AerE310: Incompressible Aerodynamics

Homework Problem Set #1:

<u>Due</u>: 5:00 PM, Friday, 02/02/2024

1. Expand following terms:

a). $\frac{d}{dt}(\vec{A}\cdot(\vec{B}\times\vec{C}))$

b). $\frac{d}{dt}(\vec{A}\times(\vec{B}\times\vec{C}))$

Solution:

a.

$$\begin{aligned}\frac{d}{dt}(\vec{A}\cdot(\vec{B}\times\vec{C})) &= \frac{d\vec{A}}{dt}\cdot(\vec{B}\times\vec{C}) + \vec{A}\cdot\frac{d}{dt}(\vec{B}\times\vec{C}) \\ &= \frac{d\vec{A}}{dt}\cdot(\vec{B}\times\vec{C}) + \vec{A}\left(\frac{d\vec{B}}{dt}\times\vec{C}\right) + \vec{A}\cdot\left(\vec{B}\times\frac{d\vec{C}}{dt}\right)\end{aligned}$$

b.

$$\frac{d}{dt}(\vec{A}\times(\vec{B}\times\vec{C})) = \frac{d\vec{A}}{dt}\times(\vec{B}\times\vec{C}) + \vec{A}\times\left(\frac{d\vec{B}}{dt}\times\vec{C}\right) + \vec{A}\times\left(\vec{B}\times\frac{d\vec{C}}{dt}\right)$$

3. Find $\nabla\Phi$ if

a). $\Phi = \ln|\vec{r}|$

b). $\Phi = \frac{1}{r}$

(Hint: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ in Cartesian coordinate system)

Solution:

a).

$$\Phi = \ln|\vec{r}| = \frac{1}{2}\ln(x^2 + y^2 + z^2)$$

$$\nabla\Phi = \nabla[\frac{1}{2}\ln(x^2 + y^2 + z^2)] = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{x^2 + y^2 + z^2}$$

$$= \frac{\vec{r}}{r^2}$$

b).

$$\Phi = \frac{1}{r} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

$$\nabla\Phi = \nabla(\frac{1}{\sqrt{x^2 + y^2 + z^2}}) = -\frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

$$= -\frac{\vec{r}}{r^3}$$

4. Find directional derivative of $\Phi = x^2 yz + 4xz^2$ at point (1, -2, 1) in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$.

Solution:

$$\phi = x^2yz + 4xz^2 \quad \nabla\phi = \left(2xyz + 4z^2, x^2z, x^2y + 8xz\right)$$

at puint $(1, -2, 1)$, $\nabla\phi = (0, 1, 6)$ unit vector for $(2, -1, -2)$ is $\left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$ the direction dervative is $(0, 1, 6) \cdot \left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right) = -4\frac{1}{3}$

5. If $\vec{R} = \vec{R}(t) = r\hat{e}_r + z\hat{e}_z$ is the position vector of a particle in cylindrical coordinates, Obtain expression for velocity vector, $\vec{V}$, and acceleration vector, $\vec{a}$, at that point.

Solution:

$\because \vec{R} = r\hat{e}_r + z\hat{e}_r$

$\therefore \vec{V} = \frac{d\vec{R}}{dt} = \frac{d(r\hat{e}_r+z\hat{e}_r)}{dt} = \frac{dr}{dt}\hat{e}_r + r\frac{d\hat{e}_r}{dt} + \frac{dz}{dt}\hat{e}_z + z\frac{d\hat{e}_z}{dt}$

$\because \hat{e}_r, \hat{e}_z, \hat{e}_\theta$ is the functions of the position (r, θ, z), and the position (r, θ, z) are the functions of time, t.

$\therefore$

$$\frac{d\hat{e}_r}{dt} = \frac{\partial\hat{e}_r}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial\hat{e}_r}{\partial\theta}\frac{\partial\theta}{\partial t} + \frac{\partial\hat{e}_r}{\partial z}\frac{\partial z}{\partial t} = 0 + \hat{e}_\theta\frac{\partial\theta}{\partial t} + 0 = \hat{e}_\theta\frac{\partial\theta}{\partial t}$$

$$\frac{d\hat{e}_\theta}{dt} = \frac{\partial\hat{e}_\theta}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial\hat{e}_\theta}{\partial\theta}\frac{\partial\theta}{\partial t} + \frac{\partial\hat{e}_\theta}{\partial z}\frac{\partial z}{\partial t} = 0 + (-\hat{e}_r)\frac{\partial\theta}{\partial t} + 0 = -\hat{e}_r\frac{\partial\theta}{\partial t}$$

$$\frac{d\hat{e}_z}{dt} = \frac{\partial\hat{e}_z}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial\hat{e}_z}{\partial\theta}\frac{\partial\theta}{\partial t} + \frac{\partial\hat{e}_z}{\partial z}\frac{\partial z}{\partial t} = 0 + 0 + 0 = 0$$

Defining: $V_r = \frac{dr}{dt}$; $V_\theta = \frac{d\theta}{dt}$; $V_Z = \frac{dz}{dt}$ Then:

$$\vec{V} = \frac{d\vec{R}}{dt} = \frac{dr}{dt}\hat{e}_r + r\frac{d\hat{e}_r}{dt} + \frac{dz}{dt}\hat{e}_z + z\frac{d\hat{e}_z}{dt}$$
$$= V_r\hat{e}_r + rV_\theta\hat{e}_\theta + V_z\hat{e}_z$$

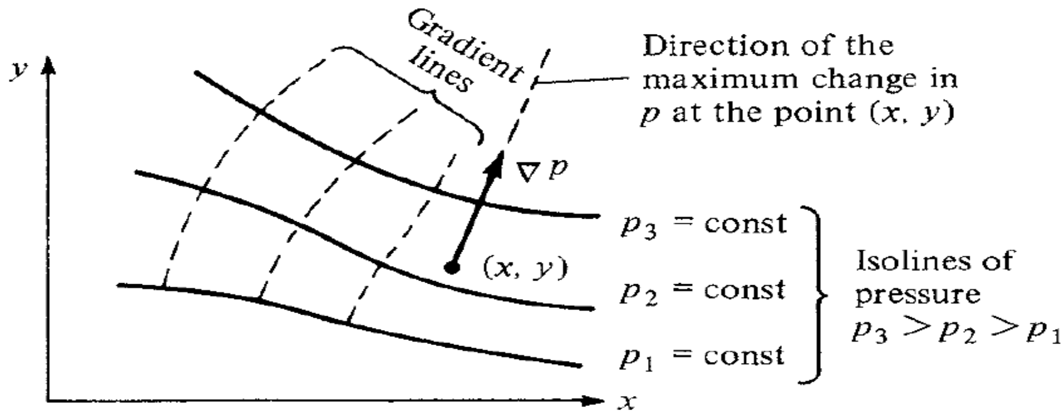
Defining: $a_r = \frac{dV_r}{dt}$; $a_\theta = \frac{dV_\theta}{dt}$; $a_Z = \frac{dV_z}{dt}$ Acceleration vector:

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{dV_r}{dt}\hat{e}_r + V_r\frac{d\hat{e}_r}{dt} + \frac{dr}{dt}V_\theta\hat{e}_\theta + r\frac{dV_\theta}{dt}\hat{e}_\theta + rV_\theta\frac{d\hat{e}_\theta}{dt} + \frac{dV_z}{dt}\hat{e}_z + V_z\frac{d\hat{e}_z}{dt}$$
$$= a_r\hat{e}_r + rV_rV_\theta\hat{e}_\theta + V_rV_\theta\hat{e}_\theta + ra_\theta\hat{e}_\theta + rV_\theta(-\hat{e}_rV_\theta) + a_z\hat{e}_z + 0$$
$$= \left[a_r - rV_\theta^2\right]\hat{e}_r + [2rV_rV_\theta + ra_\theta]\hat{e}_\theta + a_z\hat{e}_z$$

6. Show that the directions of the isoline and the gradient line at any given points in a scalar field are orthogonal to each other. (Hint: use the concept of directional derivative)

Solution:

Consider an isoline with the scalar ϕ = const. If the direction of the isoline be denoted by "s" at a point, then the directional derivative of ϕ in the direction of "s" is given by: $\frac{d\phi}{ds} = \nabla\phi \cdot \hat{e}_s$ where $\hat{l}_s$ is a unit vector in the direction of $\hat{s}$ $\frac{d\phi}{ds} = 0$, since ϕ does not vary on the isoline along which ds is taken. $\therefore$ $\nabla\phi \cdot \hat{e} = 0$. Therefore $\nabla\phi$ and $\hat{e}$ are orthogonal.



7. Spherical coordinate (R, φ, θ) are defined by the following inverse transformation:

$$x = (R\sin\varphi)\cos\theta$$
$$y = (R\sin\varphi)\sin\theta$$
$$z = R\cos\varphi$$

Where

$$0 \le R \le \infty$$
$$0 \le \theta \le 2\pi$$
$$0 \le \varphi \le 2\pi$$

(a). Obtain the scale factors for the spherical coordinate system.

(b). Obtain the unit vectors in spherical system as the function of Cartesian unit vectors.

(c). Obtain the derivatives of the unit vectors with respect to spherical coordinate directions and simplify the results to be only functions of spherical coordinates.

(d). Using vector algebra to obtain the divergence of a general vector in spherical coordinates. Simplify the results to be in conservation form.

Solution

a.

$$h_k = 1 \quad h\varphi = R \quad h_\theta = k\sin\varphi$$

b.

$$\hat{e}_R = \sin\varphi\cos\theta\hat{\imath} + \sin\varphi\sin\theta\hat{\jmath} + \cos\varphi\hat{k}$$
$$\hat{e}_\varphi = \cos\varphi\cos\theta\hat{\imath} + \cos\varphi\sin\theta\hat{\jmath} - \sin\varphi\hat{k}$$
$$\hat{e} = -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}$$

c.

$$\frac{\partial\hat{e}_n}{\partial k} \equiv 0 \quad \frac{\partial\hat{e}_n}{\partial\varphi} \equiv \hat{e}_\varphi \quad \frac{\partial\hat{e}_R}{\partial\theta} \equiv \sin\varphi\hat{e}_\theta$$
$$\frac{\partial\hat{e}_\varphi}{\partial R} \equiv 0 \quad \frac{\partial\hat{\varphi}_\varphi}{\partial\varphi} \equiv -\hat{e}_k \quad \frac{\partial\hat{e}_\varphi}{\partial\theta} \equiv \cos\varphi\hat{e}_\theta$$
$$\frac{\partial\hat{e}}{\partial R} \equiv 0 \quad \frac{\partial\hat{e}_0}{\partial\varphi} \equiv 0 \quad \frac{\partial\hat{e}_\theta}{\partial\theta} = -\left(\sin\varphi\hat{e}_k + \cos\varphi\hat{e}_\varphi\right)$$

d.

$$\nabla\cdot\vec{v} = \left(\hat{e}_n\frac{\partial}{\partial R} + \frac{\hat{e}_\varphi}{R}\frac{\partial}{\partial\varphi} + \frac{\hat{e}_\theta}{R\sin\varphi}\frac{\partial}{\partial\theta}\right)\cdot\left(V_R\hat{e}_k + V_\varphi\hat{e}_\varphi + V_\theta\hat{C}_\theta\right)$$
$$= \hat{e}_R\frac{\partial}{\partial R}\cdot\left(V_R\hat{e}_k + V_\varphi\hat{e}_\varphi + V_\theta\hat{\theta}_\theta\right) + \frac{\hat{e}_\varphi\partial}{R}\frac{\partial}{\partial\varphi}\cdot\left(v_R\hat{e}_k + v_\varphi\hat{e}_\varphi + v_\theta\hat{\theta}_\theta\right) + \frac{\hat{e}_0}{k\sin\varphi}\frac{\partial}{\partial\varphi}\cdot\left[W_R\hat{e}_n + V_\varphi\hat{e}_\varphi + v_0\hat{e}_\theta\right]$$
$$= \left(\frac{\partial V_k}{\partial R} + \frac{V_k}{R} + \frac{V_R}{R}\right) + \left(\frac{1}{R}\frac{\partial V_\varphi}{\partial\varphi} + \frac{V\varphi\cos\varphi}{R\sin\varphi}\right) + \left(\frac{1}{R\sin\varphi}\frac{\partial V_\theta}{\partial\theta}\right)$$
$$= \frac{1}{R^2}\left(\frac{\partial R^2V_R}{\partial R}\right) + \frac{1}{R\sin\varphi}\frac{\partial}{\partial\varphi}(V\varphi\sin\varphi) + \frac{1}{\mathrm{R}\sin\varphi}\frac{\partial V_\theta}{\partial\theta}$$
$$= \frac{1}{R^2\sin\varphi}\left[\frac{\partial\left(R^2\sin\varphi V_R\right)}{\partial R} + \frac{\partial\left(V_\varphi\sin\varphi\right)}{\partial\varphi} + \frac{\partial\left(RV_\theta\right)}{\partial\theta}\right]$$

8. Find the acceleration of a fluid particle at (r, θ, z) in cylindrical coordinate system.

Solution:

$$\vec{a} = \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v}$$

$$\vec{v} \cdot \nabla = (v_k \hat{e}_k + v_\theta \hat{e}_\theta + v_z \hat{e}_z) \cdot \left(\hat{e}_R \frac{\partial}{\partial R} + \frac{\hat{C}_\theta}{R} \frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z} \right)$$

$$= V_R \frac{\partial}{\partial R} + V_\theta \frac{\partial}{\partial \theta} + V_z \frac{\partial}{\partial z}$$

$$\vec{v} \cdot \nabla \vec{v} = \left(V_R \frac{\partial}{\partial R} + \frac{V_\theta}{R} \frac{\partial}{\partial \theta} + V z \frac{\partial}{\partial z} \right) (V_R \hat{e}_R + V_\theta \hat{e}_\theta + v_z \hat{e}_z)$$

$$= V_R \frac{\partial V}{\partial R} \hat{e}_k + V_R \frac{\partial V_\theta}{\partial R} \hat{e}_\theta + V_R \frac{\partial V_z}{\partial R} \hat{e}_z + 0 + 0 + 0$$

$$+ \frac{V_\theta}{R} \frac{\partial V_R}{\partial \theta} \hat{e}_R + \frac{V_\theta V_R}{R} \hat{e}_\theta + \frac{V_\theta}{R} \frac{\partial V_\theta}{\partial \theta} \hat{e}_\theta + \frac{{V_\theta}^2}{R} (-\hat{e}_\theta)$$

$$+ \frac{V_\theta}{R} \frac{\partial V_z}{\partial \theta} \hat{e}_z + 0 + V_z \frac{\partial V_k}{\partial z} \hat{e}_x + v_z \frac{\partial v_0}{\partial z} \hat{e}_\theta + V_z \frac{\partial V_z}{\partial z} \hat{e}_z + 0 + 0 + 0$$

Using the derivatives of the unit vectors.

$$a_R = \frac{\partial V_R}{\partial t} + V_R \frac{\partial V_R}{\partial R} + \frac{V_\theta}{R} \frac{\partial V_R}{\partial \theta} + V_z \frac{\partial V_R}{\partial z} - \frac{V_\theta^2}{R}$$

$$a_\theta = \frac{\partial V_\theta}{\partial t} + V_R \frac{\partial V_\theta}{\partial \gamma} + \frac{V_O V_R}{R} + \frac{V_\theta}{R} \frac{\partial V_\theta}{\partial \theta} + V_z \frac{\partial v \theta}{\partial z}$$

$$a_z = \frac{\partial v_2}{\partial t} + \psi k \frac{\partial V_z}{\partial k} + \frac{V_\theta \partial V_z}{R} + V_z \frac{\partial V_z}{\partial z}$$