Chapter 5
Simplification to the Conservation Equations

5.1 Governing Equations for Ideal Fluid Flows

Isotropic Fluid:

Isotropic, Newtonian are assumed to have linear relationship between stress and rate of strain.

A fluid is said to be isotropic when the relation between the components of stress and those of rate of train is the same in all directions. It is said to be Newtonian when this relationship is linear, that is when the fluid obeys stokes law of friction.

Ideal flow:

Non-heat conducting, inviscid, incompressible, homogeneous fluid is defined as ideal fluid.

Assumptions used are:

- Non-heat conductive
- Homogeneous
- Incompressible
- Inviscid flow

Controlling equations:

Continuity equation: $\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \vec{V}) = 0$

Momentum equation: $\frac{\partial (\rho \vec{V})}{\partial t} + \nabla \bullet (\rho \vec{V} \vec{V}) + \nabla P - \nabla \bullet \tilde{\tau} - \rho \vec{f} = 0$

Energy equation: $\Rightarrow$ drop out since non-heat conductive assumption

State equation: $P = \rho R T$

Simplification of continuity equation for in ideal flows:

$$\frac{\partial\rho}{\partial t}+\nabla\bullet(\rho\vec{V})=0\Rightarrow\frac{\partial\rho}{\partial t}+\vec{V}\bullet\nabla\rho+\rho\nabla\bullet\vec{V}=0$$

$$\Rightarrow\frac{D\rho}{Dt}+\rho\nabla\bullet\vec{V}=0$$

$$\Rightarrow\nabla\bullet\vec{V}=0$$

Simplification of momentum equation in ideal flows:

$$\frac{\partial(\rho\vec{V})}{\partial t}+\nabla\bullet(\rho\vec{V}\vec{V})+\nabla P-\nabla\bullet\tilde{\tau}-\rho\vec{f}=0$$

$$\Rightarrow\quad \vec{V}\frac{\partial\rho}{\partial t}+\rho\frac{\partial\vec{V}}{\partial t}+\vec{V}\nabla\bullet(\rho\vec{V})+(\rho\vec{V}\bullet\nabla)\vec{V}+\nabla P-\overbrace{\nabla\bullet\tilde{\tau}}^{=0\ due\ to\ inviscid}-\rho\vec{f}=0$$

$$\Rightarrow\quad \vec{V}[\frac{\partial\rho}{\partial t}+\nabla\bullet(\rho\vec{V})]+\rho[\frac{\partial\vec{V}}{\partial t}+(\vec{V}\bullet\nabla)\vec{V}]+\nabla P-\overbrace{\nabla\bullet\tilde{\tau}}^{=0\ due\ to\ inviscid}-\rho\vec{f}=0$$

$$\Rightarrow\quad \rho\frac{D\vec{V}}{Dt}=-\nabla P+\rho\vec{f}$$

$\rho\dfrac{D\vec{V}}{Dt}=-\nabla P+\rho\vec{f}$ is also called Euler equation

For non-heat-conducting, inviscid, incompressible, homogeneous fluid, density $\rho=$ constant, internal energy $e=$ constant (i.e., $T=$ constant) for the entire flow. The dependent variables become $\vec{V},P$.

For ideal fluid, the governing equations are:

1). Continuity equation: $\nabla\bullet\vec{V}=0$

2). Euler equation $\rho\dfrac{D\vec{V}}{Dt}=-\nabla P+\rho\vec{f}$

or $\dfrac{D\vec{V}}{Dt}=-\nabla(\dfrac{P}{\rho})+\vec{f}$

For the Euler equation, it can be re-written as:

$$\frac{D\vec{V}}{Dt} = -\nabla(\frac{P}{\rho}) + \vec{f}$$

$$\Rightarrow \qquad \frac{\partial \vec{V}}{\partial t} + (\vec{V} \bullet \nabla)\vec{V} = -\nabla(\frac{P}{\rho}) + \vec{f}$$

According to vector identity relation:

$$\nabla(\vec{A} \bullet \vec{B}) = (\vec{A} \bullet \nabla)\vec{B} + (\vec{B} \bullet \nabla)\vec{A} + \vec{B} \times (\nabla \times \vec{A}) + \vec{A} \times (\nabla \times \vec{B})$$

Using $\vec{A} = \vec{B} = \vec{V}$, then

$$\nabla(\vec{V} \bullet \vec{V}) = (\vec{V} \bullet \nabla)\vec{V} + (\vec{V} \bullet \nabla)\vec{V} + \vec{V} \times (\nabla \times \vec{V}) + \vec{V} \times (\nabla \times \vec{V}) = 2[(\vec{V} \bullet \nabla)\vec{V} + \vec{V} \times (\nabla \times \vec{V})]$$

$$\Rightarrow \qquad (\vec{V} \bullet \nabla)\vec{V} = \nabla(\frac{\vec{V} \bullet \vec{V}}{2}) - \vec{V} \times (\nabla \times \vec{V})$$

Therefore, Euler equation can be expressed as:

$$\frac{\partial \vec{V}}{\partial t} + \nabla(\frac{\vec{V} \bullet \vec{V}}{2}) - \vec{V} \times (\nabla \times \vec{V}) = -\nabla(\frac{P}{\rho}) + \vec{f}$$

Rearrange the above equation, we get:

$$\frac{\partial \vec{V}}{\partial t} + \nabla(\frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho}) - \vec{V} \times (\nabla \times \vec{V}) = \vec{f}$$

Let us only consider body forces that are conservative only.
A necessary and sufficient condition for $\vec{f}$ to be conservative is that the force field $\vec{f}$ can be represent as the gradient of a scalar field U, i.e., $\vec{f} = \nabla U$

Substitute for $\vec{f} = \nabla U$ into the Euler equation, it becomes:

$$\frac{\partial \vec{V}}{\partial t} + \nabla(\frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho}) - \vec{V} \times (\nabla \times \vec{V}) = \nabla U$$

Rearrange the equation, it can express as:

$$\frac{\partial \vec{V}}{\partial t} + \nabla(\frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho}) - \vec{V} \times (\nabla \times \vec{V}) = \nabla U \quad \Rightarrow \quad \frac{\partial \vec{V}}{\partial t} + \nabla(\frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho} - U) - \vec{V} \times (\nabla \times \vec{V}) = 0.$$

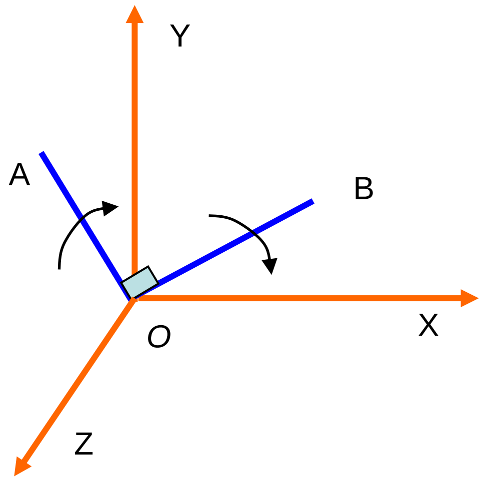
This is a differential equation which holds at every point in an ideal fluid (i.e. non-heat conducting, homogeneous, incompressible, inviscid flow).

There are two special cases for which this equation can be integrated in closed form:

1). Irrotational flow
2). Steady flow along streamline

5.2 Rotation, Vorticity and Circulation:

Fluid rotation



Definition: Fluid rotation at a point *O* (also called the angular velocity at the point) is the average angular velocity of two infinitesimal and mutually perpendicular fluid lines *OA* and *OB* instantaneously passing thorough point *O*.

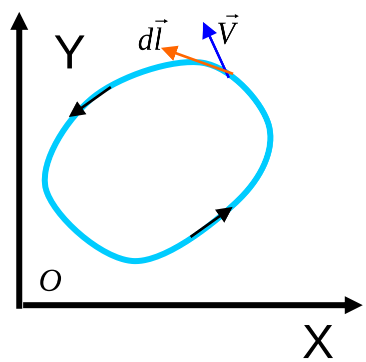
A fluid line is a line passing through a set of fluid particles of fixed identity.

The average fluid rotation at a point is given by $\vec{\varpi} = \frac{1}{2}(\nabla \times \vec{V})$.

Vorticity

Definition: Vorticity is defined to be $\vec{\Omega} = 2\vec{\varpi} = \nabla \times \vec{V}$.
In an irrotational velocity field as know as irrotational flow, i.e., $\nabla \times \vec{V} = 0$ and $\vec{\Omega} = 0$.

Circulation



Definition: Circulation is defined as $\Gamma = -\oint_C \vec{V}\, d\vec{l}$, where C is a closed path.

Circulation is the value of the line integral of the velocity around a closed curve *C*.

By Stokes' theorem:

$$\Gamma = -\oint_C \vec{V}\, d\vec{l} = -\iint_S (\nabla \times \vec{V}) \bullet d\vec{A}$$

Or

$$\Gamma = -\oint_C \vec{V}\, d\vec{l} = -\iint_S \vec{\Omega} \bullet d\vec{A}$$ where S is the area enclosed by curve *C*.

5.3 Integral of Euler Equation in Irrotational Flows

The Euler equation $\frac{\partial \vec{V}}{\partial t} + \nabla(\frac{\vec{V} \bullet \vec{V}}{2}) - \vec{V} \times (\nabla \times \vec{V}) = -\nabla(\frac{P}{\rho}) + \vec{f}$

If $\vec{f}$ is conservative, then, $\vec{f}$ can be represented as the gradient of a scalar field U, i.e.,

$\vec{f} = \nabla U$

Therefore, the above equation can be expressed as

$$\frac{\partial \vec{V}}{\partial t} + \nabla(\frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho} - U) - \vec{V} \times (\nabla \times \vec{V}) = 0 .$$

If the flow is irrotational, then $\nabla \times \vec{V} = 0$. The above equation can be simplified as:

$$\frac{\partial \vec{V}}{\partial t} + \nabla(\frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho} - U) = 0$$

For steady flows: $\frac{\partial \vec{V}}{\partial t} = 0$, then, the equation can be simplified as:

$$\nabla(\frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho} - U) = 0$$

By taking a dot product with $d\vec{l}$, an element length along any arbitrary path, we get:

$$\nabla(\frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho} - U) \bullet d\vec{l} = 0$$

By definition, $\nabla() \bullet d\vec{l} = d()$, and therefore,

$$\nabla(\frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho} - U) \bullet d\vec{l} = d(\frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho} - U) = 0$$

Integrating the above equation leads to the equation known as unsteady Bernoulli equation.

$$\int d(\frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho} - U) = 0 \quad \Rightarrow \quad \frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho} - U = const\ C$$

If we choose a Cartesian coordinate system with the Z-axial to be positive when pointing upward and normal to the surface of the earth, The force on a body of mass *m* is given by *(0,0,-mg).* Thus for $\vec{f}$ which is body force vector per unit mass. It becomes $\vec{f} = (f_x, f_y, f_z) = (0, 0, -g)$.

$$\frac{\partial U}{\partial Z} = -g \quad \Rightarrow \quad U = -g\,Z + \text{Constant } C$$

By choosing reference as $U = 0$ at $Z = 0$ leads to Constant $C=0$.

Then, $U = -g\,Z$.

Therefore, Bernoulli equation for irrotational flow becomes:

$$\frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho} + gZ = C$$

C is a constant all over the flow field.

5.4 Potential Flow and Potential Function

Potential Flow

Definition: A non-heat conducting, homogeneous, inviscid, incompressible (i.e., ideal fluid), and irrotational flow is defined as potential flow.

For potential flows, the governing equations of the Fluid flow are:

1). Continuity equation: $\nabla \bullet \vec{V} = 0$

2). Euler equation

$$\frac{\partial \vec{V}}{\partial t} + \nabla\left(\frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho} - U\right) - \vec{V} \times (\nabla \times \vec{V}) = 0$$

$$\Rightarrow \quad \frac{\partial \vec{V}}{\partial t} + \nabla\left(\frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho} - U\right) = 0$$

Velocity Potential (ϕ)

Definition: Velocity potential is defined only for ideal irrotational flow for either steady or unsteady flows as:

$\vec{V} = \nabla \phi$

Since

$$\vec{V} = V_1 \hat{e}_1 + V_2 \hat{e}_2 + V_3 \hat{e}_3$$

$$\nabla \phi = \frac{\partial \phi}{h_1 \partial q_1} \hat{e}_1 + \frac{\partial \phi}{h_2 \partial q_2} \hat{e}_2 + \frac{\partial \phi}{h_3 \partial q_3} \hat{e}_3$$

Therefore: $V_1 = \frac{1}{h_1} \frac{\partial \phi}{\partial q_1}$; $V_2 = \frac{1}{h_2} \frac{\partial \phi}{\partial q_2}$; $V_3 = \frac{1}{h_3} \frac{\partial \phi}{\partial q_3}$

Note: Potential function ϕ is defined for either 2-D or 3-D, steady or unsteady flow. As long as the flow is irrotational, the potential function ϕ exists.

Since $\nabla \bullet \vec{V} = 0$ and $\vec{V} = \nabla \phi$ $\Rightarrow$ $\nabla \bullet \vec{V} = \nabla \bullet \nabla \phi = \nabla^2 \phi = 0$

$\nabla^2 \phi = 0$ $\Rightarrow$ Laplace Equation.

Therefore, the potential function satisfies the Laplace Equation.

5.5 Path Lines, Streak Line and Streamlines

Pathlines

This is the path traced out by any one particle of the fluid in motion. It is a history of the particle's location of the same particle.

Streaklines

Streakline is defined as an instantaneous line whose points are occupied by all the particles originating from some specific points in the fluid field.

Streamlines

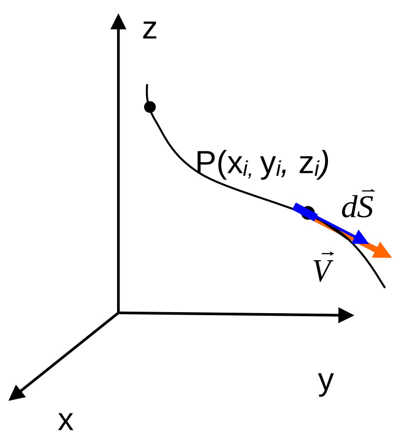
A streamline is defined as an imaginary line drawn in the fluid whose tangent at any point is in the direction of the velocity vector at that point.

- By definition there is no flow across it at any point.
- Any streamline may be replaced by a solid boundary without modifying the flow.
- Any solid boundary is itself a streamline of the flow around it.

Streakline, pathlines and streamlines are in general different in unsteady flows, while in steady flow both are identical.

According to the definition of streamlines, no fluid can cross a stream surface or the walls of a stream tube.

By definition, the equation of a streamline is $\vec{V} \times d\vec{S} = 0$, where $d\vec{S}$ is an elemental length along a streamline. $\vec{V}$ is the flow velocity vector.



$$\vec{V} \times d\vec{S} = 0$$

$$\vec{V} \times d\vec{S} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ V_1 & V_2 & V_3 \\ h_1 dq_1 & h_2 dq_2 & h_3 dq_3 \end{vmatrix}$$

$$= \hat{e}_1 (V_2 h_3 dq_3 - V_3 h_2 dq_2) + \hat{e}_2 (V_3 h_1 dq_1 - V_1 h_3 dq_3)$$
$$+ \hat{e}_3 (V_1 h_2 dq_2 - V_2 h_1 dq_1) = 0$$

In differential form:

$$V_2 h_3 dq_3 - V_3 h_2 dq_2 = 0$$
$$V_3 h_1 dq_1 - V_1 h_3 dq_3 = 0$$
$$V_1 h_2 dq_2 - V_2 h_1 dq_1 = 0$$

5.6 Integral of Euler Equation along a Streamline

In the body force is conservative; the Euler equation can be expressed as:

$$\frac{\partial \vec{V}}{\partial t} + \nabla(\frac{\vec{V} \bullet \vec{V}}{2}) - \vec{V} \times (\nabla \times \vec{V}) = -\nabla(\frac{P}{\rho}) + \nabla U$$

For the steady flow $\frac{\partial \vec{V}}{\partial t} = 0$

Then: $\nabla(\frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho} - U) = \vec{V} \times (\nabla \times \vec{V})$

Multiple (dot product) both sides of the above equation by $d\vec{S}$, which is an element length alone a streamline.

$$\nabla(\frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho} - U) \bullet d\vec{S} = \vec{V} \times (\nabla \times \vec{V}) \bullet d\vec{S}$$

According to vector identity: $\vec{C} \bullet (\vec{A} \times \vec{B}) = \vec{B} \bullet (\vec{C} \times \vec{A}) = \vec{A} \bullet (\vec{B} \times \vec{C})$

If $\vec{C} = d\vec{S}$; $\vec{A} = \vec{V}$; $\vec{B} = \nabla \times \vec{V}$,

Then $\vec{V} \times (\nabla \times \vec{V}) \bullet d\vec{S} = d\vec{S} \bullet [\vec{V} \times (\nabla \times \vec{V})] = (\nabla \times \vec{V}) \bullet (d\vec{S} \times \vec{V}) = 0$

Therefore, $\nabla(\frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho} - U) \bullet d\vec{S} = 0 \quad \Rightarrow \quad d(\frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho} - U) = 0$

Integrating above equation yields $\frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho} - U = const \quad C$

Consider U is only the conservative gravity field, therefore

$\frac{\vec{V} \bullet \vec{V}}{2} + \frac{P}{\rho} + gZ = const = H_S$, where H_S is a constant only along a streamline.

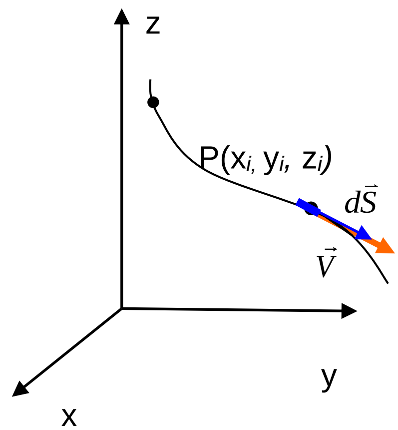
5.7 Stream Function

According to the definition of streamlines, no fluid can cross a streamline.

By definition, the equation of a streamline is $\vec{V} \times d\vec{S} = 0$. Where $d\vec{S}$ is an elemental length along a streamline. $\vec{V}$ is the flow velocity vector.

$$\vec{V} \times d\vec{S} = 0$$

$$\vec{V} \times d\vec{S} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ V_1 & V_2 & V_3 \\ h_1 dq_1 & h_2 dq_2 & h_3 dq_3 \end{vmatrix}$$
$$= \hat{e}_1 (V_2 h_3 dq_3 - V_3 h_2 dq_2) + \hat{e}_2 (V_3 h_1 dq_1 - V_1 h_3 dq_3)$$
$$+ \hat{e}_3 (V_1 h_2 dq_2 - V_2 h_1 dq_1) = 0$$



In differential form:

$$V_2 h_3 dq_3 - V_3 h_2 dq_2 = 0$$
$$V_3 h_1 dq_1 - V_1 h_3 dq_3 = 0$$
$$V_1 h_2 dq_2 - V_2 h_1 dq_1 = 0$$

or

$$\frac{V_1}{h_1 dq_1} = \frac{V_2}{h_2 dq_2} = \frac{V_3}{h_3 dq_3} = const$$

$$\left.\begin{aligned} V_2 h_3 dq_3 - V_3 h_2 dq_2 &= 0 \\ V_3 h_1 dq_1 - V_1 h_3 dq_3 &= 0 \\ V_1 h_2 dq_2 - V_2 h_1 dq_1 &= 0 \end{aligned}\right\} \quad \Rightarrow \quad \text{only two independent equations.}$$

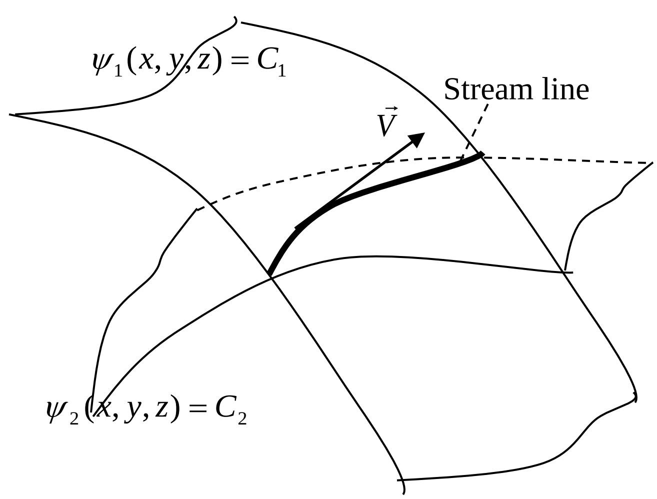
For general purpose, in Cartesian coordinate system, the two independent equations can be written as

$$a_1(x,y,z)dx + a_2(x,y,z)dy + a_3(x,y,z)dz = 0$$
$$b_1(x,y,z)dx + b_2(x,y,z)dy + b_3(x,y,z)dz = 0$$

The solutions of the above equations can be expressed as

$\psi_1(x,y,z) = C_1$, where C_1 is a constant.

$\psi_2(x,y,z) = C_2$, where C_2 is a constant.



It can be seen that the above equation represents a one parameter family of surface. When constant C_1 and C_2 change their values, the above equations represent different stream surfaces.

A streamline in space can be expressed as the curve of intersection of two surfaces.

As shown in the figure, along the streamline, the velocity $\vec{V}$ lies in both the surface of $\psi_1(x,y,z)=C_1$ and $\psi_2(x,y,z)=C_2$. Since direction of the gradients of $\psi_1(x,y,z)$ and $\psi_2(x,y,z)$ (i.e., $\nabla\psi_1$ and $\nabla\psi_2$) being parallel to the direction that normal to the their respective surfaces, thus, $\vec{V}$ is normal to $\nabla\psi_1$ and $\nabla\psi_2$.

$\vec{V}\bullet\nabla\psi_1=0$ and $\vec{V}\bullet\nabla\psi_2=0$

This shows that $\vec{V}$ is normal to the plane formed by the vectors $\nabla\psi_1$ and $\nabla\psi_2$. In other words, $\vec{V}$ should be parallel to the cross product of $\nabla\psi_1$ and $\nabla\psi_2$, i.e.,

$$\mu\vec{V}=\nabla\psi_1\times\nabla\psi_2=\begin{vmatrix}\hat{e}_1 & \hat{e}_2 & \hat{e}_3\\ \dfrac{\partial\psi_1}{h_1\partial q_1} & \dfrac{\partial\psi_1}{h_2\partial q_2} & \dfrac{\partial\psi_1}{h_3\partial q_3}\\ \dfrac{\partial\psi_2}{h_1\partial q_2} & \dfrac{\partial\psi_2}{h_2\partial q_2} & \dfrac{\partial\psi_2}{h_3\partial q_3}\end{vmatrix}$$

where μ is a scalar function of position.

Since $\nabla\bullet(\mu\vec{V})=\nabla\bullet(\nabla\psi_1\times\nabla\psi_2)=0$, therefore, μ can be arbitrary scalar to satisfied $\nabla\bullet(\mu\vec{V})=0$.

For incompressible flows, the continuity equation can be expressed as $\nabla\bullet\vec{V}=0$, therefore, we usually choose $\mu=1$ for incompressible flow applications.

5.7.1 Stream Function for 2-D Flows

When a flow is called 2-D flow, it means the flow quantities are independent of distance along a certain fixed direction. It we designate Z axis as the direction, we shall have:

$\frac{\partial()}{\partial Z} = \frac{\partial}{\partial Z}$ of any quantities $\equiv 0$.

In Cartesian coordinate system

In Cartesian system, we will have:

$d\vec{S} = (dx, dy, dz)$

$\vec{V} = (u, v, 0)$

$u = u(x, y)$

$v = v(x, y)$

Along a streamline, it can be written as

$$\frac{dx}{u(x,y)} = \frac{dy}{v(x,y)} = \frac{dz}{0}$$

It immediately follows that $dz = 0$ or $Z = const$

For the steam function ψ_1 and ψ_2, we can choose stream function ψ_2 is simply *z* (i.e., $\psi_2 = z$), the ψ_1 is the only one unknown function which is denoted as ψ . ψ is a function of *x* and *y* only. i.e., $\psi_1 = \psi(x, y)$ and $\psi_2 = z$.

Since $\mu\vec{V} = \nabla\psi_1 \times \nabla\psi_2$, then we have

$$\mu\vec{V} = \mu u\,\hat{i} + \mu v\,\hat{j} + 0\hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial y} & 0 \\ 0 & 0 & 1 \end{vmatrix} \Rightarrow \begin{cases} \mu u = \frac{\partial\psi}{\partial y} \\ \mu v = -\frac{\partial\psi}{\partial x} \end{cases}$$

For incompressible flows, $\mu = 1$, $\Rightarrow$
$$\begin{cases} u = \frac{\partial\psi}{\partial y} \\ v = -\frac{\partial\psi}{\partial x} \end{cases}$$

For compressible flows, $\mu = \rho$, $\Rightarrow$
$$\begin{cases} \rho u = \frac{\partial\psi}{\partial y} \\ \rho v = -\frac{\partial\psi}{\partial x} \end{cases}$$

In Cylindrical coordinate system

$d\vec{S} = (dr, rd\theta, dz)$

$\vec{V} = (V_r, V_\theta, 0)$

$V_r = V_r(r,\theta)$

$V_\theta = V_\theta(r,\theta)$

Along a streamline, it can be written as

$$\frac{dr}{V_r(r,\theta)} = \frac{d\theta}{V_\theta(r,\theta)} = \frac{dz}{0}$$

It immediately follows that $dz = 0$ or $Z = const$

For the steam function ψ_1 and ψ_2, we can choose stream function ψ_2 is simply *z* (i.e., $\psi_2 = z$), the ψ_1 is the only one unknown function which is denoted as ψ. ψ is a function of *r* and θ only. i.e., $\psi_1 = \psi(r,\theta)$ and $\psi_2 = z$.

Since $\mu\vec{V} = \nabla\psi_1 \times \nabla\psi_2$, then we have

$$\mu\vec{V} = \mu V_r\,\hat{e}_r + \mu V_\theta\,\hat{e}_\theta + 0\hat{e}_Z = \begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{e}_z \\ \dfrac{\partial\psi}{\partial r} & \dfrac{\partial\psi}{r\partial\theta} & 0 \\ 0 & 0 & 1 \end{vmatrix} \Rightarrow \begin{cases} \mu V_r = \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta} \\ \mu V_\theta = -\dfrac{\partial\psi}{\partial r} \end{cases}$$

For incompressible flows, $\mu = 1$, $\Rightarrow$ $\begin{cases} V_r = \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta} \\ V_\theta = -\dfrac{\partial\psi}{\partial r} \end{cases}$

For compressible flows, $\mu = \rho$, $\Rightarrow$ $\begin{cases} \rho V_r = \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta} \\ \rho V_\theta = -\dfrac{\partial\psi}{\partial r} \end{cases}$

Another Commonly-Used Method to Determine Stream Function:

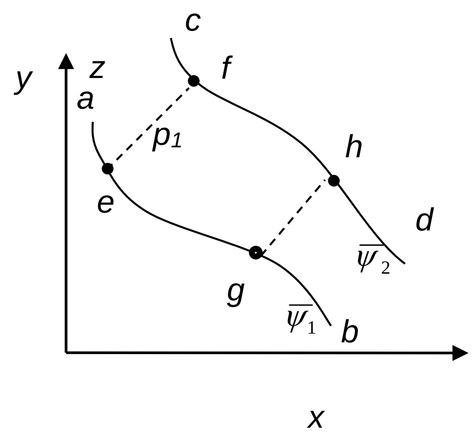
In a 2-D flow, $\dfrac{V_1}{V_2} = \dfrac{h_1 dq_1}{h_2 dq_2}$

On the integration of the above equation yields

$q_2 = f(h_1, h_2, q_1)$ or $F = F(h_1, h_2, q_1, q_2)$

Let's call this F as stream function $\overline{\psi}$, then $\overline{\psi} = \overline{\psi}(h_1, h_2, q_1, q_2)$.

Different constant of integration yield different stream lines.

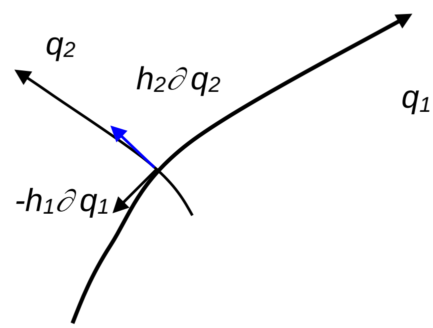


Let *a*, *b*, *c* and *d* represent two streamlines, by definition, not fluid passes *ab* or *cd*. Therefore, same mass of fluid must cross *ef* and *gh*.

If the streamline *ab* is arbitrarily chosen as a base, every other streamline in the field can be identified by assigning to it a number $\Delta\overline{\psi}$ equal to the mass of fluid passing, per second per unit depth perpendicular to the plane containing the base streamline and the streamline in question.

$$\Delta\overline{\psi} = \overline{\psi}_2 - \overline{\psi}_1 = C_2 - C_1$$
$$= \int_e^f \rho\vec{V} \bullet d\vec{A} = \int_e^f (\rho\vec{V} \bullet \hat{e}_n) dA = \int_e^f (\rho\vec{V} \bullet \hat{e}_n)(dl \cdot 1)$$
$$= \int_e^f \rho\vec{V} \bullet \hat{e}_n dl = \int_e^f \rho V_n dl$$

Where *dl* is the elemental length along the line *ef*.



In the limit *ef* is small, we can write this as :

$$\Delta\overline{\psi} = \rho V_n \Delta l$$

Where Δl is the normal distance between streamlines.

Or $$\frac{\Delta\overline{\psi}}{\Delta l} \cong \rho V_n$$

In the limit, $\Delta l \to 0$, this equation becomes

$$\lim_{\Delta l \to 0} \frac{\Delta\overline{\psi}}{\Delta l} = \frac{\partial\overline{\psi}}{\partial l} = \rho V_n$$

Thus, the velocity component in any direction is obtained by differencing $\overline{\psi}$ with respect to distance at right angles to the left of that direction.

Therefore:

$$\rho V_1 = \frac{\partial \overline{\psi}}{h_2 \partial q_2}$$
$$\rho V_2 = -\frac{\partial \overline{\psi}}{h_1 \partial q_1}$$

Accepting the above equation as the equations describing $\overline{\psi}$, we can show that $\overline{\psi}$ is constant along a streamline.

Proof: Along a stream, $\rho \vec{V} \times d\vec{S} = 0$ by definition, therefore

$$V_1 h_2 dq_2 - V_2 h_1 dq_1 = 0$$
$$\Rightarrow \quad \rho V_1 h_2 dq_2 - \rho V_2 h_1 dq = 0$$
$$\Rightarrow \quad \frac{\partial \overline{\psi}}{h_2 \partial q_2}(h_2 dq_2) + \frac{\partial \overline{\psi}}{h_1 \partial q_1}(h_1 dq_1) = 0$$
$$\Rightarrow \quad \frac{\partial \overline{\psi}}{\partial q_2} dq_2 + \frac{\partial \overline{\psi}}{\partial q_1} dq_1 = 0$$
$$\Rightarrow \quad d\overline{\psi} = 0$$
$$\Rightarrow \quad \overline{\psi} = const$$

Therefore, $\overline{\psi}$ is a constant on a streamline, that is, streamlines are isolines with constant $\overline{\psi}$.

- Since streamlines and stream functions are related to the mass flow rate $\dot{m}$, we can define stream function with continuity equation for steady flows.

- The stream function is a point function which satisfies the equation of continuity for steady flow identically.

- A necessary condition that a steady flow be physical possible is that a stream function exists.

- For incompressible flows, $\overline{\psi}$ is changed to ψ to denote that stream function, where $\Delta\psi$ denote the volume flow rate. i.e., $V_n = \frac{\partial(\overline{\psi}/\rho)}{\partial l} = \frac{\partial \psi}{\partial l}$

Note: The stream function ψ is defined only for 2-D flow. Stream function ψ exists whether the flow is rotational or irrotational.

5.8 Irrotational 2-D Ideal Fluid Flow

Since flow is 2-D and irrotational ideal flow, then

In Cartesian system, $\vec{\Omega}=0 \Rightarrow \quad \Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$

Since $\quad u = \frac{\partial \psi}{\partial y}; \quad v = -\frac{\partial \psi}{\partial x}$

Thus, $\quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial(-\frac{\partial \psi}{\partial x})}{\partial x} - \frac{\partial(\frac{\partial \psi}{\partial y})}{\partial y} = -[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}] = 0$

i.e., $\quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad$ or $\quad \nabla^2 \psi = 0 \quad$ Laplace Equation

Laplace equation has solutions which are called as harmonic functions.

For 2-D flow

- Any irrotational and incompressible flow has a velocity potential ϕ and stream function ψ that both satisfy Laplace equation.
- Conversely any solution represents the velocity potential ϕ or stream function ψ for an irrotational and incompressible flow.
- A powerful procedure for solving irrotational flow problems is to represent ϕ and ψ by linear combinations of known solutions of Laplace equation.

$$\phi = \sum_{i=1}^{N} C_i \phi_i; \qquad \psi = \sum_{i=1}^{N} C_i \psi_i$$

Finding the coefficients C_i so that the boundary conditions are satisfied both far from the body and the body surface.

Say ϕ_1 and ϕ_2 are solutions of $\nabla^2 \phi = 0$
i.e., $\nabla^2 \phi_1 = 0; \qquad \nabla^2 \phi_2 = 0$
Then $\phi = A_1\phi_1 + A_2\phi_2$ is also a solution of the Laplace equation

A complicated flow pattern for an irrotational and incompressible flow can be synthesized by adding together a number of elementary flows which are also irrotational and incompressible.

The Advantages of using stream function and potential function to solve aerodynamics problems

1. It can replace the nonlinear flow controlling equations (such as N-S equations) by Linear equations (Laplace equation).
2. It reduces the number of unknown numbers, i.e., $\left.\begin{matrix} p \\ u \\ v \\ w \\ \rho \end{matrix}\right\} \Rightarrow \quad \psi \quad or \quad \phi$
3. The solutions of the Laplace equation are adjustable.
4. A complex flow can be decomposed as the summation of many simple flows.