2024 Spring Semester AerE310 Course
Homework Problem Set #04:

Problem#1: Consider the incompressible, irrotational, 2-D flow where the potential functions is:

$$\phi = K\, ln(\sqrt{x^2+y^2})\text{, where K is a constant.}$$

a) What is the velocity field for this flow? Please verify the flow is irrotational. What is the magnitude and direction of the velocity vector at (2,0), at ($\sqrt{2}$, $\sqrt{2}$) and at (0, 2)?
b) What is the stream function for this flow field? Sketch the streamline pattern.
c) Sketch the lines with constant potential values. How do the lines of equipotential related to the streamlines?

The flow is incompressible, irrotational, and two dimensional. Thus, the flow can be described both by a potential function and a stream function. We are given:

$$\phi = K \ln\sqrt{x^2+y^2}$$

(a) Note that, if one converts from Cartesian coordinates to cylindrical coordinates where $r = \sqrt{x^2+y^2}$, we see that

$$\phi = K \ln r$$

Referring to Table 3.2, we see that this is the potential function for flow from a source. Comparing the current potential function with that given in Table 3.2, it is clear that the 2π is included in the constant K for this problem. We could, therefore, work this problem using cylindrical coordinates. However, let us work with Cartesian coordinates as would be expected based on what is given.

$$u = \frac{\partial\phi}{\partial x} = K\frac{1}{\sqrt{x^2+y^2}}\frac{(\frac{1}{2})(2x)}{\sqrt{x^2+y^2}} = \frac{Kx}{x^2+y^2}$$

Note that along the x-axis ($y=0$), $u = \frac{K}{x}$, i.e., the velocity varies inversely with distance from the origin. Furthermore, $u>0$ for $x>0$ and $u<0$ for $x<0$, since the source flow is always directed away from the origin.

$$v = \frac{\partial\phi}{\partial y} = K\frac{1}{\sqrt{x^2+y^2}}\frac{(\frac{1}{2})(2y)}{\sqrt{x^2+y^2}} = \frac{Ky}{x^2+y^2}$$

Thus, $\vec{V} = u\hat{i} + v\hat{j} = \frac{Kx}{x^2+y^2}\hat{i} + \frac{Ky}{x^2+y^2}\hat{j}$

Let us verify that the flow is irrotational.

Is $\nabla\times\vec{V} = 0$?

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{Kx}{x^2+y^2} & \dfrac{Ky}{x^2+y^2} & 0 \end{vmatrix} = \hat{k}\left[\frac{\partial}{\partial x}\left(\frac{Ky}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{Kx}{x^2+y^2}\right)\right]$$

$$= \hat{k}\left[\frac{Ky(-2x)}{(x^2+y^2)^2} - \frac{Kx(-2y)}{(x^2+y^2)^2}\right] = 0$$

Q.E.D. that the flow is irrotational. Let us now calculate the magnitude and the direction of the flow at specific points. Since

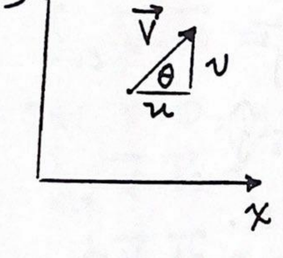
$$\vec{V} = \frac{Kx}{x^2+y^2}\,\hat{i} + \frac{Ky}{x^2+y^2}\,\hat{j}$$

Thus, at $x=2;\ y=0$: $\vec{V} = \frac{K}{2}\hat{i}$. Obviously, the magnitude of the velocity is $\frac{1}{2}K$ and the flow is in the direction of the positive x-axis.

At $x=\sqrt{2};\ y=\sqrt{2}$: $\vec{V} = \frac{K\sqrt{2}}{4}\hat{i} + \frac{K\sqrt{2}}{4}\hat{j}$

The magnitude of the velocity is

$$|\vec{V}| = \sqrt{u^2+v^2} = \frac{K}{4}\sqrt{2+2} = \frac{K}{2}$$



The direction of the flow, θ, is given by:

$$\theta = \tan^{-1}\left\{\frac{\frac{K\sqrt{2}}{4}}{\frac{K\sqrt{2}}{4}}\right\} = \tan^{-1} 1 = 45^\circ$$

At $x=0;\ y=2$: $\vec{V} = \frac{K}{2}\hat{j}$. Clearly, the magnitude of the velocity is $\frac{1}{2}K$ and it is in the direction of the y-axis.

Note that the magnitude of the velocity is the same at all three points. This should not be surprising, since the three points are the same radial distance from the origin, i.e., $r=2$ for all three.

(b) For the stream function, recall that

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$

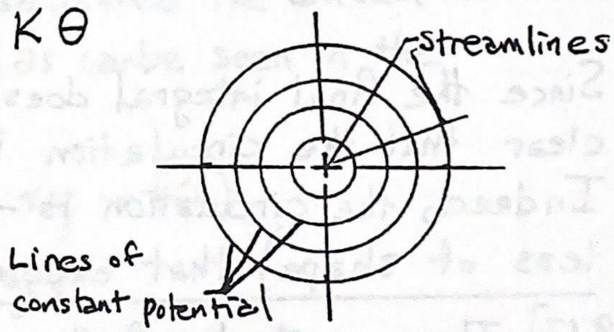
for a two-dimensional, incompressible flow. Thus, the stream function must satisfy both of the following expressions:

$$\psi = \int u\,dy + f(x) \quad \text{and} \quad \psi = -\int v\,dx + g(y)$$

so that $\psi = \int \frac{Kx}{x^2+y^2}\,dy + f(x)$ and $\psi = -\int \frac{Ky}{x^2+y^2}\,dx + g(y)$

Thus, $\psi = K \tan^{-1} \frac{y}{x} = K\theta$

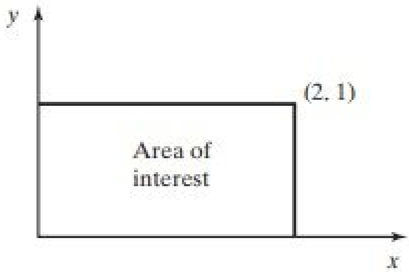
(c) The streamlines are perpendicular to the lines of constant potential.



Problem#2: The equation of the streamlines in a two-dimensional velocity field are given by the expression of $\psi = xy + y^2 + const$.

a) Pease find the expression of the flow velocity vector and the magnitude of the velocity vector.

b) Please find the integral over the surface shown in the figure for the normal component of vector of $\nabla \times \vec{V}$ by two methods.



$$u = \frac{\partial \psi}{\partial y} = x + 2y \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} = -y$$

To verify that the magnitude of the velocity corresponds to that given in the problem statement:

$$|\vec{V}| = \sqrt{u^2 + v^2} = \sqrt{x^2 + 4xy + 4y^2 + y^2} = \sqrt{5y^2 + x^2 + 4xy}$$

Let us evaluate the two sides of the equation representing Stokes's Theorem:

$$\oint \vec{V} \cdot \vec{ds} = \int_0^2 x\,dx + \int_0^1 (-y)\,dy + \int_2^0 (x+2)\,dx + \int_1^0 (-y)\,dy$$

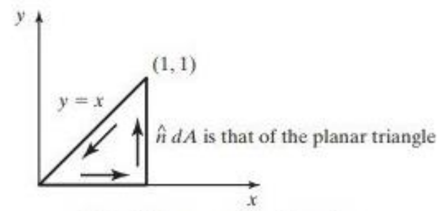
$$= \frac{x^2}{2}\Big|_0^2 - \frac{y^2}{2}\Big|_0^1 + \left(\frac{x^2}{2} + 2x\right)\Big|_2^0 - \frac{y^2}{2}\Big|_1^0 = 2 - \frac{1}{2} - 2 - 4 + \frac{1}{2}$$

$$= -4$$

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y & -y & 0 \end{vmatrix} = \hat{k}\left[\frac{\partial}{\partial x}(-y) - \frac{\partial}{\partial y}(x+2y)\right] = -2\hat{k}$$

$$\iint (\nabla \times \vec{V}) \cdot \hat{n}\, dA = \int_0^2 dx \int_0^1 (-2)\, dy = -4$$

Problem#3: Given an incompressible, steady flow, where the flow velocity is $\vec{V} = (x^2 y - xy^2)\hat{i} + (\frac{y^3}{3} - xy^2)\hat{j}$



Circulation around the triangle

(a) Does the flow filed satisfy the continuity equation? Does a stream function exist? If a stream function exists, what is it?
(b) Does a potential function exist? If yes, what is it?
(c) For the region shown in the figure, evaluate $\iint (\nabla \times \vec{V}) \bullet d\vec{A} = ?$ and $\Gamma = \oint \vec{V} \bullet d\vec{S} = ?$ to demonstrate that Stokes's theorem is valid.

$$\psi = \int u\, dy + f(x) = \frac{x^2 y^2}{2} - \frac{x y^3}{3} + f(x)$$

and $$\psi = -\int v\, dx + g(y) = -\frac{x y^3}{3} + \frac{x^2 y^2}{2} + g(y)$$

For these two expressions to be consistent, $f(x) = g(y)$ = constant. Thus,

$$\psi = \frac{x^2 y^2}{2} - \frac{x y^3}{3} + \text{constant}$$

(b) In order for a potential function to exist, the flow must be irrotational, i.e., $\nabla \times \vec{V} = 0$. Does it?

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2 y - xy^2) & (\frac{y^3}{3} - xy^2) & 0 \end{vmatrix} = \hat{k}(-y^2 - x^2 + 2xy) \neq 0$$

Therefore, a potential function does not exist for this flow.

(c) To evaluate $\iint (\nabla \times \vec{V}) \cdot \hat{n}\, dA$ for the triangle shown:

$$\iint (\nabla \times \vec{V}) \cdot \hat{n}\, dA = \int_0^1 dx \int_0^x (-y^2 - x^2 + 2xy)\, dy$$

$$= \int_0^1 dx \left[-\frac{y^3}{3} - x^2 y + x y^2 \right|_0^x = \int_0^1 dx \left(-\frac{x^3}{3} - x^3 + x^3 \right)$$

$$= -\int_0^1 \frac{x^3}{3}\, dx = -\frac{x^4}{12}\Big|_0^1 = -\frac{1}{12}$$

To evaluate the circulation around the closed path bounding the triangle

$$\oint \vec{V} \cdot \vec{ds} = \int_0^1 (0)\, dx + \int_0^1 \left(\frac{y^3}{3} - y^2 \right) dy + \int_1^0 (x^3 - x^3)\, dx$$

$$+ \int_1^0 \left(\frac{y^3}{3} - y^3 \right) dy = \left[\frac{y^4}{12} - \frac{y^3}{3} \right|_0^1 + \left[\frac{y^4}{12} - \frac{y^4}{4} \right|_0^1 = -\frac{1}{12}$$

Problem#4: What is the stream function that represents the potential flow about a cylinder whose radius is 1.0m and which is located in an air stream where the free-stream velocity is 50m/s? What is the change in pressure from the free-stream value to the value at the top of the cylinder (i.e., $\theta = 90^o$)? What is the change in pressure from the free-stream value to the value at the stagnation point (i.e., $\theta = 180^o$)?

Since the potential flow around a cylinder is represented by the superposition of a uniform flow and a doublet, the stream function is:

$$\psi = U_\infty r \sin\theta - \frac{B}{r}\sin\theta$$

where $R = \sqrt{B/U_\infty}$. Thus, $B = R^2 U_\infty = 50 m^3/s$, so that:

$$\psi = 50 r \sin\theta - \frac{50}{r}\sin\theta$$

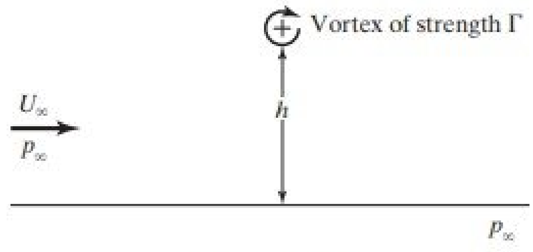
To calculate the change in pressure from the free-stream value to that at a point on the surface of the cylinder where $\theta = 90°$, use Eq. (3.44):

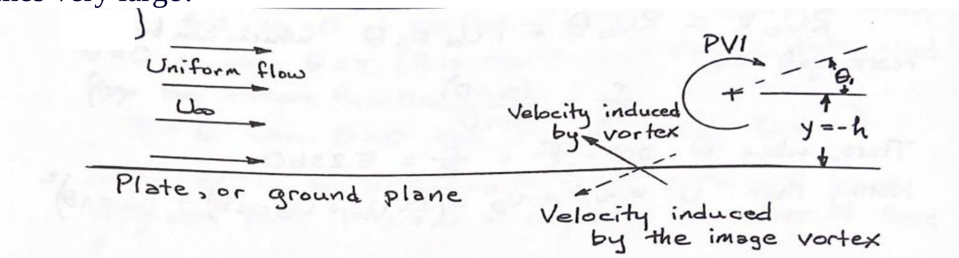
$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} = 1 - 4\sin^2\theta = -3$$

Thus, $p - p_\infty = -\frac{3}{2}\rho_\infty U_\infty^2 = -\frac{3}{2}(1.225 \frac{kg}{m^3})(50 \frac{m}{s})^2$

$= -4.594 \times 10^3 \ N/m^2$

Problem#5: A two-dimensional free vortex is located near an infinite plane at a distance h above the plane as shown in the figure. The pressure at infinity is P_∞, and the velocity at infinity is U_∞, parallel to the plane. Please find the total force (per unit depth normal to the paper) acting on the plane if the pressure on the underside of the plane is P_∞. The strength of the vortex is Γ. The fluid is incompressible and perfect. To what expression does the force simplify if h becomes very large?





For the two-dimensional, free, potential vortex (PV1) which is located a distance h above the plate (or ground plane) in a uniform flow:

$$\phi = \phi_{UF} + \phi_{PV1} = U_\infty x - \frac{\Gamma}{2\pi}\tan^{-1}\left(\frac{y}{x}\right)$$

where the x, y coordinate system is located at the origin of the vortex, as shown in the sketch. The velocity induced by the vortex at a representative point on the plate is represented by the solid arrow in the sketch. Clearly, the sum of the uniform flow and the single potential vortex (PV1) do not result in a flow parallel to the plate (which is a boundary condition: that is, the inviscid flow must be parallel to a plate surface). In order for the plate to be a streamline, we need to add an "image" vortex of equal strength, located a distance h below the plate. This image vortex induces a velocity which is represented by the broken arrow. The normal component of the velocity induced by the image vortex is equal and opposite to that induced by the original vortex. Thus, the resultant velocity induced by these three potential functions is parallel to the plate. To calculate the resultant velocity, let us calculate first the velocity induced by PV1:

$$u = \frac{\partial \phi_{PV1}}{\partial x} = \frac{\partial}{\partial x}\left[-\frac{\Gamma}{2\pi}\tan^{-1}\left(\frac{y}{x}\right)\right]$$

$$u = -\frac{\Gamma}{2\pi}\,\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = +\frac{\Gamma y}{2\pi(x^2+y^2)}$$

We shall not worry about the y-component of velocity (v)

since that is cancelled by the image vortex. However, the horizontal velocity component (u) induced by the image vortex at the plate is equal to that induced by the original vortex (PV1), so that the velocity induced by the original vortex and its image at this plate is twice this value. Further, note that $y = -h$ for the plate. Thus, adding the free-stream velocity, the net velocity at the surface of the plate is:

$$U = U_\infty - \frac{\Gamma h}{\pi (x^2 + h^2)}$$

The pressure along the upper surface is:

$$p_u = p_\infty + \frac{1}{2}\rho_\infty U_\infty^2 - \frac{1}{2}\rho_\infty U^2$$

Since the pressure acting on the underside of the plate is p_∞, the net pressure force acting on the plate is:

$$\Delta p = p_\ell - p_u = \frac{1}{2}\rho_\infty (U^2 - U_\infty^2)$$

$$= \frac{1}{2}\rho_\infty \left[\frac{\Gamma^2}{\pi^2} \frac{h^2}{(x^2+h^2)^2} - \frac{2U_\infty \Gamma}{\pi} \frac{h}{x^2+h^2} \right]$$

To find the net lift force acting on the plate per unit depth, we integrate the net pressure force from $x = -\infty$ to $x = +\infty$. Thus, the force per unit depth is:

$$\ell = \frac{\rho_\infty}{2} \int_{-\infty}^{+\infty} \left\{ \frac{\Gamma^2 h^2}{\pi^2} \frac{dx}{(x^2+h^2)^2} - \frac{2U_\infty \Gamma h}{\pi} \frac{dx}{(x^2+h^2)} \right\}$$

$$\ell = \frac{\rho_\infty}{2} \left\{ \frac{\Gamma^2 h^2}{\pi^2} \left[\frac{1}{2h^2} \frac{x}{(x^2+h^2)} + \frac{1}{2h^2} \frac{1}{h} \tan^{-1}\left(\frac{x}{h}\right) \right] - \frac{2U_\infty \Gamma h}{\pi} \frac{1}{h} \tan^{-1}\left(\frac{x}{h}\right) \right\} \Bigg|_{-\infty}^{+\infty}$$

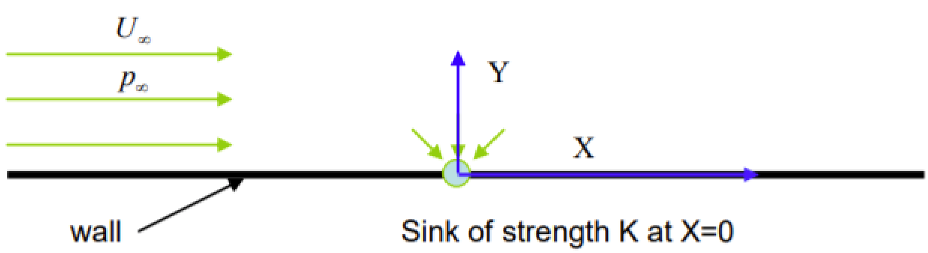
$$l = \frac{\rho_\infty}{2} \frac{\Gamma^2}{2\pi^2 h} \left[\frac{\pi}{2} - \frac{3\pi}{2} \right] - \frac{\rho_\infty}{2} \frac{2 U_\infty \Gamma}{\pi} \left[\frac{\pi}{2} - \frac{3\pi}{2} \right]$$

$$l = \rho_\infty U_\infty \Gamma - \frac{\rho_\infty \Gamma^2}{4\pi h}$$

Note that as $h \to \infty$, the lift per unit span becomes equal to $\rho_\infty U_\infty \Gamma$.

Problem#6: In an ideal, 2-D incompressible irrotational flow field, the fluid is flowing past a wall with a sink of strength K per unit length at the origin as shown in the Figure. The potential function of a 2-D sink is $\phi = -\frac{k}{2\pi}\ln\sqrt{x^2+y^2}$. At infinity the flow is parallel to wall and of uniform velocity U_∞ .

a. Determine the location of the stagnation point X0 at the wall in terms of U_∞ and K.
b. Find the pressure distribution along the wall as a function of X. Taking the free stream static pressure at infinity to be P_∞, express the pressure coefficient as a function of 0 X/X_0 .
c. Sketch the resulting pressure distribution.



For the component of the flow represented by the sink,

$$v_r = \frac{\partial \phi}{\partial r} = -\frac{K}{2\pi r}$$

Note that v_r is negative at all points, i.e., it is directed toward the origin. Since we are concerned only with the pressure distribution along the wall (where $y=0$), we can write that the component of the velocity parallel to the wall due to the sink alone is:

$$u_{sink} = -\frac{K}{2\pi x}$$

Note that u_{sink} is positive (directed to the right) when x is negative. Further, u_{sink} is negative (directed to the left) when x is positive. This is as it should be.

The total velocity parallel to the wall is the sum of that induced by the sink and that due to the uniform flow:

$$u = u_{UF} + u_{sink} = U_\infty - \frac{K}{2\pi x} \qquad (i)$$

The stagnation point corresponds to the x location where $u = 0$. Thus,

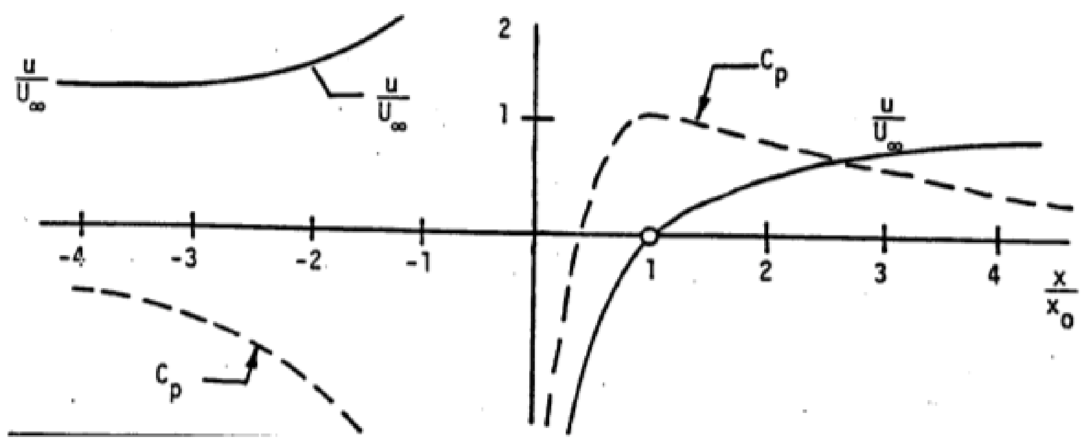
$$x_{stag} = x_o = \frac{K}{2\pi U_\infty}$$

where x_o is the x coordinate of the stagnation point. Note that the terms in the velocity expression [eq. (i)] are of opposite sign only for positive x. Thus, the stagnation point is located to the right of the origin (i.e., for positive x). We can write the velocity along the wall as:

$$u = U_\infty - \frac{K}{2\pi x} = U_\infty - \frac{2\pi x_o U_\infty}{2\pi x} = U_\infty\left(1 - \frac{x_o}{x}\right)$$

The pressure coefficient along the wall is:

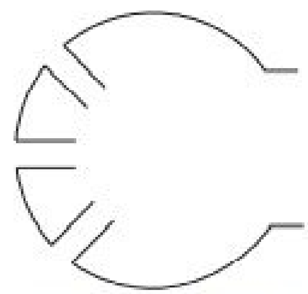
$$C_p = 1 - \frac{u^2}{U_\infty^2} = 1 - \left[1 - \frac{2x_o}{x} + \frac{x_o^2}{x^2}\right] = \frac{2x_o}{x} - \frac{x_o^2}{x^2}$$



Problem#7: A cylindrical tube with three radially drilled orifices, as shown in the figure below can be used as a flow-direction indicator. Whenever the pressure on the two side holes is equal, the pressure at the center hole is the stagnation pressure. The instrument is called a direction-finding Pitot tube, or a cylindrical yaw probe.

a) If the orifices of a direction-finding Pitot tube were to be used to measure the freestream static pressure, where would they have to be located if we use our solution for flow around a cylinder?

b) For a direction-finding Pitot tube with orifices located as calculated in part (a), what is the sensitivity? Let the sensitivity be defined as the pressure change per unit angular change (i.e., $\partial p / \partial \theta$)



For a cylindrical yaw probe, the static pressure around the cylinder where the pressure orifices are may be approximated by Eq. (3.44):

$$C_p = \frac{p - p_\infty}{q_\infty} = 1 - 4\sin^2\theta$$

(a) For the orifices located where $p = p_\infty$, or $C_p = 0$. Thus, we can solve our expression for C_p to find that, if

$$1 - 4\sin^2\theta = 0 \quad , \quad \text{then} \quad \theta = 30° \text{ or } 150°$$

(i.e., these orifices should be $\pm 30°$ from the stagnation point)

(b) To find $\frac{dp}{d\theta}$, note that $p = p_\infty + q_\infty(1 - 4\sin^2\theta)$

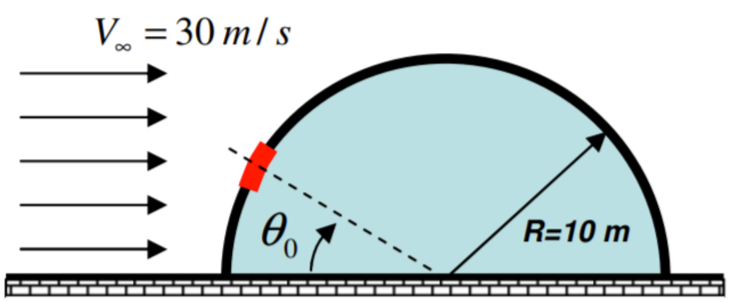
Thus, $\frac{dp}{d\theta} = -8q_\infty \sin\theta\cos\theta$

which, at $\theta = 30°$, becomes $\frac{dp}{d\theta} = -3.464\,q_\infty$

Problem#8: Consider the flow around the Quonset shown in the figure to be represented by superimposing a uniform flow and 2-D a doublet. Assume steady, incompressible, potential flow. The ground plane is represented by the plane of symmetry and the hut by the upper half of the cylinder. The free-stream wind velocity is 30 m/s; the radius of the hunt is 10m. The door of the hut is not well sealed, and the leakage opening is very small compared with the radius of the hut R. Therefore, the static pressure inside the hunt is equal to that the outer surface of the hunt where the door is located. Density of air ρ=1.2kg/m^3.

a) If the door to the hunt is located at ground level (i.e., at the stagnation point), what is the net lift acting on the hunt?

b) Where should the door be located (i.e., at what angle θ_0 relative to the ground) so that the net force on the hut will vanish?



Solution:

The hut can be considered as a semi-cylinder, and the pressure distribution on the outer surface of a cylinder is

$$C_p = \frac{P - P_\infty}{\frac{1}{2}\rho V_\infty^2} = \frac{P - P_\infty}{q_\infty} = 1 - 4\sin^2\theta \quad \Rightarrow \quad P = P_\infty + q_\infty(1 - 4\sin^2\theta).$$

Therefore, the pressure on the outer surface of the cylinder will be $P = P_\infty + q_\infty(1 - 4\sin^2\theta)$

Then, the total force per unit span along Y direction at the outer surface of the cylinder will be:

$$L_{out} = -\int_0^\pi PR\sin\theta\, d\theta = -R\int_0^\pi \{P_\infty + q_\infty(1 - 4\sin^2\theta)\}\sin\theta\, d\theta$$

$$= -R[\int_0^\pi P_\infty \sin\theta\, d\theta - \int_0^\pi q_\infty(-3 + 4\cos^2\theta)\, d(\cos\theta)\ ]$$

$$= -R[P_\infty 2 - q_\infty(-3\cos\theta + \frac{4}{3}\cos^3\theta)\Big|_0^\pi] = -2P_\infty R + \frac{10}{3}R q_\infty$$

The pressure inside the hut is P_i, then, the total life force per unit span at the inner surface will be: $L_{inner} = +\int_0^\pi P_i R\sin\theta\, d\theta = P_i R\int_0^\pi \sin\theta\, d\theta = 2P_i R$

Therefore, the net lift acting on the hut will be: $L = L_{inner} + L_{out} = 2P_i R - 2P_\infty R + \frac{10}{3}R q_\infty$

For the question A, when door is on the ground, $P_i = P_{\theta=\pi} = P_\infty + q_\infty$, then, the total lift forceper unit span will be: $L = 2(P_\infty + q_\infty)R - 2P_\infty R + \frac{10}{3}R q_\infty = \frac{16}{3}R q_\infty = \frac{16}{3}R\frac{1}{2}\rho V^2{}_\infty$

$$= 24{,}000 * \rho = 24{,}000 * 1.2 = 28{,}800N$$

For the question B, when lift vanish, $L = 2P_i R - 2P_\infty R + \frac{10}{3}R q_\infty = 0$. Since

$P_i = P_\infty + q_\infty(1 - 4\sin^2\theta)$ then, $1 - 4\sin^2\theta + \frac{5}{3} = 0 \Rightarrow \sin^2\theta = \frac{2}{3} \quad \Rightarrow \quad \begin{cases} \theta = 54.74^o \\ \theta = 180 - 54.74^o \end{cases}$