_{\text{Only a function of r}}$$

The only way the above equation is true if the two side are both equal to a constant.

Therefore:

$$\frac{\partial p}{\partial Z}=C$$

$$\mu\frac{\partial(r\frac{\partial V_Z}{\partial r})}{r\partial r}=C$$

where C is a constant.

Boundary conditions:

- At the wall, $r = r_0$, $V_Z = 0$.
- At the centerline, slope of velocity profile is zero. $r = 0$, $\frac{\partial V_Z}{\partial r} = 0$

$$\mu \frac{\partial (r \frac{\partial V_Z}{\partial r})}{r \partial r} = C \quad \Rightarrow \frac{\partial (r \frac{\partial V_Z}{\partial r})}{\partial r} = \frac{Cr}{\mu}$$

$$\Rightarrow \int \partial (r \frac{\partial V_Z}{\partial r}) = \int \frac{Cr}{\mu} \partial r \frac{Cr}{\mu} \partial r \quad \Rightarrow r \frac{\partial V_Z}{\partial r} = \frac{Cr^2}{2\mu} + B_1$$

Since $r = 0$, $\frac{\partial V_Z}{\partial r} = 0 \quad \Rightarrow \quad B_1 = 0$

$$r \frac{\partial V_Z}{\partial r} = \frac{Cr^2}{2\mu} \quad \Rightarrow \frac{\partial V_Z}{\partial r} = \frac{Cr}{2\mu}$$

$$\Rightarrow \int \partial V_Z = \int \frac{Cr}{2\mu} \partial r \quad \Rightarrow V_Z = \frac{Cr^2}{4\mu} + B_2$$

Since $r = r_0$, $V_Z = 0 \quad \Rightarrow \quad B_2 = -\frac{C r_0^{\ 2}}{4\mu}$

Therefore: $V_Z = \frac{C}{4\mu}(r^2 - r_0^{\ 2})$

Since $\frac{\partial p}{\partial Z} = C$ and $P = P(Z)$

Therefore: $V_Z = \frac{1}{4\mu} \frac{dP}{dZ} (r^2 - r_0^{\ 2})$

Mass flow rate:

$$\dot{m} = \int \rho V_Z dA = \int_0^{r_0} \rho \frac{1}{4\mu} \frac{dP}{dZ} (r^2 - r_0^{\ 2})(2\pi r dr)$$

$$= \rho \frac{\pi}{2\mu} \frac{dP}{dZ} (\frac{r^4}{4} - \frac{r_0^{\ 2} r^2}{2}) \Big|_0^{r_0}$$

$$= \rho \frac{\pi}{2\mu} \frac{dP}{dZ} (-\frac{r_0^{\ 4}}{4})$$

$$= -\rho \frac{\pi}{8\mu} \frac{dP}{dZ} r_0^{\ 4}$$

Or

$$\frac{dP}{dZ} = -\frac{8\mu \dot{m}}{\rho \pi r_0^{\ 4}}$$