AerE310: Incompressible Aerodynamics

Homework Problem Set #02:

<u>Due</u>: 5:00 PM, Friday, 02/16/2024

1. A ball is being inflated with an air supply of 0.6 m^3/s. Find the rate of growth of the radius at the instant when R=0.5 m. (Hint: the air flow is assumed to be incompressible)

Solution:

Because of mass conservation and the flow is in-compressible.

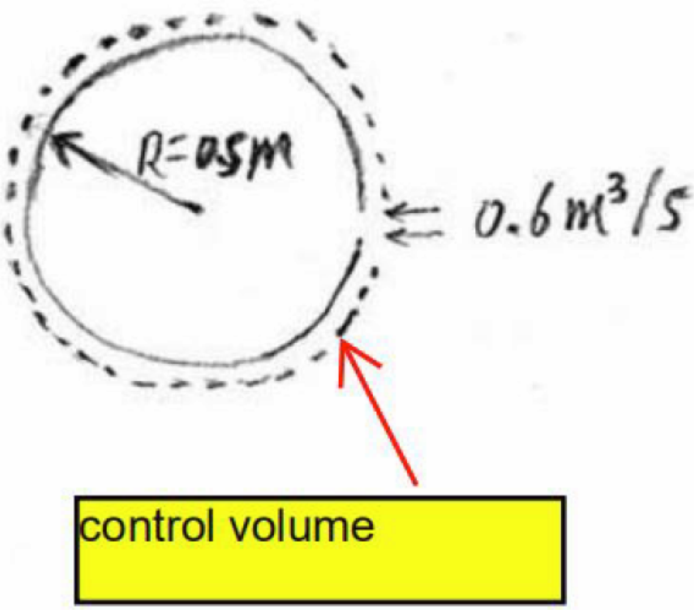
$$\frac{D}{Dt}(m_s) = 0 = \frac{1}{Dt}\int_{c_r} \rho d\forall + \int_{cs} \rho \vec{v} d\vec{A}$$

$$\rho \frac{\partial \forall}{\partial t} - \rho \vec{V}\vec{A} = 0 \qquad \frac{\partial \forall}{\partial t} = \vec{V} \cdot \vec{A}$$

$$\vec{v} \cdot \vec{A} = 0.6 \text{ m}^3/\text{s} \qquad \frac{\partial \forall}{\partial t} = 0.6 \text{ m}^3/\text{s}$$

$$\forall = \frac{4}{3}\pi r^3 \qquad \frac{\partial \forall}{\partial t} = 4\pi r^2 \frac{\partial r}{\partial t}$$

$$\frac{\partial r}{\partial t} = \frac{0.6}{4\pi(as)^2} = 0.191(\text{ m/s})$$



2. Which of the following flows are physically possible, that is, satisfy the continuity equation? Substitute the expression for density and for the velocity field into the continuity equation to substantiate your answer.

(a). Water, which has a density of ρ =1.0 g/cm^3, is following radically outward from a source in a plan such that $\vec{V} = (K/2\pi r)\hat{e}_r$. Note that $u_\theta = u_z = 0$. Note also that, in cylindric coordinates,

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \frac{\hat{e}_\theta}{r}\frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}$$

(b). A gas is flowing at relatively low speeds (so that its density may be assumed constant) where the velocity can be expressed as:

$$u = -\frac{2xyz}{(x^2+y^2)^2}U_\infty L$$

$$v = \frac{(x^2-y^2)z}{(x^2+y^2)^2}U_\infty L$$

$$w = \frac{y}{(x^2+y^2)}U_\infty L$$

Solution:

(a)

If the flow is steady and incompressible, the continuity equation can be written as $\nabla \cdot \vec{V} = 0$, in cylinder coordinates:

$$\frac{1}{r}\frac{\partial(rV_r)}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} = 0$$

since $\vec{y} = \frac{k}{2\pi r}\hat{e}_r$. we know that $v_r = \frac{k}{2\pi r}$; $v_\theta = 0$; $V_z = 0$. Substituting these components into the continuity equation $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{k}{2\pi r}\right) = 0$, Continuity is satisfied.

Let us use the continuity equation for a three-dimensional flow, i.e., equation (2.1):

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \overset{?}{=} 0$$

For constant density flow, this equation becomes:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Thus, $\frac{\partial}{\partial x}\left\{-\frac{2xyz}{(x^2+y^2)^2}U_\infty L\right\} + \frac{\partial}{\partial y}\left\{\frac{(x^2-y^2)z}{(x^2+y^2)^2}U_\infty L\right\}$

$$+ \frac{\partial}{\partial z}\left\{\frac{y}{x^2+y^2}U_\infty L\right\} = 0$$

Since U_∞ and L are constants and since they appear in every term, they can be divided out leaving:

$$-\frac{2yz}{(x^2+y^2)^2} - \frac{2xyz(-2)(2x)}{(x^2+y^2)^3} - \frac{2yz}{(x^2+y^2)^2} + \frac{(x^2-y^2)z(-2)(2y)}{(x^2+y^2)^3}$$

$$= -\frac{4yz}{(x^2+y^2)^2} - \frac{-8x^2yz + 4x^2yz - 4y^3z}{(x^2+y^2)^3}$$

$$= \frac{-4x^2yz - 4y^3z + 8x^2yz - 4x^2yz + 4y^3z}{(x^2+y^2)^3} = 0$$

Therefore, the continuity equation is satisfied.

3. Consider the velocity field of $\vec{V} = -\frac{x}{2t}\hat{i}$ in a compressible flow where $\rho = \rho_0 xt$. What is the total acceleration of a fluid particle at (1, 1, 1) at the time of $t = 10$?

Solution:

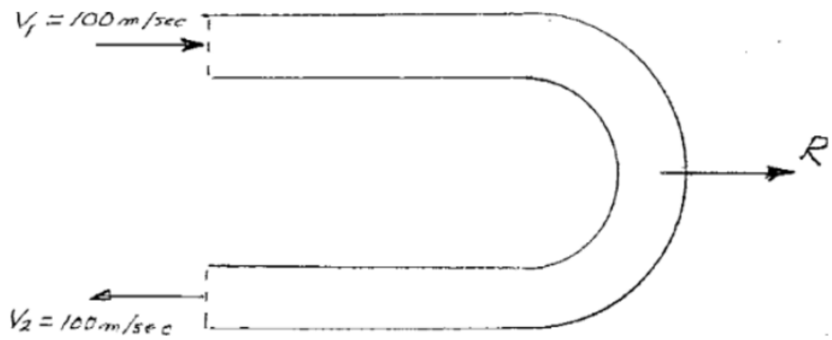
$$\vec{v} = -\frac{x}{2t}\hat{i}; \quad \rho = \rho x t$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + u\frac{\partial \vec{v}}{\partial x} + v\frac{\partial \vec{v}}{\partial y} + w\frac{\partial \vec{v}}{\partial z}$$

$$\vec{a} = \frac{x}{2t^2}\hat{i} + \left[-\frac{x}{2t}\right]\left[-\frac{1}{2t}\hat{i}\right] = \frac{x}{2t^2}\hat{i} + \frac{x}{4t^2}\hat{i} = \frac{3x}{4t^2}\hat{i}$$

$$textat(1,1,1) \text{ at time } t = 10 \quad \vec{a} = 0.0075\hat{i}$$

4. Consider a length of pipe bent into a U-shape. The inside diameter of the pipe is 0.5m. Air enters on leg of the pipe at a mean velocity of 100 m/s and exit the other leg at the same magnitude of the velocity but moving in the opposite direction. The pressure of the flow at the inlet and exit is the ambient pressure of the surroundings. Calculate the magnitude and direction of the force exerted on the pipe by the airflow. The air density is 1.23kg/m^3.



Solution:

$$\Sigma F = \frac{\partial}{\partial t}\int_{cv} \rho \vec{v} dv + \int_{cs} \vec{v}\rho\vec{v} d\vec{A}$$

$$\Sigma F_X = \int_{A_1} \vec{v}\rho\vec{v} dA + \int_{A_2} \vec{v}\rho\vec{v} dA$$

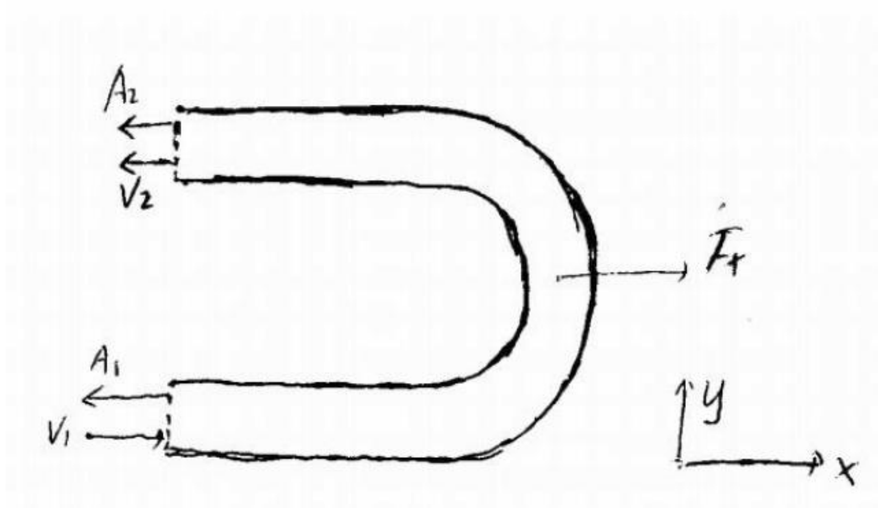
$$F_x = -v_1^2\rho_1 A_1 - v_2^2\rho_2 A_2$$

$$= -2v_1^2\rho_1 A_1$$

$$= -2 \times 100^2 \times 1.23 \times \frac{\pi}{4}(0.5)^2$$

$$= -4830.2 \text{ N}$$

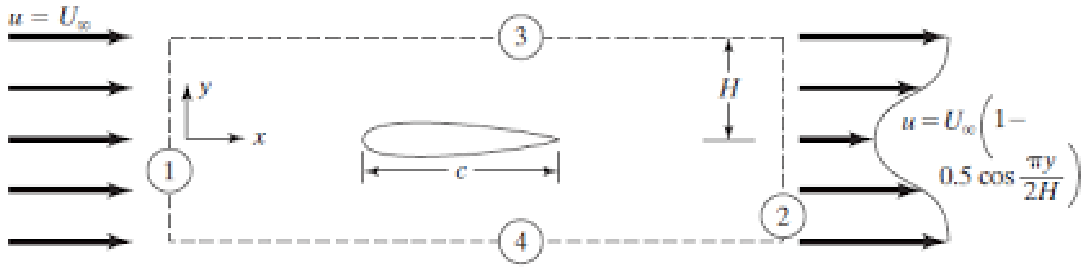
$$F_x = 4830.2 \text{ N} \quad (\longleftarrow)$$



5. Velocity profiles are measured at the upstream end (surface 1) and at the downstream end (surface 2) of a rectangular control volume, as shown in the figure below. If the flow is incompressible, two dimensional, and steady, please show you work to answer following questions:

(a). What is the total volumetric flow rate (i.e., $\iint \vec{V} \bullet \hat{n} dA$) across the horizontal surfaces (i.e., surfaces 3 and 4).

(b). What is the drag coefficient for the airfoil? The vertical dimension H is 0.025c (i.e., H = 0.025c). The pressure is P_∞ (a constant) over the entire surface of the control volume.



Solution:

(a).

The flow is steady and incompressible. As a result, the integral continuity equation becomes

$$\oiint \vec{V}\cdot\hat{n}\,dA = 0$$

$$\underset{①}{\int_{-H}^{H}(U_\infty\hat{i})\cdot[-\hat{i}dy]} + \underset{②}{\int_{-H}^{+H}[U_\infty(1-0.5\cos\tfrac{\pi y}{2H})\hat{i} + v\hat{j}]\cdot[\hat{i}dy]}$$

$$+ \underset{③}{\int_0^L [U_\infty\hat{i} + V_\infty\hat{j}]\cdot[\hat{j}dx]} + \underset{④}{\int_0^L [U_\infty\hat{i} - V_\infty\hat{j}][-\hat{j}]dx} = 0 \qquad 2/4$$

$$-U_\infty(2H) + U_\infty\left[y - 0.5\frac{2H}{\pi}\sin\frac{\pi y}{2H}\right]\Big|_{-H}^{+H} + \int_0^L V_\infty dx + \int_0^L V_\infty dx = 0$$

So the total volumetric flow across surfaces ③ and ④ is

$$\left[2\int_0^L V_\infty dx\right] = 2HU_\infty - 2HU_\infty + \frac{HU_\infty}{\pi}[1-(-1)]$$

$$2\int_0^L V_\infty dx = \frac{2HU_\infty}{\pi}$$

(b).

2.21 The integral form of the momentum equation.

$$\Sigma F_x = \frac{\partial}{\partial t}\iiint \rho V_x d(Vol) + \oiint \rho (\vec{V}\cdot\hat{n}dA) V_x$$

Since the only force acting on the fluid particles within the control volume is the negative of the drag and the flow is steady.

$$-d = +\rho\int_{-H}^{H} [(U_\infty\hat{i})\cdot(-\hat{i}dy)] U_\infty + \rho\int_{-H}^{H}\{[U_\infty(1-0.5\cos\frac{\pi y}{2H})\hat{i} \pm V\hat{j}]\cdot(\hat{i}dy)\} [U_\infty(1-0.5\cos\frac{\pi y}{2H})] + 2\rho\int_0^L (U_\infty\hat{i}+V_\infty\hat{j})\cdot(\hat{j}dx)] U_\infty$$

$$-d = \rho U_\infty^2 y|_{-H}^{H} + \rho U_\infty^2\int_{-H}^{H} [1-\cos\frac{\pi y}{2H} + 0.25\cos^2\frac{\pi y}{2H}] dy + 2\rho U_\infty\int_0^L V_\infty dx$$

from 2.9. we get $2\rho\int_0^L V_\infty dx = \rho U_\infty \frac{2H}{\pi}$. Thus

$$-d = -\rho U_\infty^2(2H) + \rho U_\infty^2 [y|_{-H}^{H} - \frac{2H}{\pi}\sin\frac{\pi y}{2H}|_{-H}^{H} + \frac{y}{8}|_{-H}^{H} + \frac{H}{8\pi}\sin\frac{\pi y}{H}|_{-H}^{H}] + \rho U_\infty^2\frac{2H}{\pi}$$

$$-d = -\rho U_\infty^2 H(\frac{4}{\pi} - \frac{1}{4} - \frac{2}{\pi}) = -\rho U_\infty^2\frac{C}{40}[\frac{2}{\pi} - \frac{1}{4}]$$

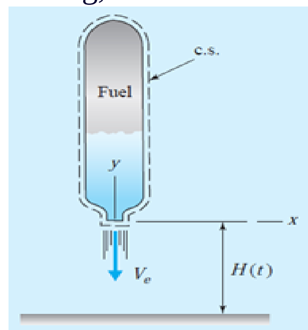
$$C_d = \frac{d}{\frac{1}{2}\rho U_\infty^2 C} = 0.01933$$

2. As shown in the following figure, a rocket with an initial mass of 150 kg, burns fuel at the rate of 10 kg/s with a constant exhaust velocity of 700 m/s. Please show your work to determine

   a). What is the initial acceleration of the rocket?

   b). What is the velocity after 1 s?

   Note: Neglect the drag on the rocket.



Solution:

The control volume is sketched and includes the entire rocket. The reference frame attached to the rocket is accelerating upward at d^2H/dt^2. Newton's second law is written as, using z upward,

$$\Sigma F_z - (F_I)_z = \frac{d}{dt}\int_{\text{c.v.}} \rho V_z \, dV + \int_{\text{c.s}} \rho V_z \mathbf{V}\cdot\hat{\mathbf{n}}\, dA$$

$$\therefore -W - \frac{d^2H}{dt^2} m_{\text{c.v}} = \rho_e(-V_e)V_eA_e$$

where $\quad \frac{d}{dt}\int_{\text{cv.}} \rho V_z \, dV \approx 0$

since V_z is the velocity of each mass element $\rho\, dV$ relative to the reference frame attached to the control volume; the only vertical force is the weight W; and $m_{\text{c.v.}}$ is the mass of the control volume. From continuity we see that

$$m_{\text{c.v.}} = 150 - \dot{m}t = 150 - 10t$$

$$\therefore W = (150 - 10t) \times 9.81$$

The momentum equation becomes

$$-(150-10t)\times 9.81 - \frac{d^2H}{dt^2}(150-10t) = -\dot{m}_eV_e = -10\times 700 = -7000$$

This is written as

$$\frac{d^2H}{dt^2} = \frac{700}{15-t} - 9.81$$

The initial acceleration is found by letting $t = 0$:

$$\left.\frac{d^2H}{dt^2}\right|_{t=0} = \frac{700}{15} - 9.81 = 36.9\ \text{m/s}^2$$

Integrate the expression for d^2H/dt^2 and obtain

$$\frac{dH}{dt} = -700\ln(15-t) - 9.81t + C$$

The constant $C = 700\ln 15$ since $dH/dt = 0$ at $t = 0$. Thus at $t = 1$ s the velocity is

$$\frac{dH}{dt} = 700\ln\frac{15}{14} - 9.81\times 1 = 38.5\ \text{m/s}$$