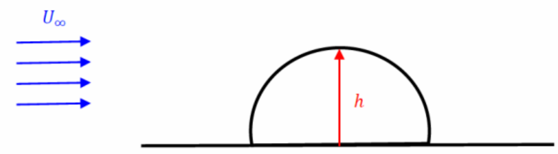
2021 Fall Semester AerE310 Course
Homework Problem Set #05:

***Due**: Midnight, Sunday, 11/07/2021*

Problem#1: A hill with the height h has the shape of a half circle as shown in figure below. The wind approaching the hill has a constant velocity parallel to the ground, $U\infty$. Assume the stream function is in the form of $\psi(r,\theta) = r^n \sin\theta$.

- Find the value of n such that the flow is irrotational.
- Use the surface (hill) streamline (ψ=0) and the fact that at far field ($r\rightarrow\infty$), the stream function approaches to that of a freestream ($U\infty r\sin\theta$) to formulate the final form of stream function in terms of the given parameters (h, $U\infty$).



Solution:

a. $\psi(r,\theta) = r^n \sin\theta$

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = r^{n-1}\cos\theta$$

$$v_\theta = -\frac{\partial\psi}{\partial r} = -nr^{n-1}\sin\theta$$

For irrotationality $\omega_z = 0$

$$\frac{1}{r}\left[\frac{\partial(rv_\theta)}{\partial r} - \frac{\partial v_r}{\partial\theta}\right] = 0 \rightarrow \frac{\partial}{\partial r}(-nr^n \sin\theta) - \frac{\partial}{\partial\theta}(r^{n-1}\cos\theta) = 0$$

$$\rightarrow -n^2 r^{n-1}\sin\theta + r^{n-1}\sin\theta = 0 \rightarrow (1-n^2)r^{n-1}\sin\theta = 0$$

For this to be always zero

$$1 - n^2 = 0 \rightarrow n = \pm 1$$

b. There are two valid solutions, therefore the stream function can be written as the superposition of the two. In a general form

$$\psi = Ar\sin\theta + Br^{-1}\sin\theta = \left(Ar + \frac{B}{r}\right)\sin\theta$$

With A and B as constants that must be determined.

$$r \rightarrow \infty:\ \psi \rightarrow U_\infty r\sin\theta\ \ \text{(Uniform flow)}$$

$$\rightarrow Ar\sin\theta = U_\infty r\sin\theta \rightarrow A = U_\infty$$

To find the ψ value for the streamline representing the hill, evaluate ψ at a point on the surface:

$$r = h, \theta = 0 \rightarrow \left(U_\infty h + \frac{B}{h}\right)\sin(0) = 0 \rightarrow \psi_{hill} = 0$$

Therefore $\left(U_\infty h + \frac{B}{r}\right)\sin\theta = 0$ is the surface (hill) streamline.

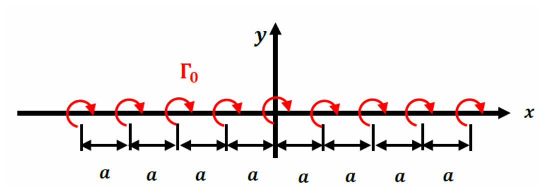
Now evaluate at $r = h, \theta = \frac{\pi}{2}$

$$\left(U_\infty h + \frac{B}{h}\right)\sin\frac{\pi}{2} = 0 \rightarrow B = -U_\infty h^2$$

Then the final form of stream function is

$$\psi(r,\theta) = U_\infty\left(r - \frac{h^2}{r}\right)\sin\theta$$

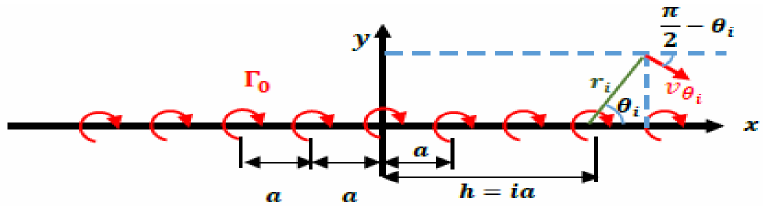
Problem#2: Consider a row of vortices of equal strength, Γ_0 and equal spacing, a, as shown. The number of vortices is N+1.



- Write the stream function for the resulting flow field and determine velocity components u,v in the Cartesian system.
- Calculate u and v for a=1, Γ_0=10π at points (x, y) = (1,10) and (1, −10) for N+1=1, 101 and 1001. Assume these vortices are symmetrically placed with respect to y axis.

This problem illustrates the concept of a vortex sheet. For sufficiently large y values, the flow above a vortex sheet is essentially uniform. There is no velocity normal to the sheet at sheet surface and horizontal component of velocity changes sign across the sheet.

Solution:



Consider a single vortex at location $h = ia,\ i = -\frac{N}{2}, \dots, \frac{N}{2}$ and write the induced velocity at an arbitrary point (x, y).

$$v_{\theta_i} = -\frac{\Gamma_0}{2\pi r_i}, \qquad v_{r_i} = 0$$

In Cartesian coordinates

$$u_i = v_{\theta_i}\cos\left(\frac{\pi}{2} - \theta_i\right) = v_{\theta_i}\sin\theta_i$$

$$v_i = -v_{\theta_i}\sin\left(\frac{\pi}{2} - \theta_i\right) = -v_{\theta_i}\cos\theta_i$$

Also

$$r_i^2 = (x-h)^2 + y^2 = (x-ia)^2 + y^2$$

$$\sin\theta_i = \frac{y}{r_i}, \cos\theta_i = \frac{x-ia}{r_i}$$

$$\rightarrow u_i = -\frac{\Gamma_0}{2\pi r_i}\frac{y}{r_i} = -\frac{\Gamma_0}{2\pi}\frac{y}{(x-ia)^2 + y^2}$$

$$v_i = \frac{\Gamma_0}{2\pi r_i}\frac{x-ia}{r_i} = \frac{\Gamma_0}{2\pi}\frac{x-ia}{(x-ia)^2 + y^2}$$

Now add the contribution from all the vortices by summing over index i:

$$u(x, y) = \sum_{i=-N/2}^{N/2} u_i = -\frac{\Gamma_0}{2\pi}\sum_{i=-N/2}^{N/2}\frac{y}{(x-ia)^2 + y^2}$$

$$v(x, y) = \sum_{i=-N/2}^{N/2} v_i = \frac{\Gamma_0}{2\pi}\sum_{i=-N/2}^{N/2}\frac{x-ia}{(x-ia)^2 + y^2}$$

Substitute $a = 1, \Gamma_0 = 10\pi$ and evaluate the summations for various Ns. The numerical value of u and v for points (1,10) and (1,-10) are shown in the table below.

Point	(1,10)		(1,-10)	
Velocity	u	v	u	v
N=10	5.3745	-0.4396	-5.3745	-0.4396
N=100	13.7711	-0.1888	-13.7711	-0.1888
N=1000	15.5084	-0.0200	-15.5084	-0.0200
N=10,000	15.688	-0.0020	-15.688	-0.0020

For large N values, the flow is nearly fully unidirectional along x axis and $v \rightarrow 0$. The horizontal component of velocity is symmetric with respect to x axis. This is a vortex sheet!

Also note: $u(1,10) - u(1,-10) = 15.688 + 15.688 = 31.376 \approx 10\pi = \Gamma_0$

Problem#3: Based on the thin airfoil theory, the vorticity distribution along the mean camber line of a symmetrical airfoil can be expressed as:

$$\gamma(\theta) = 2\alpha V_\infty \frac{1+\cos\theta}{\sin\theta}, \textit{where} \left| \begin{array}{l} \theta = 0 \ \textit{for leading edge} \\ \theta = \pi \ \textit{for leading edge} \end{array} \right. .$$

- **a)** Prove that the Kutta condition for airfoil trailing edge is satisfied
- **b)** What is the physical significance of $2\gamma / V_\infty$?
- **c)** What angle of attack is required for a symmetrical airfoil to develop a section of $C_l = 0.5$? Sketch the distribution $2\gamma / V_\infty$ as a function of *x/c* for a section lift coefficient of $C_l = 0.5$.
- d) Using the vorticity distribution to calculate the pitch moment about 0.75 chord from the leading edge. Verify your answer using the fact that the center of pressure (Xcp) is at the quarter chord for all angles of attack and the definition for lift.

Solution:

Question – a:

The Kutta condition states that that for a given airfoil of a given angle of attack, the value of circulation (Γ) around the airfoil is such that the flow leaves the trailing edge of the airfoil smoothly. i.e., $\gamma(TE) = 0.0$

For a symmetrical airfoil, $\gamma(\theta) = \dfrac{2\,\alpha\,V_\infty(1+\cos\theta)}{\sin\theta}$ where $x = \dfrac{c}{2}(1-\cos\theta)$

At the trailing edge: $x = c \quad \Rightarrow \quad \theta = \pi$

Therefore at trailing edge: $\gamma(\theta)\big|_{\theta=\pi} = \left.\dfrac{2\,\alpha\,V_\infty(1+\cos\theta)}{\sin\theta}\right|_{\theta=\pi} = \dfrac{0}{0} = ?$

According to L'Hospital's role:

$$\gamma(\theta)\big|_{\theta=\pi} = \left.\frac{2\,\alpha\,V_\infty(1+\cos\theta)}{\sin\theta}\right|_{\theta=\pi} = \frac{0}{0} = \left(\left.\frac{2\,\alpha\,V_\infty(1+\cos\theta)}{\sin\theta}\right|_{\theta=\pi}\right)' = \left.\frac{-2\,\alpha\,V_\infty\sin\theta)}{\cos\theta}\right|_{\theta=\pi} = \frac{0}{-1} = 0$$

$$\Rightarrow \qquad \gamma(\theta)\big|_{\theta=\pi} = 0$$

Therefore, the Kutta condition is satisfied.

Question – b:

Note that γ has the units of velocity. The incremental lift (per unit span) acting on an infinitesimal chordwise element is given by the Kutta-Joukowski theorem as:

$$dl = \rho_\infty U_\infty \gamma$$

But the increment of lift (per unit span) acting on an infinitesimal chordwise element is equal to the difference in the pressure acting on the lower surface and that acting on the upper surface:

$$dl = \Delta p = p_l - p_u = (p_l - p_\infty) - (p_u - p_\infty)$$

Equating the two expressions for the incremental lift:

$$\rho_\infty U_\infty \gamma = (p_l - p_\infty) - (p_u - p_\infty)$$

$$\frac{2\gamma}{U_\infty} = C_{p_l} - C_{p_u} = 4\alpha \left[\frac{1 + \cos\theta}{\sin\theta}\right]$$

Thus, the parameter $\frac{2\gamma}{U_\infty}$ represents the difference between the pressure coefficient for the lower surface and that for the upper surface at a given chordwise station.

Question – c:

Since $C_l = 2\pi\alpha$, the angle of attack required to develop a section lift coefficient of 0.5 is 4.56° = 0.0796 radians. Thus, the load distribution for a flat-plate airfoil which develops a section lift coefficient of 0.5 is that given in the following table.

θ	$\frac{x}{c} = \frac{1}{2}(1-\cos\theta)$	$\frac{2\gamma}{U_\infty}$	θ	$\frac{x}{c} = \frac{1}{2}(1-\cos\theta)$	$\frac{2\gamma}{U_\infty}$
0°	0.000	∞	120°	0.750	0.1837
30°	0.067	1.1879	135°	0.854	0.1318
45°	0.146	0.7684	150°	0.933	0.0853
60°	0.250	0.5513	180°	1.000	0.0000
90°	0.500	0.3183			

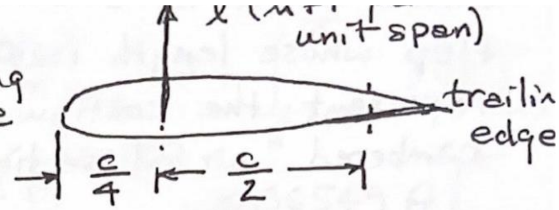
Question – d:

Let us calculate the section moment coefficient about a point 0.75 chord (0.75c) from the leading edge. A nose up pitching moment is positive.

$$m_{0.75c} = \int_0^{0.75c} \Delta p \, d\xi \, (0.75c - \xi) - \int_{0.75c}^{1.0c} \Delta p \, d\xi \, (\xi - 0.75c)$$

$$m_{0.75c} = 0.75c \int_0^c \Delta p \, d\xi - \int_0^c \Delta p \, \xi \, d\xi \qquad \text{(P. 6.1a)}$$

$$l = \int_0^c \Delta p \, d\xi$$



and that the pitching moment about the leading edge is:

$$m_0 = -\int_0^c \Delta p \, \xi \, d\xi$$

Thus, equation (P.6.1a) can be written:

$$m_{0.75c} = 0.75c \, l + m_0 = (0.75c)(\pi \rho_\infty U_\infty^2 \alpha c) - \frac{\pi}{4} \rho_\infty U_\infty^2 \alpha c^2 = \frac{\pi}{2} \rho_\infty U_\infty^2 \alpha c^2$$

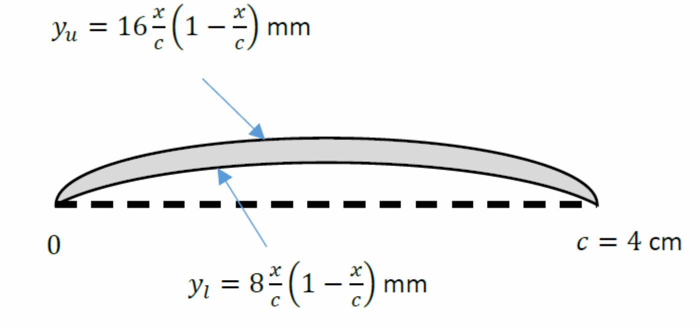
Since the theoretical location of the center of pressure for a flat-plate airfoil is at the quarter chord, we can calculate the moment about the three-quarter chord point as:

$$m_{0.75c} = l \, (0.5c)$$

where l, the lift per unit span, is a force that acts effectively at the quarter chord and 0.5c is the moment arm. A positive (nose-up) moment results when the lift is positive. Thus,

$$m_{0.75c} = \frac{\pi}{2} \rho_\infty U_\infty^2 \alpha c^2$$

Problem#4: For the parabolic-arc airfoil shown below, find the equations for lift coefficient c_l and moment coefficient at quarter chord, $c_{m,c/4}$.



Solution:

The equation of mean camber line is:

$$\eta_c = \frac{1}{2}(y_u + y_l) = 12\frac{x}{c}\left(1-\frac{x}{c}\right)\ mm$$

The derivative

$$\frac{d\eta_c}{dx} = \frac{12}{c}\left(1-\frac{x}{c}-\frac{x}{c}\right) = \frac{12}{c}\left(1-\frac{2x}{c}\right)$$

Change of variable to θ

$$\frac{x}{c} = \frac{1}{2}(1-\cos\theta) \rightarrow \frac{d\eta_c}{dx}(\theta) = \frac{12}{c}[1-1(1-\cos\theta)] = \frac{12}{c}\cos\theta$$

$$A_0 = \frac{1}{\pi}\int_0^{\pi}\frac{d\eta_c(\theta)}{dx}d\theta = \frac{12}{\pi c}\int_0^{\pi}\cos\theta\, d\theta = 0$$

$$A_1 = \frac{2}{\pi}\int_0^{\pi}\frac{d\eta_c(\theta)}{dx}\cos\theta\, d\theta = \frac{24}{\pi c}\int_0^{\pi}\cos^2\theta\, d\theta = \frac{12}{\pi c}\int_0^{\pi}(1+\cos 2\theta)d\theta = \frac{12}{\pi c}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi} = \frac{12}{c}$$

$$A_2 = \frac{2}{\pi}\int_0^{\pi}\frac{d\eta_c(\theta)}{dx}\cos 2\theta\, d\theta = \frac{24}{\pi c}\int_0^{\pi}\cos\theta\cos 2\theta\, d\theta = 0$$

Now calculate $\alpha_{L=0}$ and c_l

$$\alpha_{L=0} = A_0 - \frac{A_1}{2} = 0 - \frac{12}{2c} = -\frac{6}{c} = -\frac{6}{40} = -\frac{3}{20}$$

$$c_l = 2\pi(\alpha - \alpha_{L=0}) = 2\pi\left(\alpha + \frac{3}{20}\right)$$

Moment coefficient

$$c_{m_{c/4}} = \frac{\pi}{4}(A_2 - A_1) = \frac{\pi}{4}\left(0 - \frac{12}{c}\right) = -\frac{3\pi}{c} = -\frac{3\pi}{40}$$

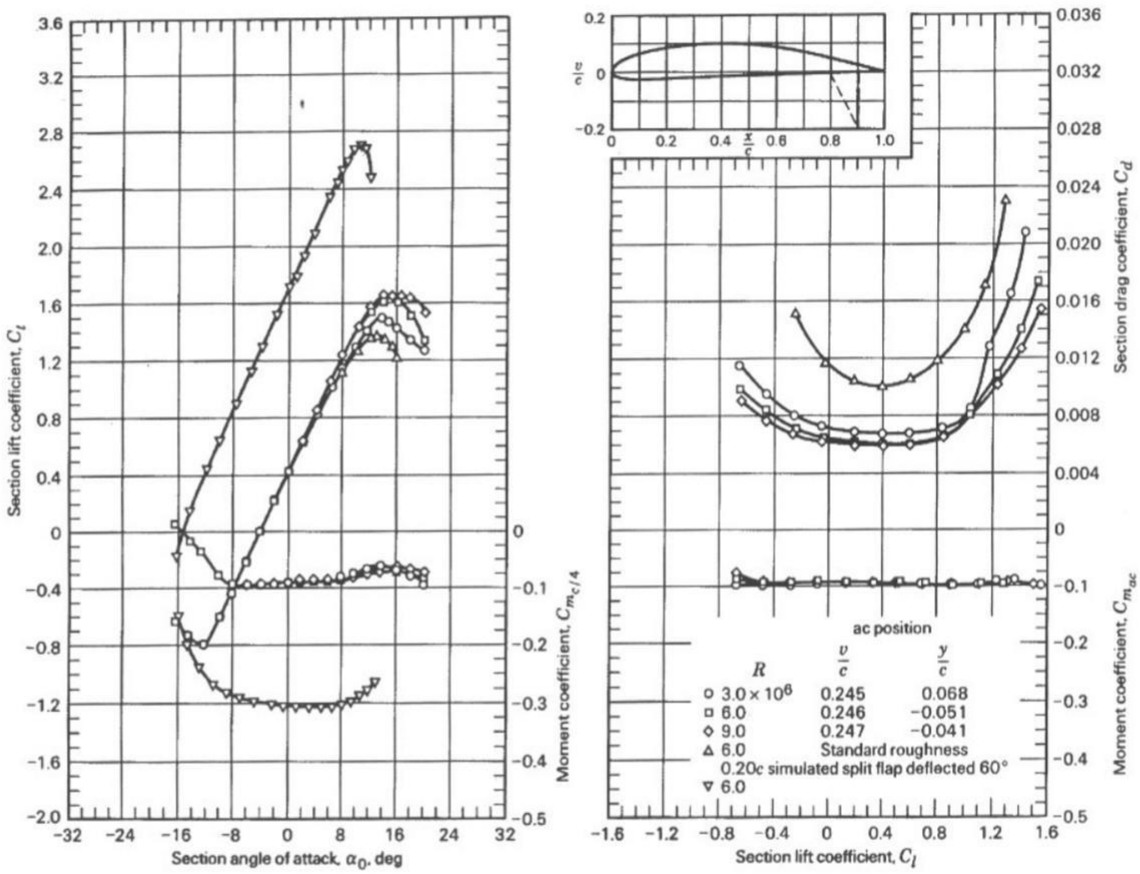
$$c_{m_{c/4}} = -0.2356$$

Problem#5: The NACA 4412 airfoil has a mean camber line given by:

$$\frac{\eta_c}{c} = \begin{cases} 0.25\left[0.8\frac{x}{c} - \left(\frac{x}{c}\right)^2\right] & \text{for } 0 \le \frac{x}{c} \le 0.4 \\ 0.111\left[0.2 + 0.8\frac{x}{c} - \left(\frac{x}{c}\right)^2\right] & \text{for } 0.4 \le \frac{x}{c} \le 1 \end{cases}$$

Using thin airfoil theory, calculate

- $\alpha_{L=0}$ and c_l when α=3°.
- $c_{m,c/4}$ and x_{cp}/c for α=3°.
- Compare the results of part (a) and (b) with experimental data of NACA 4412 airfoil given below.
- Lift per unit length of span and circulation for an airfoil with chord length of 2m flying at a standard altitude of 3 km and velocity of 60 m/s (same angle of attack of 3°).



Aerodynamic characteristics of the NACA 4412 airfoil.

Solution: **(a).**

$$\frac{d\eta_c}{dx}=\frac{d(\eta_c/c)}{d(x/c)}=\begin{cases}0.25\left[0.8-2\left(\frac{x}{c}\right)\right]=0.2-0.5\frac{x}{c} & \text{for } 0\le\frac{x}{c}\le 0.4\\ 0.111\left[0.8-2\left(\frac{x}{c}\right)\right]=0.089-0.222\frac{x}{c} & \text{for } 0.4\le\frac{x}{c}\le 1\end{cases}$$

For $\frac{x}{c}=0.4$ & $\frac{x}{c}=\frac{1}{2}(1-\cos\theta)\to\cos\theta=0.2\to\theta=1.369\,rad$

$$\frac{d\eta_c}{dx}=\begin{cases}0.2-0.25(1-\cos\theta)=-0.05+0.25\cos\theta & \text{for } 0\le\theta\le 1.369\\ 0.089-0.111(1-\cos\theta)=-0.0223+0.111\cos\theta & \text{for } 1.369\le\theta\le\pi\end{cases}$$

$$A_0=\frac{1}{\pi}\int_0^{\pi}\frac{d\eta_c(\theta)}{dx}d\theta$$
$$=\frac{1}{\pi}\int_0^{1.369}(-0.05+0.25\cos\theta)d\theta$$
$$+\frac{1}{\pi}\int_{1.369}^{\pi}(-0.0223+0.111\cos\theta)d\theta=0.0089$$

$$A_1=\frac{2}{\pi}\int_0^{\pi}\frac{d\eta_c(\theta)}{dx}\cos\theta\,d\theta$$
$$=\frac{1}{\pi}\int_0^{1.369}(-0.05+0.25\cos\theta)\cos\theta\,d\theta$$
$$+\frac{1}{\pi}\int_{1.369}^{\pi}(-0.0223+0.111\cos\theta)\cos\theta\,d\theta=0.163$$

$$A_2=\frac{2}{\pi}\int_0^{\pi}\frac{d\eta_c(\theta)}{dx}\cos 2\theta\,d\theta$$
$$=\frac{1}{\pi}\int_0^{1.369}(-0.05+0.25\cos\theta)\cos 2\theta\,d\theta$$
$$+\frac{1}{\pi}\int_{1.369}^{\pi}(-0.0223+0.111\cos\theta)\cos 2\theta\,d\theta=0.0277$$

$$\alpha_{L=0}=A_0-\frac{A_1}{2}=0.0089-\frac{0.163}{2}=-0.0726\,rad$$
$$\alpha_{L=0}=-4.16^\circ$$

$$c_l=2\pi(\alpha+0.0726)$$

For $\alpha=3^\circ=0.0524\,rad\to c_l=0.7854$

b.

$$c_{m_{c/4}} = \frac{\pi}{4}(A_2 - A_1) = \frac{\pi}{4}(0.0277 - 0.163) = -0.1063$$

$$\frac{x_{cp}}{c} = \frac{1}{4}\left[1 + \frac{\pi}{c_l}(A_1 - A_2)\right] = \frac{1}{4}\left[1 + \frac{\pi}{0.7854}(0.163 - 0.0277)\right]$$

$$\frac{x_{cp}}{c} = 0.3853$$

c.

Comparison with experimental data

	Inviscid theory	**Experiment**	**Error (%)**
c_l	0.7854	0.76	3.3
$c_{m_{c/4}}$	-0.1063	-0.095	11.9

d.

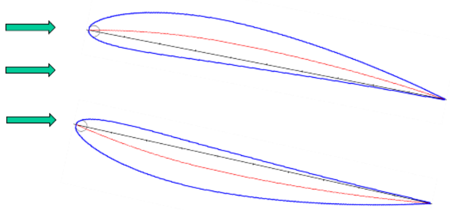
$$L' = \frac{1}{2}\rho_\infty V_\infty^2 c_l c, \qquad \rho_\infty = 0.9093\frac{kg}{m^3} \quad \text{(standard atmosphere h=3 km)}$$

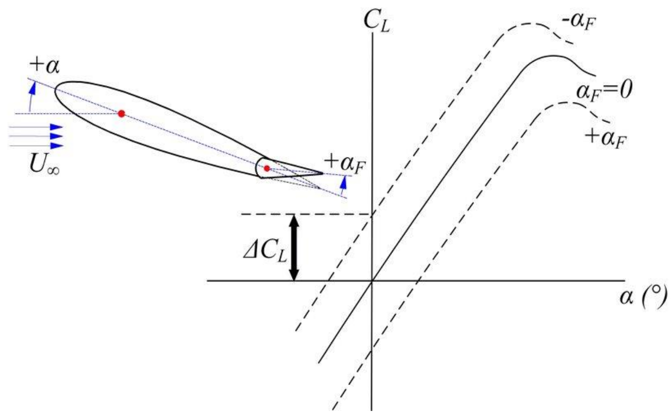
$$L' = \frac{1}{2}(0.9093)(60^2)(0.7854)(2) = 2571\frac{N}{m}$$

$$L' = \rho_\infty V_\infty \Gamma \rightarrow \Gamma = \frac{L'}{\rho_\infty V_\infty} = \frac{2571}{0.9090 \times 60} = 47.12\frac{m^2}{s}$$

Problem#6: The question is often asked: Can an airfoil fly upside-down? To answer this, make the following calculation. Consider a positively cambered airfoil with a zero-lift angle of -3°. The lift slope is about 0.1 per degree.

a) Calculate the lift coefficient at an angle of attack of 5°.
b) Now imagine the same airfoil turned upside-down, but at the same 5° angle of attack as part (a). Calculate its lift coefficient.
c) At what angle of attack must the upside-down airfoil be set to generate the same lift as that when it is right-side-up at a 5° angle of attack?





Solution:

a. The lift coefficient for standard airfoil is

$$c_l = 0.1(\alpha + 3), \qquad \alpha \text{ in degrees}$$

$$\alpha = 5^\circ \rightarrow c_l = 0.1(5+3) \rightarrow c_l = 0.8$$

b. For upside-down airfoil lift coefficient is

$$c_l = -0.1(-\alpha + 3)$$

$$\alpha = 5^\circ \rightarrow c_l = -0.1(-5+3) = 0.2 \rightarrow c_l = 0.2$$

c. To generate same 0.8 lift coefficient

$$0.8 = -0.1(-\alpha + 3) \rightarrow \alpha = 11^\circ$$